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The based ring of the lowest two-sided cell of an affine Weyl group, III

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REPORT No. 29, 2014/2015, spring

ISSN 1103-467X
ISRN IML-R- -29-14/15- -SE+spring

THE BASED RING OF THE LOWEST TWO-SIDED CELL OF AN AFFINE WEYL GROUP, III

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ABSTRACT. We show that Lusztig's homomorphism from an affine Hecke algebra to the direct summand of its asymptotic Hecke algebra corresponding to the lowest two-sided cell is related to the homomorphism constructed by Chriss and Ginzburg using equivariant K-theory by a matrix over the representation ring of the associated algebraic group.

Let W be the extended affine Weyl group associated to a simply connected simple algebraic group G over $\mathbb{C}$. Let J_0 be the based ring of the lowest two-sided cell of W and let H be the Hecke algebra of W over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in an indeterminate v with integer coefficients. Lusztig defined an $\mathcal{A}$-algebra homomorphism $\varphi_0 : H \to J_0 \otimes_{\mathbb{Z}} \mathcal{A}$ (see [L3]). In [CG, Corollary 5.4.34], Chriss and Ginzburg defined an $\mathcal{A}$-algebra homomorphism $\psi_0 : H \to J_0 \otimes_{\mathbb{Z}} \mathcal{A}$ using equivariant K-theory. We will show that φ_0 is essentially the conjugacy of ψ_0 by an invertible $W_0 \times W_0$ matrix with entries in the representation $R_{G\times\mathbb{C}^*}$ of $G \times \mathbb{C}^*$, see Theorem 2.5.

1. Preliminaries

1.1. Let G be a simply connected simple algebraic group over the complex number field $\mathbb{C}$. The Weyl group W_0 of G acts naturally on the character group X of a maximal tours of G. The semidirect product $W = W_0 \ltimes X$ with respect to the action is called an (extended) affine Weyl group. Let H be the associated Hecke algebra over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ (v an indeterminate) with parameter v^2. Thus H has an $\mathcal{A}$-basis $\{T_w \mid w \in W\}$ and its multiplication is defined by the relations $(T_s - v^2)(T_s + 1) = 0$ if s is a simple reflection and $T_w T_u = T_{wu}$ if $l(wu) = l(w) + l(u)$, here l is the length function of W.

We shall write the operation in X multiplicatively. Let $X^+ = \{x \in X \mid l(w_0 x) = l(w_0) + l(x)\}$ be the set of dominant weights in X, where w_0 is the longest element of W_0. For any $z \in X$, one can find dominant weights x and y such that $z = xy^{-1}$. Then define $\theta_z = v^{l(y)-l(x)} T_x T_y^{-1}$. It is known that θ_z is well defined, that is, it only depends on z and independent of the choice of x, y. Moreover, the elements $\theta_z T_w$, $z \in$

The work was partially supported by Natural Sciences Foundation of China (No. 11321101).

$X,\ w\in W_0$, form an $\mathcal{A}$-basis of H and the the elements $T_w\theta_z,\ w\in W_0$, $z\in X$, form an $\mathcal{A}$-basis of H as well. See [L1] or [L4].

The $\mathcal{A}$-subalgebra Θ of H generated by all $\theta_x,\ x\in X$, is commutative and is isomorphic to the group algebra $\mathcal{A}[X]$ of X. Let $\theta:\mathcal{A}[X]\to\Theta$, $\sum_{x\in X}a_x x\to\sum_{x\in X}a_x\theta_x$ be the isomorphism, where $a_x\in\mathcal{A}$ are 0 except for finitely many x in X. For $\gamma\in\mathcal{A}[X]$, we shall also write θ_γ for $\theta(\gamma)$.

Let R be the root system. For simple reflection s in W_0, we shall denote by α_s for the simple root which defines s. For $x\in X$, we have (see [L5, 1.19])

(a) $T_s\theta_x=\theta_{s(x)}T_s-(v^2-1)\theta_{\frac{s(x)\alpha_s-x\alpha_s}{\alpha_s-1}}$.

1.2. For y,w in W, let $P_{y,w}$ be the Kazhdan-Lusztig polynomial. Set $C_w=v^{l(w)}\sum_{y\le w}(-1)^{l(w)-l(y)}P_{y,w}(v^{-2})T_y$, where $\le$ is the Bruhat order on W. Then the elements $C_w,\ w\in W$, form an $\mathcal{A}$-basis of H and is called a Kazhdan-Lusztig basis of H.

Recall that w_0 is the longest element in W_0. We shall simply write C for C_{w_0}. We have $T_sC=-C$ for all simple reflections s in W_0. Therefore, the $\mathcal{A}$-linear map π: $\mathcal{A}[X]\to HC$ defined by $x\to\theta_xC$ for all $x\in X$ is an isomorphism of $\mathcal{A}$-module. Using the formula 1.1(a) we see that for any simple reflection s in W_0 and x,y in X, we have

$$T_s\theta_xC=\frac{(\theta_{s(x)}-\theta_x\theta_{\alpha_s})+v^2(\theta_x\theta_{\alpha_s}-\theta_{s(x)}\theta_{\alpha_s})}{\theta_{\alpha_s}-1}C,$$

$$\theta_y\theta_xC=\theta_{yx}C.$$

Thus the natural H-module structure on HC leads an H-module structure on $\mathcal{A}[X]$ as follows (see [L5, Lemma 4.7]):

$$T_s\circ x=\frac{(s(x)-x\alpha_s)+v^2(x\alpha_s-s(x)\alpha_s)}{\alpha_s-1},$$

$$\theta_y\circ x=yx,$$

where $s\in W_0$ is a simple reflection and $x,y\in X$.

1.3. Let R^+ be the set of positive roots, $R^-=-R^+$ and Δ be the set of simple roots in R. For $\alpha\in\Delta$, let x_α be the corresponding fundamental weight. For $w\in W_0$, let

$$e_w=w(\prod_{\substack{\alpha\in\Delta\\ w(\alpha)\in R^-}}x_\alpha)\in X.$$

According to [St] we have

(a) $\mathcal{A}[X]$ is a free $\mathcal{A}[X]^{W_0}$-module with a basis $e_w,\ w\in W_0$.

Note that $\mathcal{A}[X]^{W_0}$ is naturally isomorphic to $R_{G\times\mathbb{C}^*}=R_G\otimes_{\mathbb{Z}}\mathcal{A}$, where $R_{\mathcal{G}}$ and R_G are the representation ring of $\mathcal{G}=G\times\mathbb{C}^*$ and G respectively, and we identify $R_{\mathbb{C}^*}$ with $\mathcal{A}$ by regarding v as the identity representation $\mathbb{C}^* \to \mathbb{C}^*$. We also use v for the element in $R_{\mathcal{G}}$ defined by the natural projection $G \times \mathbb{C}^* \to \mathbb{C}^*$.

According to Bernstein (see [L1]), we know that

(b) The $\theta(\mathcal{A}[X]^{W_0})$ is the center $Z(H)$ of H and the center $Z(H)$ of H is isomorphic to the representation ring of $R_{\mathcal{G}}$.

Therefore we have

(c) The H-module HC is a free $Z(H)$-module with a basis $\theta_{e_w}C,\ w \in W_0$.

1.4. For each $w \in W_0$, set $d_w^{-1} = (\prod_{\substack{\alpha\in\Delta \\ w(\alpha)\in R^-}} x_\alpha)w^{-1} \in W$. According to [Sh1, Sh2], the lowest two-sided cell of W c_0 can be described as

$$c_0 = \{d_w w_0 x d_u^{-1} \,|\, w, u \in W,\ x \in X^+\}.$$

For $x \in X^+$, let $V(x)$ be an irreducible rational G-module of highest weight x. For $x, y, z \in X^+$, let $m_{x,y,z}$ be the multiplicity of $V(z)$ appearing in $V(x) \otimes V(y)$. The based ring J_0 of c_0 defined by Lusztig in [L3] is a free $\mathbb{Z}$-module with a basis $t_w,\ w \in c_0$ and the structure constants are given by (see [X1, Theorem 1.10])

$$t_{d_w w_0 x d_u^{-1}} t_{d_v w_0 y d_p^{-1}} = \delta_{u,v} \sum_{z\in X^+} m_{x,y,z} t_{d_w w_0 z d_p^{-1}}.$$

For $x \in X^+$, let S_x be the element in Θ corresponding to $V(x)$. By abuse notation, we also use S_x for the element in R_G or $R_{\mathcal{G}}$ corresponding to $V(x)$. Then we have

(d) The map $S_x \to \sum_{w\in W_0} t_{d_w w_0 x d_w^{-1}}$ defines an ring isomorphism from R_G to the center of J_0.

So we have

(e) The map $t_{d_w w_0 x d_u^{-1}} \to (S_x)_{w,u}$ defines an isomorphism φ_1 of R_G-algebra from J_0 to the $W_0 \times W_0$ matrix ring $M_{W_0}(R_G)$ over R_G, where $(S_x)_{w,u}$ stands for the $W_0 \times W_0$ matrix with entry S_x at the position (w, u) and with 0 entries at other positions. See [X1, Theorem 1.10].

Note that we have $S_x C_{d_w w_0} = C_{d_w w_0 x}$, see [X1, Theorem 2.9] and [L1, 8.6, 6.12]. It is known that the elements $C_{d_w w_0 x},\ w \in W_0,\ x \in X^+$, form an $\mathcal{A}$-basis of HC (see [L2]). Hence we have

(f) The elements $C_{d_w w_0},\ w \in W_0$, form a $Z(H)$-basis of HC.

Now we have two $Z(H)$-bases for HC, one is $\{\theta_{e_w}C \,|\, w \in W_0\}$, the other is $\{C_{d_w w_0} \,|\, w \in W_0\}$. For each $u \in W$, we then have

$$C_{d_u w_0}C = \sum_{w\in W_0} a_{w,u}\theta_{e_w}, \quad a_{w,u} \in Z(H) = R_{\mathcal{G}}.$$

Let $A = (a_{w,u})$ be the $W_0 \times W_0$ matrix. Then A is the transfer matrix from the basis $\{\theta_{e_w} C \mid w \in W_0\}$ to the basis $\{C_{d_w w_0} \mid w \in W_0\}$. That is

$$(C_{d_w w_0})_{w\in W_0} = (\theta_{e_w} C)_{w\in W_0} A,$$

where $(\cdot)_{w\in W_0}$ are row vectors.

1.5. For $w, u \in W$, write $C_w C_u = \sum_{v\in W} h_{w,u,v} C_v,\ h_{w,u,v} \in \mathcal{A}$. Let $\mathcal{D}$ be the set consisting of all $d_w w_0 d_w^{-1},\ w \in W_0$. Lusztig showed that the map

$$\varphi_0 : H \to J_0 \otimes \mathcal{A}, \quad C_w \to \sum_{\substack{d\in\mathcal{D}\\ u\in W}} h_{w,d,u} t_u,$$

is an $\mathcal{A}$-algebra homomorphism (see [L3]). Note that the restriction of φ_0 to the center $Z(H)$ gives an isomorphism from $Z(H)$ to the center of $J_0 \otimes \mathcal{A}$. If we identify $Z(H)$ with the representation ring $R_{\mathcal{G}}$ of $\mathcal{G} = R \times \mathbb{C}^*$, then φ_0 is in fact an $R_{\mathcal{G}}$-algebra homomorphism.

Let

$$\varphi = (\varphi_1 \otimes \mathrm{id}_{\mathcal{A}})\varphi_0 : H \to J_0 \otimes \mathcal{A} \to M_{W_0}(R_G) \otimes \mathcal{A} = M_{W_0}(R_{\mathcal{G}}),$$

see 1.4 (e) for the definition of φ_1. Then φ is a homomorphism of $R_{\mathcal{G}}$-algebra.

2. Equivariant K-theory Construction

Chriss and Ginzburg constructed a homomorphism of $R_{\mathcal{G}}$-algebra from H to the matrix ring $M_{W_0}(R_{\mathcal{G}})$, see [CG, Corollary 5.4.34, 7.6.8]. In this section we recall this construction. We shall follow the approach in [L5] for explicit calculation.

2.1. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$ and $\mathcal{B}$ the variety of all Borel subalgebras of $\mathfrak{g}$. The Steinberg variety Z is the subvariety of $\mathcal{N} \times \mathcal{B} \times \mathcal{B}$ consisting of all triples $(n, \mathfrak{b}, \mathfrak{b}'),\ n \in \mathfrak{b} \cap \mathfrak{b}' \cap \mathcal{N},\ \mathfrak{b}, \mathfrak{b}' \in \mathcal{B}$. Let $\Lambda = \{(n, \mathfrak{b}) \mid n \in \mathcal{N} \cap \mathfrak{b}, \mathfrak{b} \in \mathcal{B}\}$ be the cotangent bundle of $\mathcal{B}$. Clearly Z can be regarded as a subvariety of $\Lambda \times \Lambda$ by the imbedding $Z \to \Lambda \times \Lambda$, $(n, \mathfrak{b}, \mathfrak{b}') \to (n, \mathfrak{b}, n, \mathfrak{b}')$. Define a $\mathcal{G} = G \times C^*$-action on Λ by $(g, z) :\ (n, \mathfrak{b}) \to (z^{-2}\mathrm{ad}(g)n, \mathrm{ad}(g)\mathfrak{b})$. Let $G \times C^*$ acts on $\Lambda \times \Lambda$ diagonally, then Z is a $G \times C^*$-stable subvariety of $\Lambda \times \Lambda$. Let $K_{G\times\mathbb{C}^*}(Z) = K_{G\times\mathbb{C}^*}(\Lambda \times \Lambda; Z)$ be the Grothendieck group of the category of $G \times \mathbb{C}^*$-equivariant coherent sheaves on $\Lambda \times \Lambda$ with support in Z.

Let

$$\Lambda^{aab} = \{(n, \mathfrak{b}, n', \mathfrak{b}', n'', \mathfrak{b}'') \in \Lambda^3 \mid n = n'\},$$

$$\Lambda^{abb} = \{(n, \mathfrak{b}, n', \mathfrak{b}', n'', \mathfrak{b}'') \in \Lambda^3 \mid n' = n''\},$$

$$\Lambda^{aaa} = \{(n, \mathfrak{b}, n', \mathfrak{b}', n'', \mathfrak{b}'') \in \Lambda^3 \mid n = n' = n''\}.$$

Let $\mathcal{G}$ act on Λ^3 diagonally, then Λ^{aab}, Λ^{abb} and Λ^{aaa} are $\mathcal{G}$-stable subvarieties of Λ^3. Define $\pi_{12}:\Lambda^{aab}\to Z$, $\pi_{23}:\Lambda^{abb}\to Z$, $\pi_{13}:\Lambda^{aaa}\to Z$ as follows,

$$\pi_{12}(n,\mathfrak{b},n,\mathfrak{b}',n,\mathfrak{b}'')\to(n,\mathfrak{b},\mathfrak{b}').$$
$$\pi_{23}(n,\mathfrak{b},n',\mathfrak{b}',n'',\mathfrak{b}'')\to(n',\mathfrak{b}',\mathfrak{b}''),$$
$$\pi_{13}(n,\mathfrak{b},n,\mathfrak{b}',n,\mathfrak{b}'')\to(n,\mathfrak{b},\mathfrak{b}'').$$

Note that π_{12}, π_{23} are smooth and π_{13} is proper. Following [L5, 7.9], one defines the convolution product

$$*:K_{G\times\mathbb{C}^*}(Z)\times K_{G\times\mathbb{C}^*}(Z)\to K_{G\times\mathbb{C}^*}(Z),$$
$$\mathscr{F}*\mathscr{G}=(\pi_{13})_*(\pi_{12}^*\mathscr{F}\otimes^L_{\mathcal{O}_{\Lambda^3}}p_{23}^*\mathscr{G}),$$

where $\mathcal{O}_{\Lambda^3}$ is the structure sheaf of Λ^3. This endows with $K_{G\times\mathbb{C}^*}(Z)$ an associative algebra structure over the representation ring $R_{G\times\mathbb{C}^*}$ of $G\times\mathbb{C}^*$. Recall that we regard the indeterminate v as the representation $G\times\mathbb{C}^*\to\mathbb{C}^*$, $(g,z)\to z$. Then $R_{G\times\mathbb{C}^*}$ is identified with $R_G\otimes_{\mathbb{Z}}\mathcal{A}$. In particular, $K_{G\times\mathbb{C}^*}(Z)$ is an $\mathcal{A}$-algebra. Moreover, as $\mathcal{A}$-algebras, $K^{G\times\mathbb{C}^*}(Z)$ is isomorphic to the Hecke algebra H, see [G1, G2, KL2] or [CG, L5]. We shall identify $K_{G\times\mathbb{C}^*}(Z)$ with H.

2.2. We shall simply write $\mathcal{G}$ for $G\times\mathbb{C}^*$. Consider the $\mathcal{G}$-equivariant injections

$$j:\mathcal{B}\times\mathcal{B}\to Z,\quad(\mathfrak{b},\mathfrak{b}')\to(0,\mathfrak{b},\mathfrak{b}'),$$

and

$$k:Z\to\Lambda\times\mathcal{B},\quad(n,\mathfrak{b},\mathfrak{b}')\to(n,\mathfrak{b},\mathfrak{b}').$$

We then have two $K_{\mathcal{G}}(Z)$-linear maps:

$$j_*:K_{\mathcal{G}}(\mathcal{B}\times\mathcal{B})\to K_{\mathcal{G}}(Z),$$
$$k_*:K_{\mathcal{G}}(Z)\to K_{\mathcal{G}}(\Lambda\times\mathcal{B}).$$

Here the $K_{\mathcal{G}}(Z)$-module structure on $K_{\mathcal{G}}(\mathcal{B}\times\mathcal{B})$ and on $K_{\mathcal{G}}(\Lambda\times\mathcal{B})$ are defined as follows.

Let

$$r_{12}:Z\times\mathcal{B}\to Z,\quad(n,\mathfrak{b},\mathfrak{b}',\mathfrak{b}'')\to(n,\mathfrak{b},\mathfrak{b}'),$$
$$r_{23}:\Lambda\times\mathcal{B}\times\mathcal{B}\to\mathcal{B}\times\mathcal{B},\quad(n,\mathfrak{b},\mathfrak{b}',\mathfrak{b}'')\to(\mathfrak{b}',\mathfrak{b}''),$$
$$r_{13}:\mathcal{B}\times\mathcal{B}\times\mathcal{B}\to\mathcal{B}\times\mathcal{B},\quad(\mathfrak{b},\mathfrak{b}',\mathfrak{b}'')\to(\mathfrak{b},\mathfrak{b}''),$$
$$q_{23}:\Lambda^2\times\mathcal{B}\to\Lambda\times\mathcal{B},\quad(n,\mathfrak{b},n',\mathfrak{b}',\mathfrak{b}'')\to(n',\mathfrak{b}',\mathfrak{b}''),$$
$$q_{13}:Z\times\mathcal{B}\to\Lambda\times\mathcal{B},\quad(n,\mathfrak{b},n,\mathfrak{b}',\mathfrak{b}'')\to(n,\mathfrak{b},\mathfrak{b}'').$$

Define the $R_{\mathcal{G}}$-bilinear pairings

$$\star:K_{\mathcal{G}}(Z)\times K_{\mathcal{G}}(\mathcal{B}\times\mathcal{B})\to K_{\mathcal{G}}(\mathcal{B}\times\mathcal{B}),$$
$$\mathscr{F}\star\mathscr{G}=r_{13_*}(r_{12}^*\mathscr{F}\otimes^L_{\Lambda^3}r_{23}^*\mathscr{G})\in K_{\mathcal{G}}(\mathcal{B}\times\mathcal{B});$$

and

$$\circ:K_{\mathcal{G}}(Z)\times K_{\mathcal{G}}(\Lambda\times\mathcal{B})\to K_{\mathcal{G}}(\Lambda\times\mathcal{B}),$$
$$\mathscr{F}\circ\mathscr{G}=q_{13_*}(r_{12}^*\mathscr{F}\otimes^L_{\Lambda^2\otimes\mathcal{B}}q_{23}^*\mathscr{G})\in K_{\mathcal{G}}(\Lambda\times\mathcal{B});$$

respectively. These $R_{\mathcal{G}}$-pairings give $K_{\mathcal{G}}(Z)$-module structure on $K_{\mathcal{G}}(\mathcal{B}\times\mathcal{B})$ and $K_{\mathcal{G}}(\Lambda\times\mathcal{B})$ respectively.

2.3. According to [KL2,1.6] and its proof, we have

(a) The external tensor product in K_G-theory

$$\boxtimes : K_G(\mathcal{B})\otimes_{R_G} K_G(\mathcal{B})\to K_G(\mathcal{B}\times\mathcal{B})$$

is an isomorphism of R_G-module. For $x\in X$, we shall also denote the corresponding line bundle on $\mathcal{B}$ by x.

Note that $K_G(\mathcal{B}) = R_B = \mathbb{Z}[X]$. Define the pairing $(,) : \mathbb{Z}[X]\times\mathbb{Z}[X]\to R_G$ by

$$(x,y)=\sum_{w\in W_0}(-1)^{l(w)}w(xy\rho)/\sum_{w\in W_0}(-1)^{l(w)}w(\rho),$$

where ρ is the product of all fundamental weights.

(b) There exist elements $e'_w\in\mathbb{Z}[X],\ w\in W_0$, such that $(e_w,e'_u)=\delta_{w,u}$. (See the proof of [KL2,1.6].)

For $1\le i<j\le 3$, let p_{ij} be the projection from $\mathcal{B}^3=\mathcal{B}\times\mathcal{B}\times\mathcal{B}$ to its (i,j)-factor. Define the convolution $\cdot$ on $K_G(\mathcal{B}\times\mathcal{B})$ as follows:

$$V\cdot V'=(p_13)_*(p_{12}^*V\otimes^L_{\mathcal{B}^3}p_{23}^*V').$$

Let w,u,t,v be elements in W_0 and ξ,η be elements in R_G. Then we have

$$(e_w\xi\boxtimes e'_u)\cdot(e_t\boxtimes e'_v)=(e_t,e'_u)(e_w\xi\boxtimes\eta e'_v),$$

see [X2, 5.16.1] or [CG, Lemma 5.2.28]. Therefore the map

$$e_w\xi\boxtimes e'_u\to(\xi)_{w,u},\quad w,u\in W_0,\ \xi\in R_G,$$

defines an isomorphism of R_G-algebra

$$\psi_1 : K_G(\mathcal{B}\times\mathcal{B})\to M_{W_0}(R_G),$$

where $(\xi)_{w,u}$ stands for the matrix whose entries are ξ at position (w,u) and zero otherwise.

2.4. The projection $pr:\Lambda\times\mathcal{B}\to\mathcal{B}\times\mathcal{B}$, $(n,\mathfrak{b},\mathfrak{b}')\to(\mathfrak{b},\mathfrak{b}')$, is $\mathcal{G}$-equivariant. The Thom isomorphism says that $pr^*:K_{\mathcal{G}}(\mathcal{B}\times\mathcal{B})\to K_{\mathcal{G}}(\Lambda\times\mathcal{B})$ is isomorphism of $R_{\mathcal{G}}$-module. Composing its inverse with $k_*:K_{\mathcal{G}}(Z)\to K_{\mathcal{G}}(\Lambda\times\mathcal{B})$ we get an $R_{\mathcal{G}}$-linear map

$$(pr^*)^{-1}k_*:K_{\mathcal{G}}(Z)\to K_{\mathcal{G}}(\mathcal{B}\times\mathcal{B}).$$

The following result is showed in [CG, Corollary 5.4.34].

(a) The map $(pr^*)^{-1}k_*:K_{\mathcal{G}}(Z)\to K_{\mathcal{G}}(\mathcal{B}\times\mathcal{B})$ is a homomorphism of $R_{\mathcal{G}}$-algebra.

We explain this. Through inverse image $pr_2^*:K_{\mathcal{G}}(\mathcal{B})\to K_{\mathcal{G}}(\Lambda)$ of the projection $pr_2:\Lambda\to\mathcal{B}$, we regard the trivial bundle $\mathbb{C}$ on $\mathcal{B}$

as an $\mathcal{G}$-equivariant bundle on Λ. Let $r : \Lambda \to Z$ be the inclusion $(n, \mathfrak{b}) \to (n, \mathfrak{b}, \mathfrak{b})$. Then $r_*(\mathbb{C})$ is the unit in $K_{\mathcal{G}}(Z)$ (see [L5, 7.10].

Let $\tilde{j} : \Lambda \to \Lambda \times \mathcal{B}$ be the inclusion $(n, \mathfrak{b}) \to (n, \mathfrak{b}, \mathfrak{b})$ and let $i : \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ be the diagonal map. Then we have a cartesian diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\tilde{j}} & \Lambda \times \mathcal{B} \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle pr} \\ \mathcal{B} & \xrightarrow[i]{} & \mathcal{B} \times \mathcal{B} \end{array}$$

So we have $\tilde{j}_* pr_2^*(\mathbb{C}) = pr^* i_*(\mathbb{C})$ (see for example [FK, Chapter I, 6.1 Theorem]). Therefore, $(pr^*)^{-1}\tilde{j}_* pr_2^*(\mathbb{C}) = i_*(\mathbb{C})$. This means that $(pr^*)^{-1}k_* r_*(\mathbb{C})$ is $i_*(\mathbb{C})$ in $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$. But it is known that $i_*(\mathbb{C}) = \sum_{w \in W_0} e_w \boxtimes e'_w$ is the unit in $K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$ (see [KL2, 1.7]). Since k_* is $K_{\mathcal{G}}(Z)$-linear, we see easily that the map in (a) is a homomorphism of $R_{\mathcal{G}}$-algebra.

Define $\psi : H = K_{\mathcal{G}}(Z) \to M_{W_0}(R_{\mathcal{G}}) = M_{W_0}(R_G) \otimes \mathcal{A}$ to be the homomorphism of $R_{\mathcal{G}}$-algebra $(\psi_1 \otimes \mathrm{id}_{\mathcal{A}})(pr^*)^{-1}k_*$, see 2.3 for definition of ψ_1. Recall that we defined in subsection 1.5 the homomorphism $\varphi : H \to M_{W_0}(R_{\mathcal{G}})$ of $R_{\mathcal{G}}$-algebra and defined the matrix $A \in M_{W_0}(R_{\mathcal{G}})$ in the last part of subsection 1.4. Now we can state the main result of the paper.

Theorem 2.5. For $h \in H$ we have $\varphi(h) = A^{-1}\psi(h)A$.

Proof. Let $h \in H$. According to [L5, 7.14], we have

$$(pr^*)^{-1}k_*(h) = (pr^*)^{-1}k_*(h)(pr^*)^{-1}k_*(1).$$

In subsection 2.4 we have seen that

$$(pr^*)^{-1}k_*(1) = \sum_{w \in W_0} e_w \boxtimes e'_w.$$

According to [L5, Lemma 7.21], for $h \in H$, we have

$$(pr^*)^{-1}k_*(h) = h \circ \sum_{w \in W_0} e_w \boxtimes e'_w = \sum_{w \in W_0} (h \circ e_w) \boxtimes e'_w.$$

Let $h \circ e_w = \sum_{u \in W_0} h_{u,w} e_u$. Then we have

$$\psi(h) = (h_{u,w}) \in M_{W_0}(R_{\mathcal{G}}).$$

Note that we also have

$$h\theta_{e_w} C = \sum_{u \in W_0} h_{u,w} \theta_{e_u} C.$$

Therefore the homomorphism comes from the natural module structure on HC of the $R_{\mathcal{G}}$-algebra H with respect to the $R_{\mathcal{G}}$-basis $\theta_{e_w} C$, $w \in W_0$.

By definition, for $r \in W$, we have

$$\varphi_0(C_r) = \sum_{d \in \mathcal{D}} h_{r,d,r'} t_{r'},\ \ h_{r,d,r'} \in \mathcal{A}.$$

By [L2] and [Sh2], $h_{r,d_w w_0 d_w^{-1},r'} \neq 0$ if and only if $r' = d_u w_0 x d_w^{-1}$ for some $u \in W_0$ and $x \in X^+$. Using [X1, Theorem 2.9] we know that $h_{r,d_w w_0 d_w^{-1},r'} = h_{r,d_w w_0, d_u w_0 x}$ for $r' = d_u w_0 x d_w^{-1}$, $u, w \in W_0$, $x \in X^+$. Moreover, for $x \in X^+$, we have $S_x C_{d_u w_0 d_w^{-1}} = C_{d_u w_0 x d_w^{-1}}$ and $S_x C_{d_u w_0} = C_{d_u w_0 x}$, see [X1, Theorem 2.9] and [L1, 8.6, 6.12]. Therefore, for $h \in H$, if $\varphi(h) = (b_{u,w}) \in M_{W_0}(R_{\mathcal{G}})$, then we have $hC_{d_w w_0} = \sum_{u,w} b_{u,w} C_{d_u w_0}$. Hence, the homomorphism φ also comes from the natural module structure on HC of the $R_{\mathcal{G}}$-algebra H, but with respect to the $R_{\mathcal{G}}$-basis $C_{d_w w_0}$, $w \in W_0$.

The theorem follows.

2.6. According to [L5, 10.6], we have

$$j_*(x \boxtimes y) = (-1)^\nu \theta_{x\rho} \sum_{w \in W_0} T_w \theta_{\rho y},$$

where $\nu = |R^+|$. By the proof of Proposition 10.7 in [L5], we see that

$$(pr^*)^{-1} k_* j_* : K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B}) \to K_{\mathcal{G}}(Z) \to K_{\mathcal{G}}(\Lambda \times \mathcal{B}) \to K_{\mathcal{G}}(\mathcal{B} \times \mathcal{B})$$

is given by the following formula

$$(pr^*)^{-1} k_* j_*(x \boxtimes y) = (\prod_{\alpha \in R^+} (1 - v^2 \alpha)) x \boxtimes y.$$

Let $x = y = \rho^{-1}$. In view of the discussion in 1.2 and [L5, Lemma 7.21], the above formula implies that

$$\sum_{w \in W_0} T_w \theta_{\rho^{-1}} C = \prod_{\alpha \in R^+} (1 - v^2 \theta_\alpha) \theta_{\rho^{-1}} C.$$

This formula is proved in [L1], now has an interpretation in terms of equivariant K-theory.

Acknowledgement: The work was completed during the author's visit to the Institute Mittag-Leffler. The author is grateful to the Institute for hospitality. Part of the work was done during the author's visit to the Departement de Mathematiques, Universite de Paris VII in 2012. The author is very grateful to E. Vasserot for invitation and for very helpful discussions and comments.

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