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NONCOMMUTATIVE NOETHER'S PROBLEM FOR COMPLEX REFLECTION GROUPS

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Abstract. We solve the noncommutative Noether's problem for the reflection groups by showing that the skew field of the invariants of the Weyl algebra under the action of any reflection group is a Weyl field, that is isomorphic to a skew field of some Weyl algebra over a transcendental extension of the ground field. We also extend this result to the invariants of the ring of differential operators on any dimensional torus.The results are applied to obtain analogs of the Gelfand-Kirillov Conjecture for Cherednik algebras and Galois algebras.

1. Introduction

Let k be a field of characteristic 0. Let G be a finite subgroup of $\mathrm{GL}_n(k)$ acting naturally by linear automorphisms on $S := k[x_1, \ldots, x_n]$, and hence on the field of fractions $F := \mathrm{Frac}(S) = k(x_1, \ldots, x_n)$. It is easy to show $F^G = \mathrm{Frac}(S^G)$. Then a well-known result due to E.Artin claims that the transcendence degree of F^G over k is equal to n. Now one can ask the following natural question:

Noether's problem. Is F^G is a purely transcendental extension of k or equivalently is $F^G \cong F$?

This problem is been studied by many authors. Let us briefly recall some of the results. For more detailed discussion see [2].

It is known that the answer is positive for all $n \geq 1$ if G is abelian and k algebraically closed. For a general k this is no longer true, necessary and sufficient conditions were found in [7] The answer is also positive for all $n \geq 1$ when G is a complex reflection group, since it is a consequence of the Chevalley-Shepard-Todd theorem: $S^G \cong S$.

In general, for $n = 1$, the positive answer is a consequence of the classical theorem of Luroth. For $n = 2$, the positive answer is a consequence of Castelnuovo's theorem, while for $n = 3$ it was proved by Burnside using Miyata's theorem for k algebraically closed. However, some counterexamples were produced for $n > 3$ (see [2]).

Now, we will discuss non commutative version of Noether's problem. Let $A_n = A_n(k)$ be the n-th Weyl algebra with usual generators $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_n$, and let F_n be its skew field of fractions. Then the action of G on S naturally extends to A_n and to F_n. The following question was originally posed by J.Alev and F.Dumas (see [1], Section 1.2.2)

Noether's problem for A_n: Do we have $F_n^G \cong F_n$?
Let V be a finite dimensional vector space of dimension n over k. Fixing a basis in V one can identify $S(V^*)$ with $k[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ is the dual basis in V^*. If G be a finite subgroup of $GL(V)$ (more generally V is a G-module) then it acts on $S(V^*)$ by linear automorphisms: $g.f(v) = f(g^{-1}v)$, $g \in G$, $f \in S(V^*)$,

$v \in V$. This action can be naturally extended to the ring of differential operators $\mathcal{D}(k[x_1, \ldots, x_n])$ on $S(V^*)$. This induces a linear automorphism of the Weyl algebra A_n.

Theorem 1. [1] (a) *Let V be a representation of G which is a direct sum of n representations of dimension one. Then $F_n^G \cong F_n$.*
(b) *For any 2-dimensional representation of G, we have $F_2^G \cong F_2$.*

It follows from part (a) that we have the positive answer for all $n \geq 1$ if G is abelian and k is algebraically closed. We should also point out that in [1], this problem was discussed for the case when G is not necessarily a finite group. In this case the above question should be slightly modified.

To our knowledge, the only other case for which the Noether's problem for A_n has been considered is when $G \cong S_n$, a symmetric group, acting on S by permuting variables x_i. In [6], it was shown among other things that $F_n^{S_n} \cong F_n$.

Our first main result in this paper is

Theorem 2. *Let V be an n-dimensional vector space over $\mathbb{C}$ and W be a finite complex reflection subgroup of* $\mathrm{GL}(V)$*. Then $F_n^W \cong F_n$.*

Perhaps, this result is known to specialists but we could not find any proof. We are aware of an unpublished manuscript by I.Gordon where a similar statement is claimed without a proof.

Next we extend this technique to the study of the skew field of invariants for classical reflection groups in the case of any dimensional torus.

Our second main result is

Theorem 3. *Let $X = T^n$ be an n-dimensional torus, $D(X)$ the ring of differential operators on X, $F(D(X))$ the skew filed of fractions of $D(X)$, W a classical complex reflection group. Then there exists a natural action of W on $D(X)$ which extends to $F(D(X))$ and $F(D(X))^W \cong F_n$.*

As one of the applications of Theorem 2, we will show that an analogue of the Gelfand-Kirillov conjecture for Lie algebras holds for spherical subalgebras of rational Cherednik algebras $H_k := H_k(W)$ associated to W. We also discuss the Gelfand-Kirillov conjecture for a class of linear Galois algebras which includes the universal enveloping algebras of gl_n and sl_n.

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2. Invariant differential operators

In this section we assume that all varieties and algebras are defined over $\mathbb{C}$.

Let B be a commutative algebra. The ring of differential operators $\mathcal{D}(B)$ is defined to be $\mathcal{D}(B) = \cup_{n=0}^{\infty} \mathcal{D}(B)_n$, where $\mathcal{D}(B)_0 = B$ and

$$\mathcal{D}(B)_n = \{ d \in \mathrm{End}_{\mathbb{C}}(B) \, : \, d\,b - b\,d \in \mathcal{D}(B)_{n-1} \text{ for all } b \in B \}.$$

The ring $\mathcal{D}(B)$ is filtered. If G is a group acting on B, then this action can be extended naturally to $\mathcal{D}(B)$ and we can define $\mathcal{D}(B)^G$.

For an algebraic variety X with the ring of functions $\mathcal{O}(X)$, we define $\mathcal{D}(X) = \mathcal{D}(\mathcal{O}(X))$. Next, if $\phi : X \to Y$ be a dominant morphism, then we have $\mathcal{O}(Y) \hookrightarrow \mathcal{O}(X)$ and can define

$$\mathcal{D}(X,Y) := \{d \in \mathcal{D}(X) \,|\, d(\mathcal{O}(Y)) \subset \mathcal{O}(Y)\}.$$

We have the following important result by Knopp:

Theorem 4. [8, Theorem 3.1] *Let G be a finite group acting on X. Then*

$$\mathcal{D}(X, X/G) = \mathcal{D}(X)^G.$$

Definition. Let X and Y be normal varieties and $\phi : X \to Y$ a finite surjective morphism. Let $D \subset Y$ be a prime divisor and consider the divisor $\phi^{-1}(D) = r_1E_1 + \ldots + r_sE_s$, where the E_i are pairwise distinct prime divisors and $r_i > 0$. We say that ϕ is *uniformly ramified over D* if $r_1 = \ldots = r_s$. Moreover, it is uniformly ramified if it is uniformly ramified over every $D \subset Y$. If all the ramification numbers r_i are 1 for all D then we call ϕ *unramified in codimension 1.*

The simplest example of unramified in codimension 1 morphism is $X \to X/G$ where the action of G is free.

Theorem 5. [8, Proposition 3.2]. *Let $\phi : X \to Y$ be a finite dominant morphism between normal affine varieties that is unramified in codimension* 1*. Then $\mathcal{D}(X,Y) = \mathcal{D}(Y)$.*

Now combining Theorem 4 and 5 we have

Corollary 1. *Let X be a normal, irreducible, affine algebraic variety, and let G be a finite group acting freely on X. Then $\mathcal{D}(X)^G \cong \mathcal{D}(X/G)$.*

3. Proof of Theorem 2.

Let V be a finite-dimensional vector space over $\mathbb{C}$. An element $s \in \mathrm{GL}(V)$ is a *complex reflection* if it acts as identity on some hyperplane H_s in V. A finite subgroup W of $\mathrm{GL}(V)$ is called a complex reflection group if it is generated by its complex reflections. Let $(\cdot,\cdot)$ be a positive definite Hermitian form on V, which is invariant under the action of W. We may assume that $(\cdot,\cdot)$ is antilinear in the first argument and linear on its second argument: if $x \in V$, we write x^* for the linear form $V \to \mathbb{C}$, $v \mapsto (x, v)$.

Let $\mathcal{A} = \{H_s\}$ denote the set of reflection hyperplanes of W, corresponding to $s \in W$. The group W acts on $\mathcal{A}$ by permutations. If $H \in \mathcal{A}$, the (pointwise) stabilizer of H in W is a cyclic subgroup W_H of order n_H. Let α_H be a linear form for which H is the zero set. It is defined up to a constant. We set

$$\delta := \prod_{H\in\mathcal{A}} \alpha_H \quad , \quad J := \prod_{H\in\mathcal{A}} \alpha_H^{n_H-1}.$$

It is easy to show that $w.J = \det(w)J$ for any $w \in W$ (see [10, Exercise 4.3.5]). Let N be the order of W. Then $\triangle := J^N$ is an invariant polynomial.

We fix a basis $\{v_1, ..., v_n\}$ of V and let $\{x_1, ..., x_n\}$ be the corresponding dual basis of V^*. Then $S := \mathbb{C}[V] = \mathbb{C}[x_1, ..., x_n]$. Let S_δ be the localization of S by $\{1, \delta, \delta^2, ...\}$. Then $S_\delta = S_J = S_\triangle$ since they are just localizations by $\{\alpha_H^k\}_{k\geq 0, H\in\mathcal{A}}$. In fact, $S_\delta \cong \mathbb{C}[V^{\mathrm{reg}}]$, where $V^{\mathrm{reg}} := V \setminus \bigcup_{H\in\mathcal{A}} H$.

Lemma 1. *The action of W restricts to a free action on V^{reg} and on $\mathbb{C}[V^{\mathrm{reg}}]$.*

Proof. Assume that for some $w \in W$ and $v \in V^{\mathrm{reg}}$, wv belongs to a hyperplane fixed by some reflection s. Then $w^{-1}sw$ belongs to the isotropy group of v which is also a reflection group by the Steinberg's theorem. Since $v \in V^{\mathrm{reg}}$, we conclude that $s = id$, which is a contradiction. Hence, $wv \in V^{\mathrm{reg}}$ and this action is clearly free. $\square$

If we let $X := \mathrm{Spec}(S_{\triangle})$ and $Y := \mathrm{Spec}((S_{\triangle})^W)$, then both X and Y are normal, irreducible, affine algebraic varieties. Since the action of W on X is free, the dominant morphism $\phi : X \to Y$ is unramified in codimension 1. So we can use Corollary 1 to get

$$\mathcal{D}(S_{\triangle})^W \cong \mathcal{D}((S_{\triangle})^W)\,. \tag{1}$$

Proposition 1. *(i) If A is a domain and M is an Ore subset then* $\mathrm{Frac}(A_M) \cong \mathrm{Frac}(A)$.
(ii) $(S_{\triangle})^W \cong (S^W)_{\triangle}$.
(iii) $\mathcal{D}(S_{\triangle})^W \cong (\mathcal{D}(S)^W)_{\triangle}$.
(iv) $Frac(A_n)^W \simeq Frac(A_n^W)$.

Proof. *(i)* This statement is clear.
(ii) Since $\triangle$ is an invariant polynomial then $f \in (S_{\triangle})^W$ iff $\triangle^k f \in S^W$ for some $k \geq 0$ iff $f \in (S^W)_{\triangle}$.
(iii) Note that $\mathcal{D}(S_M) \cong D(S)_M$ for a multiplicative set M, [9, Theorem 15.1.25]. If $d \in \mathcal{D}(S_{\triangle})^W$ then $\triangle^k d \in \mathcal{D}(S)^W$ for some $k \geq 0$. Finally, (iv) follows from [4], Theorem 1, see also [2]. $\square$

Now $(S_{\triangle})^W \cong (S^W)_{\triangle} \cong S_{\triangle}$, where the first identity holds by part *(ii)* while the second one follows from the Chevalley-Shepard-Todd theorem. Therefore, the right hand side of (1) is isomorphic to $\mathcal{D}(S_{\triangle}) \cong \mathcal{D}(S)_{\triangle}$. Thus, using part *(ii)* we have

$$(\mathcal{D}(S)^W)_{\triangle} \cong \mathcal{D}(S)_{\triangle}\,.$$

Finally, taking the skew field of fractions on both sides, we obtain

$$F_n^W \cong F_n\,.$$

4. Gelfand-Kirillov conjecture for rational Cherednik algebras

Let us first recall the definition of rational Cherednik algebras. As before W is a finite complex reflection subgroup of $\mathrm{GL}(V)$ and $(\cdot,\cdot)$ is a W-invariant positive definite Hermitian form. For $H \in \mathcal{A}$, let $v_H \in V$ be such that $\alpha_H = v_H^*$. Next for each H from $\mathcal{A}$, we set

$$e_{H,i} := \frac{1}{n_H} \sum_{w \in W_H} (\det w)^{-i} w\,.$$

Since W_H is a cyclic group of order n_H, this is a complete set of orthogonal idempotents in $\mathbb{C}W_H$. Now, for $H \in \mathcal{A}$, we fix a sequence of non-negative integers $k_H = \{k_{H,i}\}_{i=0}^{n_H-1}$ so that $k_H = k_{H'}$ if H and H are on same orbit of W on $\mathcal{A}$.

The *rational Cherednik algebra* $H_k = H_k(W)$ is generated by elements $x \in V^*, \xi \in V$ and $w \in W$ subject to the following relations

$$[x, x'] = 0\,, \quad [\xi, \xi'] = 0\,, \quad w\,x\,w^{-1} = w(x)\,, \quad w\,\xi\,w^{-1} = w(\xi)\,,$$

$$[\xi, x] = \langle \xi, x\rangle + \sum_{H\in\mathcal{A}} \frac{\langle \alpha_H, \xi\rangle\,\langle x, v_H\rangle}{\langle \alpha_H, v_H\rangle} \sum_{i=0}^{n_H-1} n_H(k_{H,i} - k_{H,i+1})e_{H,i}\,.$$

Next, we introduce the *spherical subalgebra* U_k of H_k: by definition, $U_k := eH_ke$, where $e := |W|^{-1}\sum_{w\in W} w$ is the symmetrizing idempotent in $\mathbb{C}W \subset H_k$.

The following result of Etingof and Ginzburg describes the skew field of fractions of the spherical subalgebra:

Theorem 6. *([3, Theorem 17.7*]) The skew field of fractions of the algebra U_k is isomorphic to F_n^W.*

Combining this result with Theorem 2, we get the following analogue of the Gelfand-Kirillov conjecture for rational Cherednik algebras:

Corollary 2. *For a complex reflection group W we have* $\mathrm{Frac}(U_k(W)) \cong F_n$.

5. Noether's problem for n-dimensional torus

Denote $W_{2s+1} = B_s$ for each $s \geq 2$ and $W_{2s} = D_s$ for $s \geq 4$. Fix s and consider a space V_s with a basis $\lambda_1, \ldots, \lambda_s$. Then W_{2s+1} is a semi-direct product of the symmetric group S_s and its normal subgroup Z_2^s. The group W_{2s+1} acts naturally on V_s, where elements of S_s permute the basis vectors and the generators ε_i, $i = 1, \ldots, s$ of Z_2^s act by reflections

$$\varepsilon_i(\lambda_i) = -\lambda_i,\, \varepsilon_i(\lambda_j) = \lambda_j, j \neq i,\, i, j = 1, \ldots, n.$$

On the other hand, the group W_{2s} is generated by its subgroup S_s and its normal 2-subgroup Z_2^{s-1}, where Z_2^{s-1} consists of the transformations $(\varepsilon_1^{d_1}, \ldots, \varepsilon_s^{d_s}) \in Z_2^s$, $d_i = 0, 1, i = 1, \ldots, s$, such that $d_1 + \cdots + d_s$ is even.

Let $\tilde{A_n}$ be the localization of A_n by the multiplicative set generated by $\{x_i | i = 1, \ldots, n\}$. Let $X = \mathbb{T}^n = Specm\, k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the n-dimensional torus. Then $D(X) \simeq \tilde{A_n}$.

The involutions $\tau_i^{\pm}$ on $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, defined by $\tau_j^{\pm}(x_j) = \pm x_j^{-1}$, $\tau_j^{\pm}(x_i) = x_i$ if $i \neq j$, induce the involutions $\varepsilon_{n,j}^{\pm}$ on $\tilde{A_n}$, such that $\varepsilon_{n,j}^{\pm}(\partial_j) = \mp x_j^2\partial_j$ and $\varepsilon_{n,j}(\partial_i) = \partial_i$ if $i \neq j$. Indeed, assume $n = 1$. If we have a group action $G \times T^1 \to T^1$, then the induced action on $\tilde{A}_1$ is given as

$$x \overset{g}{\longmapsto} x^g, \partial \overset{g}{\longmapsto} \partial^g = g\partial g^{-1}.$$

Hence, if τ^- sends x to $-\dfrac{1}{x}$ then

$$\varepsilon^{-1}(\partial)(x^r) = \varepsilon^-\partial\varepsilon^-(x) = (-1)^r\varepsilon^-\partial(\frac{1}{x^r}) = (-1)^{r+1}r\varepsilon^-(\frac{1}{x^{r+1}}) = rx^{r+1},$$

implying $\varepsilon^-(\partial) = x^2\partial$. This is easily generalised to T^n and any $\tau_i^{\pm}$.

We will consider the action of W_{2n+1} on X where S_n acts by natural permutations and ε_i acts by $\varepsilon_{n,i}^-$, $i = 1, \ldots, n$.

We have

Proposition 2. (1) *The algebra of W_{2n+1}-invariants is a polynomial algebra in*

$$s_i = e_i(x_1 - x_1^{-1}, \dots, x_n - x_n^{-1}), i = 1, \dots, n.$$

In particular $T^n/W_{2n+1} \simeq A_k^n$.

(2) *Let $Z \subset T^n$ be the variety, defined by equation*

$$\Pi_{1\leq i\leq j\leq n}(x_i^2 - \frac{1}{x_j^2})\Pi_{1\leq i<j\leq n}(x_i^2 - x_j^2) = 0$$

and $U = T^n \setminus Z$. Then U is an affine W_{2n+1}-invariant subvariety of T^n and the action of W_{2n+1} on U is free. In particular, the projection $\pi : U \mapsto U/W_{2n+1}$ is etale.

Proof. The statement (1) is analogous to the case of symmetrical polynomials. We consider the lexicographical order on Laurent monomials. Let $\pi = (k_1, \dots, k_n)$ be an integral sequence with the property $(k_1 \geq k_2 \geq \dots \geq k_n \geq 0)$ and let $x_1^{k_1} \dots x_n^{k_n}$ be the corresponding monomial. Denote by $m_\pi = x_1^{k_1} \dots x_n^{k_n} + \dots$ a W_{2n+1}-invariant polynomial with a minimal number of monomials. All m_π form a basis of W_{2n+1}-invariants, since orbit of every monomial meets a unique senior one. The senior monomial $x_1^{k_1} \dots x_p^{k_p}$ of m_π coincides with the senior monomial $S_\pi = s_1^{k_1-k_2} \dots s_{n-1}^{k_{n-1}-k_n} s_n^{k_n}$ Then $m_\pi - S_\pi$ has a smaller senior monomial and we can proceed by induction.

For (2) denote

$$\Delta = \Pi_{1\leq i\neq j\leq n}(x_i^2 - \frac{1}{x_j^2})\Pi_{1\leq i<j\leq n}(x_i^2 - x_j^2)\Pi_{1\leq i<j\leq n}(\frac{1}{x_i^2} - \frac{1}{x_j^2})\Pi_{i=1}^n(x_i^2 - \frac{1}{x_i^2})^2.$$

Then Δ is W_{2n+1}-invariant and $U = T^n \setminus V(\Delta)$. □

Consider now an action of W_{2n} on $D(T^n) = \tilde{A}_1^{\otimes n}$ where S_n acts by natural permutations and ε_i acts as

$$id_{\tilde{A}_1}^{i-1} \otimes \varepsilon_{n,i}^+ \otimes \varepsilon_{n,i+1}^+ \otimes \mathrm{Id}_{\tilde{A}_1}^{p-i-1}, i = 1, \dots, n-1.$$

Proposition 3. (1) *The algebra of invariants of this action is generated by*

$$s_i = e_i(x_1 + x_1^{-1}, \dots, x_n + x_n^{-1}),\ i = 1, \dots, n-1$$

and

$$\Delta_n^\pm = \frac{1}{2}\big(\Pi_{i=1}^n(x_i + \frac{1}{x_i}) \pm \Pi_{i=1}^n(x_i - \frac{1}{x_i})\big).$$

Besides $\Delta_n^- \in k[s_1, \dots, s_{p-1}, \Delta^+]_P$, where P is some polynomial in $k[s_1, \dots, s_{n-1}, \Delta_n^+]$. In particular T^n/W_{2n} is isomorphic to a principal open subset in A_k^n.

(2) *Let $Z \subset T^n$ be the variety defined by equation $\Delta = 0$ and $U = T^n \setminus Z$. Then U is an affine W_{2n}-invariant subvariety in T^n and the action of W_{2n} on U is free. In particular, the projection $\pi : U \to U/W_{2n}$ is etale.*

Proof. Proof of (1) is similar to the proof of Proposition 2, (1). Order the Laurent monomials lexicographically. order on Let k be an integral sequence $(k_1 \geq k_2 \geq \dots \geq |k_n| \geq 0)$. Note that k_n can be negative. Set $\lambda_k = |\{g \in W_{2n} \mid g \cdot (x_1^{k_1} \dots x_n^{k_n}) = x_1^{k_1} \dots x_n^{k_n}\}|$, $m_k = \lambda_k^{-1} \sum_{g\in W_{2n}} g \cdot (x_1^{k_1} \dots x_n^{k_n})$. Then m_k's form a basis of the invariant Laurent polynomials and the senior monomial in m_k is $x_1^{k_1}\dots x_n^{k_n}$. The same senior monomial has the element

$$S_k = s_1^{k_1-k_2}\dots s_{n-1}^{k_{n-1}-k_n}(\Delta_n^{sign(k_n)})^{|k_n|}.$$

Then $m_k - S_k$ has a smaller senior monomial and we can proceed by induction. Next we show how to choose P. Note that both $s_n = \Delta_n^+ + \Delta_n^-$ and $D = \Delta_n^-\Delta_n^-$ are W_{2n}-invariant. Then D can be expressed as a polynomial in $s_1,\dots,s_n$. The senior monomial in D has a degree $(2,2,\dots,2,0)$, hence s_n can not enter in the expression for D in the degree, greater than 1, as the degree of the senior monomial in s_n is $(1,1,\dots,1,1)$. Its easy to see that polynomial part of $\Delta_n^+\Delta_n^-$ consists of the squares and hence $\Delta_n^+\Delta_n^+ \notin k[s_1,\dots,s_{n-1}]$ since it has the same senior monomial as s_{n-1}^2 and the second in lexicographical order monomial in s_{n-1}^2 has degree $(2,2,\dots,2,1,1)$. We conclude that

$$\Delta_n^+\Delta_n^- = p_0(s_1,\dots,s_{n-1}) + s_n p_1(s_1,\dots,s_{n-1}),\ \text{i.e}\ \Delta_n^- = \frac{\Delta_n^+ p_0 + p_1}{\Delta_n^+ - p_0}.$$

We choose $P = \Delta_n^+ - p_0$. For example, when $n=2$ we have $\Delta_n^+\Delta_n^- = 4(s_1^2 - s_2)$ and $P = \Delta_n^+ - 4s_1^2$.

To show (2) take the same polynomial Δ from the proof of Proposition 2. Then Δ is W_{2n}-invariant and $U = T^n \setminus V(\Delta)$.

□

5.1. **Proof of Theorem 3.** Let $X = \mathbb{T}^n = Spec\, k[x_1^{\pm1},\dots,x_n^{\pm1}]$, $\Lambda = k[x_1,\dots,x_n]$ and $\Gamma = \Lambda_f$, where $f = x_1\dots x_n$. Then $X = Spec\,\Gamma$ is an affine, regular, normal irreducible variety. Then the statement for the symmetric group S_n is analogous to Theorem 2.

We consider now the action of the Weyl group B_n. By Proposition 2, the action of B_n restricts to a free action on $U = Spec\,\Gamma_\Delta$ which is affine, irreducible, regular and normal variety. Applying Corollary 1 we have $D(U)^{B_n} \cong D(U/B_n)$ and $D(\Gamma_\Delta)^{B_n} \cong D(\Gamma_\Delta^{B_n})$. By Proposition 1 we conclude $D(\Gamma)_\Delta^{B_n} \cong D(\Gamma_\Delta^{B_n})$. Since $\Gamma^{B_n} \cong \Lambda$ we have $D(\Gamma)_\Delta^{B_n} \cong D(\Lambda_\Delta) \cong D(\Lambda)_\Delta$. Forming the skew fraction fields we conclude $Frac\, D(X)^{B_n} \cong Frac\, A_n(k) = Frac\, D(X)$.

Consider now the case of the Weyl group D_n. Repeating the same steps as above we have $D(\Gamma)_\Delta^{D_n} \cong D(\Gamma_\Delta^{D_n})$. By Proposition 3, $\Gamma^{D_n} \simeq \Lambda_P$ for some polynomial P. Therefore,

$$D(\Gamma)_\Delta^{D_n} \cong D((\Gamma_P)_{\tilde{\Delta}}) = D(\Gamma_{P\Delta}) = D(\Gamma)_{P\Delta}.$$

Taking the fraction fields we conclude $Frac\, D(X)^{D_n} \cong Frac\, D(X) \cong F_n$.

6. Galois algebras

Let Γ be an integral domain, K the field of fractions of Γ, $K \subset L$ is a finite Galois extension with the Galois group G. Let $\mathcal{M} \subset \mathrm{Aut}_{\mathbb{C}} L$ be a monoid on which G acts by conjugations,

Recall that an associative k-algebra U containing Γ is called a *Galois* Γ*-algebra* with respect to Γ if it is finitely generated Γ-subalgebra in $(L * \mathcal{M})^G$ and $KU = (L*\mathcal{M})^G, UK = (L*\mathcal{M})^G$ [5].

If U is such algebra then $S = \Gamma \setminus \{0\}$ satisfies both left and right Ore condition and the canonical embedding $U \hookrightarrow (L*\mathcal{M})^G$ induces the isomorphisms of rings of fractions $[S^{-1}]U \simeq (L*\mathcal{M})^G$, $U[S^{-1}] \simeq (L*\mathcal{M})^G$.

The following is standard.

Proposition 4. *If $L*\mathcal{M}$ is an Ore domain, then $(L*\mathcal{M})^G$ is an Ore domain. If $\mathcal{L}$ is the skew field of fractions of $L*\mathcal{M}$, then the skew field of fractions of $(L*\mathcal{M})^G$ coincides with $\mathcal{L}^G$, where the action of G on $\mathcal{L}$ is induced by the action of G on $L*\mathcal{M}$.*

We immediately have

Corollary 3. *Let U be a Galois Γ-algebra and the skew group algebra $L*\mathcal{M}$ is the left and the right Ore domain with the skew field of fractions $\mathcal{L}$. Then U is the left and right Ore domain and for its skew field of fractions $\mathcal{U}$ holds $\mathcal{U}=\mathcal{L}^G$. In particular, all Galois subalgebras with respect to Γ in $(L*\mathcal{M})^G$ have the same skew field of fractions.*

Proof. Due to Proposition 4, $U[S^{-1}]\simeq(L*\mathcal{M})^G$ is an Ore domain, $S=\Gamma\setminus\{0\}$. Hence for any $u_1,u_2\in U$ there exist $v_1,v_2\in U, s_1,s_2\in S$, such that $u_1v_1s_1^{-1}=u_2v_2s_2^{-1}$. Since S is commutative we get $u_1v_1s_2=u_2v_2s_1$. Other conditions of Ore rings are proved analogously. □

Let V be a k-vector space, $\dim_k V=N$, L the field of fractions of the symmetric algebra $S(V)$ and $G\subset \mathrm{Aut}_{\mathbb{C}k}(L)$ is a reflection group whose action on L is induced by its action by reflections on V. Then $L=k(t_1,\ldots,t_N)$ and $K=L^G$ is a purely transcendental extension of k by the Chevalley-Shephard-Todd Theorem. If $\mathcal{M}\subset\mathrm{Aut}_{\mathbb{C}k}(L)$ is a subgroup such that G normalizes $\mathcal{M}$ and U is a Galois algebra in $(L*\mathcal{M})^G$ with respect to a polynomial subalgebra Γ such that $Frac(\Gamma)\simeq K$, then such U will be called a *linear Galois algebra*. In particular, $\mathcal{K}=(L*\mathcal{M})^G$ is itself a linear Galois algebra with respect to Γ if L, G, $\mathcal{M}$, K and Γ are as above. Other examples of linear Galois algebras are the universal enveloping algebras of gl_n and sl_n with respect to their Gelfand-Tsetlin subalgebras.

We will need the following action of $\mathcal{M}$ on L. Suppose $\mathcal{M}=\mathbb{Z}^N$. We say that $\mathbb{Z}^N$ acts by shifts on L if canonical generators δ_i, $i=1,\ldots,N$ act on t_j as follows: $\delta_i(t_j)=t_j-\delta_{ij}$.

For integers $n\geq 1$, $m\geq 0$ denote $A_{n,m}$ the n-th Weyl algebra over the field of rational functions $k(z_1,\ldots,z_m)$. Then $A_{n,m}$ admits the skew field of fractions $F_{n,m}\simeq F_n\otimes k(z_1,\ldots,z_m)$. Denote $L_n=k(\lambda_1,\ldots,\lambda_s)$ and set $K=L^{W_n}$.

The free abelian group $\mathcal{M}_n$ generated by $\delta_i, i=1,\ldots,s$ of rank s acts on L_n by shifts. Thus we can construct a skew group algebra $L_n*\mathcal{M}_n$.

As a subgroup of automorphisms of L_n, W_n normalizes $\mathcal{M}_n$

$$\delta_k^{\pi}=\delta_{\pi(k)},\pi\in S_p,\delta_k^{\epsilon_i}=\begin{cases}\delta_k, & \text{if } i\neq k\\ \delta_k^{-1} & \text{otherwise,}\end{cases}$$

Hence W_n acts on $L_n*\mathcal{M}_n$. We will call this action *natural*.

Denote $L_{2s}=k(\lambda_1,\ldots,\lambda_s)$ and $K=L^{W_{2s}}$. The free abelian group $\mathcal{M}_{2s}$ generated by δ_i's of rank s acts on L_{2s} by shifts. It allows us to construct a skew group algebra $L_{2s}*\mathcal{M}_{2s}$. As above W_{2s} normalizes $\mathcal{M}_{2s}$ and W_{2s} acts on $L_{2p}*\mathcal{M}_{2s}$.

Let $R_n=k[t_1,\ldots,t_n]*<\sigma_1,\ldots,\sigma_n>$, where σ_i is an automorphism of $k[t_1,\ldots,t_n]$ such that $\sigma_i(t_j)=t_j-\delta_{ij}$. For each $i=1,\ldots,n$ and any $c\in k$ consider the involutions $\epsilon^{\pm}_{R_n,c,i}$ on R_n defined by $\epsilon^{\pm}_{R_n,c,i}(\sigma_i)=\pm\sigma_i^{-1}$, $\epsilon^{\pm}_{R_n,c,i}(\sigma_j)=\sigma_j$ if $i\neq j$, $\epsilon^{\pm}_{R_n,c,i}(t_i)=c-t_i$ and $\epsilon^{\pm}_{R_n,c,i}(t_j)=t_j$ if $j\neq i$.

Lemma 2. *The homomorphism $\phi_c^{\pm}:\tilde{A_n}\to R_n$, given by*

$$\phi_c^{\pm}(x_i)=\sigma_i,\ \phi_c^{\pm}(\partial_i)=\big(t_i+1-\frac{c}{2}\big)\sigma_i^{-1}+(1\mp\sigma_i^{-2}),$$

$i = 1, \ldots, n$ is an isomorphism of algebras with involutions.

Proof. It is sufficient to consider the case $n = 1$. Set $R = R_1$. Since $\phi_c^{\pm}(\partial x - x\partial) = t - \sigma t \sigma^{-1} = 1$, $\phi_c^{\pm}$ are homomorphisms, $(\phi_c^{\pm})^{-1}(\sigma) = x$ and $(\phi_c^{\pm})^{-1}(t) = \big(\partial x + \frac{1}{2}\big) + (\frac{1}{x} \mp x)$. We also have

$$\epsilon_{R,c}^{\pm}\phi_c^{\pm}(\partial) = \mp\big(t - 1 - \frac{c}{2}\big)\sigma + (1 \mp \sigma^2)$$

and

$$\phi_c^{\pm}\epsilon_{\tilde{A}_1,c}^{\pm}(\partial) = \mp\sigma^2\Bigg(\big(t + 1 - \frac{c}{2}\big)\sigma^{-1} + (1 \mp \sigma^{-2})\Bigg) = \mp\big(t - 1 - \frac{c}{2}\big)\sigma + (1 \mp \sigma^2).$$

□

Lemma 3. *Let $\mathcal{K} = (L * \mathcal{M})^G$ be a linear Galois algebra where $G = G_N$ is a classical Weyl group and*

- *$L = k(t_1, \ldots, t_N)$;*
- *G acts naturally on $\mathcal{K}$;*
- *$\mathcal{M} \simeq \mathbb{Z}^n$ acts by shifts on $t_1, \ldots, t_n$, $n \leq N$.*

Then $\mathcal{K}$ admits the skew field of fractions $F(\mathcal{K})$ and $F(\mathcal{K}) \simeq F_{n,N-n}$.

Proof. Since the action of $\mathcal{M}$ is trivial on $L(t_{n+1}, \ldots, t_N)$ then we have the G-equivariant embedding

$$(L(t_1, \ldots, t_n) * \mathcal{M}) \otimes L(t_{n+1}, \ldots, t_N) \hookrightarrow L * \mathcal{M}$$

and

$$((L(t_1, \ldots, t_n) \otimes L(t_{n+1}, \ldots, t_N)) * \mathcal{M})^G \hookrightarrow (L * \mathcal{M})^G.$$

Moreover, both algebras have the same skew fields. But

$$((L(t_1, \ldots, t_n) * \mathcal{M}) \otimes L(t_{n+1}, \ldots, t_N))^G \simeq ((L(t_1, \ldots, t_n) * \mathcal{M})^G \otimes L(t_{n+1}, \ldots, t_N)^G.$$

Since $(L(t_1, \ldots, t_n) * \mathcal{M} \simeq R_n$, $R_n^G = R_n^{G_n}$, $F(R_n) \simeq F(A_n)$ and $L(t_{n+1}, \ldots, t_N)^G \simeq L(z_1, \ldots z_{N-n})$ we obtain

$$F(\mathcal{K}) \simeq F(A_n)^G \otimes L(z_1, \ldots z_{N-n}).$$

The result follows from Theorem 2.

□

Theorem 7. *Let U be a linear Galois algebra in $(L * \mathcal{M})^G$ such that*

- *$L = k(t_{ij}, i = 1, \ldots, N; j = 1, \ldots, n_i; z_1, \ldots z_m)$, for some integers m, n_1, …, n_N;*
- *$G = G_1 \times \ldots \times G_N$, where G_s acts normally only on variables t_{s1}, …, t_{s,n_s}, $s = 1, \ldots, N$;*
- *$\mathcal{M} \simeq \mathbb{Z}^n$ acts by shifts on $t_{11}, \ldots, t_{N,n_N}$, $n = n_1 + \ldots + n_N$.*

Then U admits the skew field of fractions $F(U)$ and $F(U) \simeq F_{n,m}$.

Proof. Follows from Theorem 3 and Lemma 3.

□

Remark. The action of the symmetric group S_n on k^n by permutations of the coordinates is obviously linear and it normalizes the action of $\mathcal{M} = \mathbb{Z}^n$ on k^n by shifts. Recall that $U(gl_n)$ and $U(sl_n)$ are linear Galois algebras with respect to their Gelfand-Tsetlin subalgebras [5]. Then Theorem 7 implies immediately the Gelfand-Kirillov conjecture for gl_n and sl_n. In a similar manner one obtains the Gelfand-Kirillov conjecture for restricted Yangians of type A which was shown in [6].

References

[1] J. Alev and F. Dumas, *Invariants d'operateurs differentiels et probleme de Nother*, Studies in Lie Theory : A.Joseph Festschrift (eds. J.Bernstein, V.Hinich and A.Melnikov) Birkhauser, 2006.

[2] F. Dumas, *An Introduction to Non commutative polynomial invariants*, Lecture Notes, Homological methods and representations of non commutative algebras, Mar del Plata, Argentina, March 6-16, 2006.

[3] P.Etingof and V.Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. **147** (2) (2002), 243-348.

[4] Faith C., *Galois subrings of Ore domains are Ore domains*, Bull AMS, 78 (1972), no.6, 1077-1080.

[5] V.Futorny, S.Ovsienko, *Galois orders in skew monoid rings*, J. Algebra 324 (2010), 598?630.

[6] V.Futorny, A.Molev and S.Ovsienko, *Gelfand-Kirillov conjecture and Gelfand-Tsetlin modules for finite W-algebras*, Adv. Math., Advances in Mathematics 223 (2010), 773-796.

[7] H.W. Lenstra Jr., *Rational functions invariant under a finite abelian group*, Invent. Math. 25 (1974), 299-325.

[8] F. Knop, *Graded cofinite rings of differential operators*, Michigan Math. J. **54** (2006), 323.

[9] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, J. Wiley & Sons, New York, 1987.

[10] T.A. Springer, *Invariant theory*, Lecture Notes in Mahematics, 585, 1977

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