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Another exceptional poset of coherent sheaves on a Grassmannian *

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Abstract

On a complex Grassmannian defined by removing a simple root next to the end of the corresponding general linear group we construct a Karoubian complete strongly exceptional PO set of coherent sheaves from sub quotients of the Frobenius direct image of the structure sheaf of the corresponding Grassmannian defined over a field of positive characteristic. Our set is quite much different from the one discovered by Kapranov or the one recently constructed by Buchweitz, Leuschke and Van den Bergh.

Let $\mathcal{P}$ be a complex smooth homogeneous projective variety. Write $\mathcal{P} = G/P$ with a simply connected semi simple algebraic group over $\mathbb{C}$ and P a parabolic subgroup of G. Both G and P are defined over $\mathbb{Z}$, and hence have their counterparts $G_{\Bbbk}$ and $P_{\Bbbk}$ over a field $\Bbbk$ of positive characteristic p. In [K09] on the projective spaces and in [K14] on the quadrics we have constructed a Karoubian complete strongly exceptional PO set of coherent modules on $\mathcal{P}$ from certain subquotients of the Frobenius direct image of the structure sheaf of $G_{\Bbbk}/P_{\Bbbk}$. On the projective spaces the PO set consists of invertible sheaves, coinciding with the one discovered by Beilinson [Be] while the one on the quadrics is slightly different in type D from the one found by Kapranov [Kap88]. In this paper we show that the same recipe yields a Karoubian complete strongly exceptional PO set on the Grassmannian with P obtained by removing a simple root next to the end, which is quite much different from the one discovered by Kapranov [Kap83] or the one recently constructed by Buchweitz, Leuschke and Van den Bergh [BLV].

Thus let B be a Borel subgroup of G, T a maximal torus of B, Λ the character group of T, R the root system of G relative to T, R^+ the positive system of R such that the roots of B are $-R^+$, R^{s} the set of simple roots of R^+, W the Weyl group of G. Put $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$. For each simple root α let $\varpi_\alpha \in \Lambda$ denote the corresponding fundamental weight. Besides the standard action of W on Λ we let W act on Λ by shifting the origin to $-\rho$; $\forall w \in W$, $\lambda \in \Lambda$, we write $w \bullet \lambda = w(\lambda+\rho)-\rho$. Let $I \subset R^{\mathrm{s}}$ be the set of the simple roots of the standard Levi subgroup L_P of P, and let $W^P = \{w \in W | w\alpha > 0 \ \forall \alpha \in I\}$. If W_I is the Weyl group

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of L_P, $W = \sqcup_{w \in W^P} wW_I$. For each $w \in W^P$ let $\mu_w = w^{-1} \bullet (-\sum_{\substack{\alpha \in R^{\rm s} \\ w^{-1}\alpha < 0}} \varpi_\alpha)$. Let $\nabla^P = \mathrm{ind}_B^P$ denote the induction functor from the category of B-modules to the category of P-modules [J, I.3], and let $\mathcal{E}(w) = \mathcal{L}_{\mathcal{P}}(\nabla^P(\mu_w))$ be the locally free sheaf on $\mathcal{P}$ associated to the P-module $\nabla^P(\mu_w)$ of highest weight μ_w. We showed in [K09] (resp. [K14]) that if $\mathcal{P}$ is a projective space (resp. quadric), the $\mathcal{E}(w)$, $w \in W^P$, form a Karoubian complete strongly exceptional PO set such that $\mathrm{Mod}_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(y)) \neq 0$ iff $x \geq y$ in the Chevalley-Bruhat order.

Now, let E be an n-dimensional $\mathbb{C}$-linear space of basis e_i, $i \in [1, n]$, $G = \mathrm{GL}(E)$, B the Borel subgroup of G consisting of lower triangular matrices, T the maximal torus of B consisting of diagonals. Let $\varepsilon_i \in \Lambda$ such that $\mathrm{diag}(a_1, \dots, a_n) \mapsto a_i$, $i \in [1, n]$. If $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i \in [1, n[$, $R^{\rm s} = \{\alpha_i | i \in [1, n[\}$ forms a set of the simple roots for (G, T). Let P be the parabolic subgroup of G stabilizing the subspace $\mathbb{C}e_{n-1} \oplus \mathbb{C}e_n$ of E. In this paper we show on this Grassmanniann $\mathcal{P} = G/P$ that $\forall x, y \in W^P$,

(1) $$\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(y)) = 0 \quad \forall i > 0,$$

(2) $$\mathrm{Mod}_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(x)) = \mathbb{C},$$

(3) $$\mathrm{Mod}_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(y)) \neq 0 \text{ iff } x \geq y \text{ in the Chevalley-Bruhat order},$$

and that the $\mathcal{E}(w)$'s, $w \in W^P$, Karoubian generate the bounded derived category $\mathrm{D}^b(\mathrm{coh}\mathcal{P})$ of coherent sheaves on $\mathcal{P}$, i.e., the smallest triangulated subcategory of $\mathrm{D}^b(\mathrm{coh}\mathcal{P})$ closed under taking direct summands is the whole $\mathrm{D}^b(\mathrm{coh}\mathcal{P})$. As $\mathcal{P}$ is defined over $\mathbb{Z}$ and as the $\mathcal{E}(x)$'s are all defined over $\mathbb{Z}$, transferring to $\mathcal{P}_{\Bbbk}$ over an algebraically closed field $\Bbbk$ of positive characteristic p, we find for $p \gg 0$ that (1)-(3) hold on $\mathcal{P}_{\Bbbk}$ and that they Karoubian generate the bounded derived category of coherent sheaves on $\mathcal{P}_{\Bbbk}$.

1° **Some generalities**

In this section we let G denote an arbitrary simply connected semisimple algebraic group over $\mathbb{C}$.

(1.1) Let B be a Borel subgroup of G and T a maximal torus of B with the root system R, the positive system R^+ such that the roots of B are $-R^+$, and the set of simple roots $R^{\rm s}$. Let $I \subseteq R^{\rm s}$ and P the standard parabolic subgroup of G whose standard Levi L_P has the set of simple roots I. Let R^+_I be the system of positive roots for L_P. Let W (resp. W_I) denote the Weyl group of G (resp. L_P), and set $W^P = \{w \in W | w\alpha > 0 \ \forall \alpha \in I\}$. Let w_0 (resp. w_I) $\in W$ such that $w_0 R^+ = -R^+$ (resp. $w_I R^+_I = -R^+_I$). Let Λ denote the character group of T, and $\Lambda_P = \{\lambda \in \Lambda | \langle \lambda, \alpha^\vee \rangle = 0 \ \forall \alpha \in I\}$. For each simple root α let $\varpi_\alpha \in \Lambda$ denote the corresponding fundamental weight. Put $\rho = \frac{1}{2}\sum_{\alpha \in R^{\rm s}} \alpha \in \Lambda$, $\rho_P = \frac{1}{2}\sum_{\alpha \in R^+ \setminus R^+_I} \alpha \in \Lambda_P$, and $\rho_I = \rho - \rho_P = \frac{1}{2}\sum_{\alpha \in R^+_I} \alpha$. For $w \in W$ and $\lambda \in \Lambda$ we write $w \bullet \lambda$ for $w(\lambda + \rho) - \rho$. Let ∇^P denote the induction functor from the category of B-modules to P-modules. Put $\mathcal{P} = G/P$.

Recall from [J, II.5.4] that $\forall \lambda \in (-\rho_I + \Lambda^+_I) \setminus \Lambda^+_I$,

(1) $$\mathrm{H}^\bullet(P/B, \mathcal{L}_{P/B}(\lambda)) = 0.$$

and from [J, II.4.5] that $\forall \lambda \in \Lambda^+_I$, $\forall i \in \mathbb{N}$,

(2) $$\mathrm{H}^i(P/B, \mathcal{L}_{P/B}(\lambda)) \neq 0 \text{ iff } i = 0,$$

in which case $\mathrm{H}^0(P/B, \mathcal{L}_{P/B}(\lambda)) = \nabla^P(\lambda)$ is P-simple of highest weight λ.

Put $\mathcal{B} = G/B$. For $\lambda \in \Lambda_I^+$ degeneracy of the spectral sequence yields

$$(3) \qquad \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda))) \simeq \mathrm{H}^i(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\lambda)).$$

For $\lambda \in \Lambda$ recall also Bott's theorem [J, II.5.5] that

$$(4) \qquad \mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\lambda)) = 0 \text{ iff } \lambda \in W \bullet \{(-\rho + \Lambda^+) \setminus \Lambda^+\}.$$

(1.2) For each $w \in W^P$ set

$$(1) \qquad \mu_w = w^{-1} \bullet (- \sum_{\substack{\alpha \in R^{\mathrm{s}} \\ w^{-1}\alpha < 0}} \varpi_\alpha),$$

and let $\mathcal{E}(w) = \mathcal{L}_{\mathcal{P}}(\nabla^P(\mu_w))$ be the locally free sheaf on $\mathcal{P}$ associated to $\nabla^P(\mu_w)$.

Let $\Lambda^+ = \{\lambda \in \Lambda | \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in R^{\mathrm{s}}\}$ be the set of dominant weights of Λ, and put $\Lambda_I^+ = \{\lambda \in \Lambda | \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in I\}$. Recall from [K14, 1.5] that $\forall w \in W^P$,

$$(2) \qquad \mu_w \in \Lambda_I^+.$$

To compute the extension groups among the $\mathcal{E}(w)$'s, one has $\forall x, y \in W^P$,

$$(3) \quad \mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(y)) \simeq \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\mu_x)^* \otimes \nabla^P(\mu_y))) \simeq \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(-w_I\mu_x) \otimes \nabla^P(\mu_y))).$$

with $\nabla^P(-w_I\mu_x) \otimes \nabla^P(\mu_y) = \coprod_{\lambda \in \in \Lambda_I^+} \nabla^P(\lambda)^{\oplus n_\lambda}$ and the multiplicity n_λ determined by the Littlewood-Richardson rule.

(1.3) **Remark:** Over a field $\Bbbk$ of positive characteristic p, if $p \geq h$ the Coxeter number of G, $\nabla^{P_\Bbbk}(\mu_w)$ is $P_\Bbbk$-simple by [K14, 1.5], and hence we still have $\nabla^{P_\Bbbk}(\mu_w)^* \simeq \nabla^{P_\Bbbk}(-w_I\mu_w)$, and $\nabla^{P_\Bbbk}(-w_I\mu_x) \otimes \nabla^{P_\Bbbk}(\mu_y)$ admits a filtration whose subquotients are of the form $\nabla^{P_\Bbbk}(\lambda)$, $\lambda \in \Lambda_I^+$, with the same multiplicities as the ones over $\mathbb{C}$ [J, II.4.21].

(1.4) **Proposition:** *Let $w \in W^P$. $\forall i \in \mathbb{N}$, $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(e), \mathcal{E}(w)) = \delta_{i,0}\delta_{w,e}\mathbb{C}$.*

Proof: Let $w \in W^P$. $\forall i \in \mathbb{N}$,

$$\begin{aligned} \mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(e), \mathcal{E}(w)) &= \mathrm{Ext}^i_{\mathcal{P}}(\mathcal{O}_{\mathcal{P}}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\mu_w))) \simeq \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\mu_w))) \\ &\simeq \mathrm{H}^i(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mu_w)) \quad \text{by (1.1.3)} \\ &= \delta_{i,0}\delta_{w,e}\mathbb{C} \quad \text{by (1.1.1 and 2) with } P = B. \end{aligned}$$

2° Grassmannians

We now specialize into the case where $\mathcal{P}$ is a complex Grassmannian.

(2.1) Let E be an n-dimensional $\mathbb{C}$-linear space of basis e_i, $i \in [1, n]$, $G = \mathrm{GL}(E)$, B the Borel subgroup of G consisting of lower triangular matrices, T the maximal torus of B consisting of diagonals. Let Λ be the weight group of T with a basis $\varepsilon_i : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_i$, $i \in [1, n]$. If $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i \in [1, n[$, $R^{\mathrm{s}} = \{\alpha_i | i \in [1, n[\}$ constitutes the set of simple roots for (G, T). We will abbreviate as ϖ_i the fundamental weight $\varpi_{\alpha_i} = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i$ associated to the simple root α_i. For convenience we let $\varpi_0 = 0 = \varpi_n$. Let $k \in [1, n[$ and $P = \mathrm{N}_G(\coprod_{i=1}^k \mathbb{C}e_{n-k+i})$ the parabolic subgroup of G stabilizing the subspace $\coprod_{i=1}^k \mathbb{C}e_{n-k+i}$ of E. Thus the standard Levi L_P of P has its set of simple roots I such that $R^{\mathrm{s}} \setminus I = \{\alpha_{n-k}\}$. We identify the Weyl group W of G with the symmetric group $\mathfrak{S}_n$ permuting the ε_i's. Put $\mathcal{P} = G/P$.

Recall from [J, II.13.10] that

(1) $$W^P = \{w \in W \mid w(1) < w(2) < \cdots < w(n-k), \\ w(n-k+1) < w(n-k+2) < \cdots < w(n)\}$$

and that $\forall x, y \in W^P$,

(2) $x \leq y$ in the Chevalley-Bruhat order iff $x(n-k+i) \geq y(n-k+i)$ $\forall i \in [1, k]$.

Thus we may parametrize each $w \in W^P$ by the k-tuple $(w(n-k+1), w(n-k+2), \dots, w(n))$. Put $w^I = w_0 w_I$ the longest element of W^P. In the notation of (1.2)

(3) $$\mu_e = \mu_{(n-k+1, n-k+2, \dots, n)} = 0 \quad \text{and} \quad \mu_{w^I} = \mu_{(1,2,\dots,k)} = (1-n)\varpi_{n-k}.$$

We let ∇ denote the induction functor from the category of B-modules to the category of G-modules.

Proposition: *Let $w \in W^P$ and $i \in \mathbb{N}$.*

(i) $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(w), \mathcal{E}(e)) \simeq \delta_{i,0} \nabla(-w_I \mu_w)$ *with* $-w_I \mu_w \in \Lambda^+$.

(ii) $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(w^I), \mathcal{E}(w)) = \delta_{i,0} \nabla(\mu_w - \mu_{w^I})$ *with* $\mu_w - \mu_{w^I} \in \Lambda^+$.

Proof: Let $w \in W^P$ and $i \in \mathbb{N}$.

(i) One has $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(w), \mathcal{E}(e)) \simeq \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(-w_I \mu_w))) \simeq \mathrm{H}^i(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(-w_I \mu_w))$ as $-w_I \mu_w \in \Lambda^+_I$. We have now to verify that $\langle -w_I \mu_w, \alpha^\vee_{n-k} \rangle \geq 0$. We may assume $w \neq e$. Then

$$\begin{aligned} \langle -w_I \mu_w, \alpha^\vee_{n-k} \rangle &= -\langle \mu_w, \alpha^\vee_{n-k} \rangle = -\{\langle w^{-1}(\rho + \sum_{\substack{\alpha \in R^{\mathrm{s}} \\ w^{-1}\alpha < 0}} - \varpi_\alpha), \alpha^\vee_{n-k} \rangle - 1\} \\ &= 1 - \langle \sum_{\substack{\alpha \in R^{\mathrm{s}} \\ w^{-1}\alpha > 0}} \varpi_\alpha, w\alpha^\vee_{n-k} \rangle \end{aligned}$$

with $w\alpha_{n-k} = w(\varepsilon_{n-k} - \varepsilon_{n-k+1}) = \varepsilon_{w(n-k)} - \varepsilon_{w(n-k+1)}$. As $w \neq e$, we must have $w(n-k) > n-k$ while $w(n-k+1) < n-k+1$, and hence $w\alpha_{n-k} < 0$.

(ii) Likewise

$$\begin{aligned} \mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(w^I), \mathcal{E}(w)) &\simeq \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}((-\mu_{w^I}) \otimes \nabla^P(\mu_w)) = \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}((n-1)\varpi_{n-k} \otimes \nabla^P(\mu_w)) \\ &\simeq \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P((n-1)\varpi_{n-k} + \mu_w))) \quad \text{by the tensor identity} \\ &\simeq \mathrm{H}^i(\mathcal{B}, \mathcal{L}_{\mathcal{B}}((n-1)\varpi_{n-k} + \mu_w)) \quad \text{as } (n-1)\varpi_{n-k} + \mu_w \in \Lambda^+_I. \end{aligned}$$

It remains to check that $\langle \mu_w, \alpha_{n-k}^\vee \rangle \geq 1-n$. We may assume $w \neq e$ again. Then

$$\begin{aligned}\langle \mu_w, \alpha_{n-k}^\vee \rangle &= \langle \rho - \sum_{\substack{\alpha \in R^{\mathrm{s}} \\ w^{-1}\alpha < 0}} \varpi_\alpha, w\alpha_{n-k}^\vee \rangle - 1 = \langle \sum_{\substack{\alpha \in R^{\mathrm{s}} \\ w^{-1}\alpha > 0}} \varpi_\alpha, w\alpha_{n-k}^\vee \rangle - 1 \\ &\geq -\langle \rho, \alpha_0^\vee \rangle + 1 - 1 \quad \text{as} \quad \sum_{\substack{\alpha \in R^{\mathrm{s}} \\ w^{-1}\alpha > 0}} \varpi_\alpha \neq \rho. \\ &= 1-n.\end{aligned}$$

(2.2) Throughout the rest of the paper we will further restrict ourselves to the case $k=2$, and show

Theorem: *(i) All $\mathcal{E}(w)$, $w \in W^P$, are exceptional, i.e., $\forall i \in \mathbb{N}$, $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(w), \mathcal{E}(w)) = \delta_{i,0}\mathbb{C}$.*

(ii) $\forall x, y \in W^P$, $\forall i > 0$, $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(y)) = 0$ while $\mathrm{Mod}_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(y)) \neq 0$ iff $x \geq y$ in the Chevalley-Bruhat order.

(iii) The $\mathcal{E}(w), w \in W^P$, Karoubian generate the bounded derived category of coherent sheaves on $\mathcal{P}$.

3° Extensions

(3.1) In the present setup $\nabla^P(-w_I\mu_x) \otimes \nabla^P(\mu_y)$ from (1.2.3) is multiplicity-free. The assertions (i) and (ii) of Theorem 2.2 will follow from

Proposition: *Let $x, y \in W^P$ and let $\nabla^P(\lambda)$, $\lambda \in \Lambda_I^+$, be a direct summand of $\nabla^P(-w_I\mu_x) \otimes \nabla^P(\mu_y)$.*

(i) If $x = y$, there is a unique direct summand $\nabla^P(\lambda)$ with $\lambda = 0$ and all the other $\lambda \in W \bullet \{(-\rho + \Lambda^+) \setminus \Lambda^+\}$.

(ii) If $x > y$ in the Chevalley-Bruhat order, there is a unique direct summand $\nabla^P(\lambda)$ with $\lambda \in \Lambda^+$ and all the other $\lambda \in W \bullet \{(-\rho + \Lambda^+) \setminus \Lambda^+\}$.

(iii) If $x \not\geq y$ in the Chevalley-Bruhat order, all direct summands $\nabla^P(\lambda)$ have $\lambda \in W \bullet \{(-\rho + \Lambda^+) \setminus \Lambda^+\}$.

(3.2) In the rest of this section we will show (3.1). We have $W^P = \{(b, b+1), (b, b+1+a) | b \in [1, n[, a \in [1, n-b[\}$. We will sometimes denote $\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_n\varepsilon_n \in \Lambda$ by the n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. As $(1, \ldots, 1)$ gives G-module det, we will consider $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ modulo $\mathbb{Z}(1, \ldots, 1)$.

For each $w \in W^P$ let also $\lambda_w = (0, \ldots, 0, w(n) - 2, w(n-1) - 1)$. We know from [Kap83] that

$$\text{(1)} \qquad \mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_w)), \mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_w))) = \delta_{i,0}\mathbb{C} \quad \forall i \in \mathbb{N},$$

$$\text{(2)} \qquad \mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_x)), \mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_y))) = 0 \quad \forall x, y \in W^P, \forall i > 0,$$

$$\text{(3)} \qquad \mathrm{Mod}_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_x)), \mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_y))) \neq 0 \text{ iff } x \leq y \text{ in the Chevalley-Bruhat order.}$$

and that the $\mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_w))$, $w \in W^P$, Karoubian generate the bounded derived category of coherent sheaves on $\mathcal{P}$.

The relation of our μ_w's to the λ_w's is as follows: $\forall b \in [1, n-2]$,

$$\text{(4)} \qquad \mu_{(b,b+1)} = \varpi_{b-1} - (n-b)\varpi_{n-2} = (\overbrace{1, \dots, 1}^{b-1}, 0, \dots, 0, n-b, n-b)$$

does not have a counterpart in the λ_w's, while $\forall a \in [1, n-b[$,

$$\text{(5)} \qquad \begin{aligned} \mu_{(b,b+1+a)} &= -(n-b-1)\varpi_{n-2} + (a-1)\varpi_{n-1} = (0, \dots, 0, n-b-1, n-b-a) \\ &= \lambda_{(n+1-b-a, n+1-b)}. \end{aligned}$$

On the other hand, $\lambda_{(1,b+1)} = (0, \dots, 0, b-1, 0)$, $b \in [2, n[$, are missing from the μ_w's. Thus, if we let $W^P_{\mathcal{K}} = \{(n-1, n)\} \sqcup \{(b, b+1+a) | b \in [1, n-2], a \in [1, n-b[\}$,

$$\text{(6)} \qquad \{\mu_w | w \in W^P\} = \{\lambda_w | w \in W^P\},$$

$$\text{(7)} \qquad \{\mu_w | w \in W^P\} = \{\mu_w | w \in W^P_{\mathcal{K}}\} \sqcup \{\mu_{(b,b+1)} | b \in [1, n-2]\},$$

$$\text{(8)} \qquad \{\lambda_w | w \in W^P\} = \{\lambda_w | w \in W^P_{\mathcal{K}}\} \sqcup \{\lambda_{(1,b+1)} | b \in [2, n[\}.$$

Note also that $\forall (b, b+1+a), (b', b'+1+a') \in W^P_{\mathcal{K}}$,

$$\text{(9)} \qquad (b, b+1+a) \geq (b', b'+1+a') \text{ in the Chevalley-Bruhat order iff } (n+1-b'-a', n+1-b') \geq (n+1-b-a, n+1-b).$$

Thus, for $x, y \in W^P_{\mathcal{K}}$, we know already from (1)-(3) that $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(y)) = 0$ $\forall i > 0$, that $\mathrm{Mod}_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(x)) = \mathbb{C}$, and that $\mathrm{Mod}_{\mathcal{P}}(\mathcal{E}(x), \mathcal{E}(y)) \neq 0$ iff $x \geq y$.

One has $-w_I\mu_{(b,b+1)} = \varpi_{n-1-b} + (n-b-1)\varpi_{n-2}$ and $-w_I\mu_{(b,b+1+a)} = (n-b-a)\varpi_{n-2} + (a-1)\varpi_{n-1}$, and hence that

$$\text{(10)} \qquad \nabla^P(\mu_{(b,b+1)})^* \simeq \nabla^P(\varpi_{n-1-b} + (n-b-1)\varpi_{n-2}),$$

$$\text{(11)} \qquad \nabla^P(\mu_{(b,b+1+a)})^* \simeq \nabla^P((n-b-a)\varpi_{n-2} + (a-1)\varpi_{n-1}).$$

(3.3) Let θ be an automorphism of G such that $g \mapsto (n_0 g n_0^{-1})^{-\mathrm{tr}}$, where $n_0 \in \mathrm{N}_G(T)$ is a lift of w_0 and $-\mathrm{tr}$ denotes the operation taking the inverse transpose. Let $Q = \theta(P)$ be another standard parabolic subgroup of G correponding to $R^{\mathrm{s}} \setminus \{\alpha_2\}$. Let $\bar{\theta} : G/P \to G/Q$ be the isomorphism induced by θ. For a P- (resp. B-) module M let ${}^\theta M$ denote the Q- (resp. B-) module on the $\mathbb{C}$-linear space M with the action given by $g \bullet m = \theta^{-1}(g)m$, $g \in Q$, $m \in M$ [J, I.3.5]. In particular, $\forall \lambda \in \Lambda$,

$$\text{(1)} \qquad {}^\theta\lambda = -w_0\lambda.$$

One also has from [J, I.3.5.4]

$$\bar{\theta}_*\mathcal{L}_{G/P}(\nabla^P(\lambda)) \simeq \mathcal{L}_{G/Q}({}^\theta\nabla^P(\lambda)) \simeq \mathcal{L}_{G/Q}(\nabla^Q(-w_0\lambda)). \tag{2}$$

(3.4) **Proposition:** $\forall b \in [1, n[$, *all* $\mathcal{E}(b, b+1)$ *are exceptional.*

Proof: In the notation of (3.3) we have only to show that

$$\mathrm{Ext}^i_{G/Q}(\mathcal{L}_{G/Q}(\nabla^Q(-w_0\mu_{(b,b+1)})), \mathcal{L}_{G/Q}(\nabla^Q(-w_0\mu_{(b,b+1)}))) = \delta_{i,0}\mathbb{C}.$$

As $-w_0\mu_{(b,b+1)} = (b-n)\varpi_2 + \varpi_{n-b+1}$ and as $(b-n)\varpi_2$ is a Q-module,

$$\begin{aligned}
&\mathrm{Ext}^i_{G/Q}(\mathcal{L}_{G/Q}(\nabla^Q(-w_0\mu_{(b,b+1)})), \mathcal{L}_{G/Q}(\nabla^Q(-w_0\mu_{(b,b+1)}))) \\
&\quad = \mathrm{Ext}^i_{G/Q}(\mathcal{L}_{G/Q}((b-n)\varpi_2 + \varpi_{n-b+1})), \mathcal{L}_{G/Q}((b-n)\varpi_2 + \varpi_{n-b+1}))) \\
&\quad \simeq \mathrm{Ext}^i_{G/Q}(\mathcal{L}_{G/Q}(\nabla^Q(\varpi_{n-b+1})), \mathcal{L}_{G/Q}(\nabla^Q(\varpi_{n-b+1}))) \\
&\quad = \delta_{i,0}\mathbb{C} \quad \text{by [Kap83]/[K08, Prop. 2].}
\end{aligned}$$

(3.5) **Proposition:** *Let* $1 \leq b < c \leq n-1$. $\forall i > 0$, $\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(b, b+1), \mathcal{E}(c, c+1)) = 0$ *while* $\mathrm{Mod}_{\mathcal{P}}(\mathcal{E}(b, b+1), \mathcal{E}(c, c+1)) \neq 0$.

Proof: By (2.1) we may assume $b \geq 2$. As $(n-1, n) = e$, we may also assume $c \leq n-2$ by (1.4). Thus $b \in [2, n-3]$. One has

$$\begin{aligned}
&\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(b, b+1), \mathcal{E}(c, c+1)) \\
&\quad = \mathrm{Ext}^i_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{b-1} - (n-b)\varpi_{n-2})), \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{c-1} - (n-c)\varpi_{n-2}))) \\
&\quad \simeq \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{n-1-b}) \otimes \nabla^P(\varpi_{c-1}) \otimes (c-b-1)\varpi_{n-2})) \quad \text{by (3.2.10).}
\end{aligned} \tag{1}$$

The direct summands of $\nabla^P(\varpi_{n-1-b}) \otimes \nabla^P(\varpi_{c-1})$ are, as $n-1-b, c-1 \leq n-3$, of the form $\nabla^P(\lambda)$ with $\lambda = \varpi_{n-1-b} + \varpi_{c-1} - \gamma$, $\gamma \in \sum_{i=1}^{n-3} \mathbb{N}\alpha_i = \sum_{i=1}^{n-3} \mathbb{N}(\varepsilon_i - \varepsilon_{i+1})$. As the coefficient of ε_{n-2} in λ is ≥ 0, $\lambda \in \Lambda^+$, and hence also $\lambda + (c-b-1)\varpi_{n-2} \in \Lambda^+$. The assertion follows.

(3.6) Recall from [H, III.5.1] that on the projective r-space $\mathbb{P}^r$

$$\mathrm{H}^\bullet(\mathbb{P}^r, \mathcal{O}(m)) = 0 \quad \forall m \in [-r, -1]. \tag{1}$$

For $i \in [1, n[$ let $s_i \in W$ denote the reflection associated to the simple root α_i, i.e., the transposition $(i, i+1)$.

Proposition: *If* $1 \leq c < b \leq n-1$, $\mathrm{Ext}^\bullet_{\mathcal{P}}(\mathcal{E}(b, b+1), \mathcal{E}(c, c+1)) = 0$.

Proof: We may assume $b \leq n-2$ by (1.4). One has from (3.5.2)

$$\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(b, b+1), \mathcal{E}(c, c+1)) \simeq \mathrm{H}^i(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{n-1-b}) \otimes \nabla^P(\varpi_{c-1}) \otimes (c-b-1)\varpi_{n-2})).$$

Consider first the case $c = 1$. We must show

$$\mathrm{H}^\bullet(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{n-1-b} - b\varpi_{n-2}))) = 0. \tag{2}$$

One checks $s_{n-b}s_{n-b+1}\dots s_{n-2}\bullet(\varpi_{n-1-b}-b\varpi_{n-2}) = -(b-1)\varpi_{n-1} = (0,\dots,0,b-1)$. If $Q=\mathrm{N}_G(\mathbb{C}e_n)$ is the stabilizer in G of $\mathbb{C}e_n$,

$$\begin{aligned}(3)\qquad \mathrm{H}^\bullet(\mathcal{B},\mathcal{L}_\mathcal{B}(-(b-1)\varpi_{n-1}))&\simeq \mathrm{H}^\bullet(G/Q,\mathcal{L}_{G/Q}(-(b-1)\varpi_{n-1}))\simeq \mathrm{H}^\bullet(\mathbb{P}^{n-1},\mathcal{O}(-(b-1)))\\ &=0\quad \text{as } b-1\in[1,n-3].\end{aligned}$$

Then by Bott's theorem there must be some $w\in W$ such that $w\bullet(-(b-1)\varpi_{n-1})\in(-\rho+\Lambda^+)\setminus\Lambda^+$. Then $ws_{n-b}s_{n-b+1}\dots s_{n-2}\bullet(\varpi_{n-1-b}-b\varpi_{n-2})\in(-\rho+\Lambda^+)\setminus\Lambda^+$, and hence

$$0=\mathrm{H}^\bullet(\mathcal{B},\mathcal{L}_\mathcal{B}(\varpi_{n-1-b}-b\varpi_{n-2}))\simeq \mathrm{H}^\bullet(\mathcal{P},\mathcal{L}_\mathcal{P}(\nabla^P(\varpi_{n-1-b}-b\varpi_{n-2})).$$

We deal next with $\nabla^P(\varpi_{n-1-b})\otimes\nabla^P(\varpi_{c-1})$ with $c\ge 2$. Thus $2\le c<b\le n-2$. Assume first $n-1-b\ge c-1$. The P-module $\nabla^P(\varpi_{n-1-b})\otimes\nabla^P(\varpi_{c-1})$ admits a filtration whose subquotients are $\nabla^P(\varpi_{c-j}+\varpi_{n-b+j-2})$, $j\in[1,c]$, by the Littlewood-Richardson rule [FH, §25.3 and pp. 455-456]. Thus one has only to show that

$$(4)\qquad \mathrm{H}^\bullet(\mathcal{B},\mathcal{L}_\mathcal{B}(\varpi_{c-j}+\varpi_{n-b+j-2}-(b+1-c)\varpi_{n-2}))=0.$$

One checks

$$\begin{aligned}(5)\quad s_{n+c-b-1}s_{n+c-b}\dots s_{n-3}s_{n-2}\bullet(\varpi_{c-j}+\varpi_{n-b+j-2}-(b+1-c)\varpi_{n-2})&\\ =\varpi_{c-j}+\varpi_{n+j-b-2}-\varpi_{n+c-b-2}-(b-c)\varpi_{n-1}.&\end{aligned}$$

Put $\lambda=\varpi_{c-j}+\varpi_{n+j-b-2}-\varpi_{n+c-b-2}-(b-c)\varpi_{n-1}$. We show

$$(6)\qquad \mathrm{H}^\bullet(\mathcal{B},\mathcal{L}_\mathcal{B}(\lambda))=0.$$

Let $Q=\mathrm{N}_G(\coprod_{i=n+c-b-1}^n\mathbb{C}e_i)$. In view of the spectral sequence $\mathrm{E}_2^{s,t}=\mathrm{R}^s\mathrm{ind}_Q^G\mathrm{R}^t\mathrm{ind}_B^Q(\lambda)\Rightarrow \mathrm{R}^{s+t}\mathrm{ind}_B^G(\lambda)\simeq\mathrm{H}^{s+t}(\mathcal{B},\mathcal{L}_\mathcal{B}(\lambda))$, we have only to show that $\mathrm{R}^\bullet\mathrm{ind}_B^Q(\lambda)=0$. Let $L_1\times L_2$ be the standard Levi of Q, and put $B_1=L_1\cap B$, $B_2=L_2\cap B$. Then

$$\begin{aligned}\mathrm{R}^i\mathrm{ind}_B^Q(\lambda)&=\mathrm{R}^i\mathrm{ind}_B^Q(\overbrace{1,\dots,1}^{c-j},\overbrace{0,\dots,0}^{n-b-c-2+2j},\overbrace{-1,\dots,-1}^{c-j},\overbrace{0,\dots,0}^{b-c+1},b-c)\\ &\simeq\coprod_{s+t=i}\mathrm{R}^s\mathrm{ind}_{B_1}^{L_1}(\overbrace{1,\dots,1}^{c-j},\overbrace{0,\dots,0}^{n-b-c-2+2j},\overbrace{-1,\dots,-1}^{c-j})\otimes\mathrm{R}^t\mathrm{ind}_{B_2}^{L_2}(\overbrace{0,\dots,0}^{b-c+1},b-c)\\ &\qquad\qquad\text{by the Künneth formula}\\ &\simeq\mathrm{ind}_{B_1}^{L_1}(\overbrace{1,\dots,1}^{c-j},\overbrace{0,\dots,0}^{n-b-c-2+2j},\overbrace{-1,\dots,-1}^{c-j})\otimes\mathrm{R}^i\mathrm{ind}_{B_2}^{L_2}(\overbrace{0,\dots,0}^{b-c+1},b-c).\end{aligned}$$

But $\mathrm{R}^\bullet\mathrm{ind}_{B_2}^{L_2}(\overbrace{0,\dots,0}^{b-c+1},b-c)\simeq\mathrm{H}^\bullet(\mathbb{P}^{b-c+1},\mathcal{O}(-(b-c)))=0$, and hence (7). Then there is some $w\in W$ such that $w\bullet\lambda\in(-\rho+\Lambda^+)\setminus\Lambda^+$. Thus $ws_{n+c-b-1}s_{n+c-b}\dots s_{n-3}s_{n-2}\bullet(\varpi_{c-j}+\varpi_{n-b+j-2}-(b+1-c)\varpi_{n-2})\in(-\rho+\Lambda^+)\setminus\Lambda^+$, and (5) follows.

Assume next $n-1-b<c-1$, i.e., $n<b+c$. By the Littlewood-Richardson rule $\nabla^P(\varpi_{n-1-b})\otimes\nabla^P(\varpi_{c-1})$ has a filtration whose subquotients are $\nabla^P(\varpi_{n-b-j}+\varpi_{c+j-2})$, $j\in[1,n-b]$. Thus one has only to show that

$$(7)\qquad \mathrm{H}^\bullet(\mathcal{B},\mathcal{L}_\mathcal{B}(\varpi_{n-b-j}+\varpi_{c+j-2}-(b+1-c)\varpi_{n-2}))=0.$$

One checks

$$s_{n-(b-c+1)}\dots s_{n-3}s_{n-2}\bullet(\varpi_{n-b-j}+\varpi_{c+j-2}-(b+1-c)\varpi_{n-2}) \\ =\varpi_{n-j-b}+\varpi_{c+j-2}-\varpi_{n-(b-c+2)}-(b-c)\varpi_{n-1}.$$

We show

$$\mathrm{H}^\bullet(\mathcal{B},\mathcal{L}_{\mathcal{B}}(\varpi_{n-j-b}+\varpi_{c+j-2}-\varpi_{n-(b-c+2)}-(b-c)\varpi_{n-1})=0. \tag{8}$$

If $Q=\mathrm{N}_G(\coprod_{i=n-b+c+1}^n\mathbb{C}e_i)$ as above, there is a spectral sequence

$$\mathrm{E}^{s,t}=\mathrm{R}^s\mathrm{ind}_Q^G\mathrm{R}^t\mathrm{ind}_B^Q(\varpi_{n-j-b}+\varpi_{c+j-2}-\varpi_{n-(b-c+2)}-(b-c)\varpi_{n-1}) \\ \Rightarrow\mathrm{H}^{s+t}(\mathcal{B},\mathcal{L}_{\mathcal{B}}(\varpi_{n-j-b}+\varpi_{c+j-2}-\varpi_{n-(b-c+2)}-(b-c)\varpi_{n-1})$$

with

$$\mathrm{R}^t\mathrm{ind}_B^Q(\varpi_{n-j-b}+\varpi_{c+j-2}-\varpi_{n-(b-c+2)}-(b-c)\varpi_{n-1}) \\ \simeq\mathrm{ind}_{B_1}^{L_1}(\overbrace{1,\dots,1}^{n-j-b},\ \overbrace{0,\dots,0}^{c+b-n-2+2j},\overbrace{-1,\dots,-1}^{n-b-j})\otimes\mathrm{R}^t\mathrm{ind}_{B_2}^{L_2}(\overbrace{0,\dots,0}^{b-c+1},b-c).$$

As $\mathrm{R}^\bullet\mathrm{ind}_{B_2}^{L_2}(0,\dots,0,b-c)\simeq\mathrm{H}^\bullet(\mathbb{P}^{b-c+1},\mathcal{O}(-(b-c)))=0$, (8) follows. Then there is $w\in W$ such that $w\bullet(\varpi_{n-j-b}+\varpi_{c+j-2}-\varpi_{n-(b-c+2)}-(b-c)\varpi_{n-1})\in(-\rho+\Lambda^+)\setminus\Lambda^+$. Thus $ws_{n-(b-c+1)}\dots s_{n-3}s_{n-2}\bullet(\varpi_{n-b-j}+\varpi_{c+j-2}-(b+1-c)\varpi_{n-2})\in(-\rho+\Lambda^+)\setminus\Lambda^+$, and (7) follows from Bott's theorem again.

(3.7) **Proposition:** *If $1\le b\le c\le n-2$, then $\forall i\in\mathbb{N}$, $\forall a\in[1,n-c[$,*

$$\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(b,b+1),\mathcal{E}(c,c+1+a))=\delta_{i0}\nabla((c-b)\varpi_{n-2}+\varpi_{n-1-b}+(a-1)\varpi_{n-1}).$$

Proof: One has

$$\mathrm{Ext}^i_{\mathcal{P}}(\mathcal{E}(b,b+1),\mathcal{E}(c,c+1+a)) \\ \simeq\mathrm{H}^i(\mathcal{P},\mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{n-1-b}+(n-b-1)\varpi_{n-2})\otimes\nabla^P(-(n-c-1)\varpi_{n-2}+(a-1)\varpi_{n-1}))) \\ \simeq\mathrm{H}^i(\mathcal{P},\mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{n-1-b}+(n-b-1)\varpi_{n-2}-(n-c-1)\varpi_{n-2}+(a-1)\varpi_{n-1}))) \\ \simeq\mathrm{H}^i(\mathcal{B},\mathcal{L}_{\mathcal{B}}(\varpi_{n-1-b}+(c-b)\varpi_{n-2}+(a-1)\varpi_{n-1})) \\ =\delta_{i0}\nabla(\varpi_{n-1-b}+(c-b)\varpi_{n-2}+(a-1)\varpi_{n-1}).$$

(3.8) **Proposition:** *If $1\le c<b\le n-1$, then $\forall a\in[1,n-c[$,*

$$\mathrm{Ext}^\bullet_{\mathcal{P}}(\mathcal{E}(b,b+1),\mathcal{E}(c,c+1+a))=0.$$

Proof: We may assume $b\le n-2$.As $\mathrm{Ext}^\bullet_{\mathcal{P}}(\mathcal{E}(b,b+1),\mathcal{E}(c,c+1+a))\simeq\mathrm{H}^\bullet(\mathcal{B},\mathcal{L}_{\mathcal{B}}(\varpi_{n-1-b}+(c-b)\varpi_{n-2}+(a-1)\varpi_{n-1}))$, we may also assume that $c-b\le-2$, i.e., $3\le c+2\le b$. One checks

$$s_{n-(b-c)}s_{n-(b-c)+1}\dots s_{n-2}\bullet(\varpi_{n-1-b}+(c-b)\varpi_{n-2}+(a-1)\varpi_{n-1}) \\ =\varpi_{n-1-b}-\varpi_{n-(b-c+1)}-(b-c-a)\varpi_{n-1}.$$

We may then assume that $b-c-a \geq 2$.

With $Q = \mathrm{N}_G(\coprod_{i=n-(b-c+1)+1}^{n} \mathbb{C}e_i)$ one shows as in (3.6) that $\mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}_\mathcal{B}(\varpi_{n-1-b} - \varpi_{n-(b-c+1)} - (b-c-a)\varpi_{n-1})) = 0$. Then there is some $w \in W$ such that $w \bullet (\varpi_{n-1-b} - \varpi_{n-(b-c+1)} - (b-c-a)\varpi_{n-1}) \in (-\rho + \Lambda^+) \setminus \Lambda^+$. Thus $ws_{n-(b-c)}s_{n-(b-c)+1}\dots s_{n-2} \bullet (\varpi_{n-1-b} + (c-b)\varpi_{n-2} + (a-1)\varpi_{n-1}) \in (-\rho + \Lambda^+) \setminus \Lambda^+$, and the assertion follows.

(3.9) **Proposition:** *If* $1 \leq c \leq b$ *and* $c+1+a \leq b+1$, *then* $\forall i \in \mathbb{N}$, $\forall a \in [1, n-c[$,

$$\mathrm{Ext}^i_\mathcal{P}(\mathcal{E}(c, c+1+a), \mathcal{E}(b, b+1)) = \delta_{i,0}\nabla(\varpi_{b-1} + (b-c-a)\varpi_{n-2} + (a-1)\varpi_{n-1}).$$

Proof: One has

$$\begin{aligned}
&\mathrm{Ext}^i_\mathcal{P}(\mathcal{E}(c, c+1+a), \mathcal{E}(b, b+1)) \\
&\qquad \simeq \mathrm{H}^i(\mathcal{P}, \mathcal{L}_\mathcal{P}(\nabla^P((n-c-a)\varpi_{n-2} + (a-1)\varpi_{n-1}) \otimes \nabla^P(\varpi_{b-1} - (n-b)\varpi_{n-2}))) \\
&\qquad \simeq \mathrm{H}^i(\mathcal{B}, \mathcal{L}_\mathcal{B}(\varpi_{b-1} + (b-c-a)\varpi_{n-2} + (a-1)\varpi_{n-1})) \\
&\qquad = \delta_{i,0}\nabla(\varpi_{b-1} + (b-c-a)\varpi_{n-2} + (a-1)\varpi_{n-1}).
\end{aligned}$$

(3.10) **Proposition:** *If* $1 \leq b \leq c$, $\forall a \in [1, n-c[$,

$$\mathrm{Ext}^\bullet_\mathcal{P}(\mathcal{E}(c, c+1+a), \mathcal{E}(b, b+1)) = 0.$$

Proof: We may assume $b \leq n-2$. As $\mathrm{Ext}^\bullet_\mathcal{P}(\mathcal{E}(c, c+1+a), \mathcal{E}(b, b+1)) \simeq \mathrm{H}^i(\mathcal{B}, \mathcal{L}_\mathcal{B}(\varpi_{b-1} - (c+a-b)\varpi_{n-2} + (a-1)\varpi_{n-1}))$, we may also assume $c+a-b \geq 2$. One checks

(1)
$$\begin{aligned}
&s_{n-(c+a-b)}s_{n-(c+a-b)+1}\dots s_{n-2} \bullet (\varpi_{b-1} - (c+a-b)\varpi_{n-2} + (a-1)\varpi_{n-1}) \\
&\qquad\qquad = \varpi_{b-1} - \varpi_{n-(c+a-b+1)} - (c-b)\varpi_{n-1}.
\end{aligned}$$

With $Q = \mathrm{N}_G(\coprod_{i=n-(c+a-b+1)+1}^{n} \mathbb{C}e_i)$ one shows as in (3.6) again that $\mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}(\varpi_{b-1} - \varpi_{n-(c+a-b+1)} - (c-b)\varpi_{n-1})) = 0$. Then there is some $w \in W$ such that $w \bullet (\varpi_{b-1} - \varpi_{n-(c+a-b+1)} - (c-b)\varpi_{n-1}) \in (-\rho + \Lambda^+) \setminus \Lambda^+$. Thus $ws_{n-(c+a-b)}s_{n-(c+a-b)+1}\dots s_{n-2} \bullet (\varpi_{b-1} - (c+a-b)\varpi_{n-2} + (a-1)\varpi_{n-1}) \in (-\rho + \Lambda^+) \setminus \Lambda^+$, and the assertion holds.

(3.11) Finally,

Proposition: *If* $1 \leq c < b < c+a \leq n-1$,

$$\mathrm{Ext}^\bullet_\mathcal{P}(\mathcal{E}(c, c+1+a), \mathcal{E}(b, b+1)) = 0.$$

Proof: We may assume $b \leq n-2$ again. As $\mathrm{Ext}^\bullet_\mathcal{P}(\mathcal{E}(c, c+1+a), \mathcal{E}(b, b+1)) \simeq \mathrm{H}^i(\mathcal{B}, \mathcal{L}_\mathcal{B}(\varpi_{b-1} - (c+a-b)\varpi_{n-2} + (a-1)\varpi_{n-1}))$, we may also assume $c+a-b \geq 2$. By (3.10.1) one has

$$\begin{aligned}
&s_{n-(c+a-b)}s_{n-(c+a-b)+1}\dots s_{n-2} \bullet (\varpi_{b-1} - (c+a-b)\varpi_{n-2} + (a-1)\varpi_{n-1}) \\
&\qquad\qquad = \varpi_{b-1} - \varpi_{n-(c+a-b+1)} + (b-c)\varpi_{n-1} \in (-\rho + \lambda^+) \setminus \Lambda^+,
\end{aligned}$$

and the assertion follows from Bott's theorem.

4° **Generation**

Let $\langle \mathcal{E} \rangle$ denote the triangulated full subcategory Karoubian generated by the $\mathcal{E}(w)$, $w \in W^P$. As Kapranov's sheaves $\mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_w))$, $w \in W^P$, Karoubian generate $\mathrm{D}^b(\mathrm{coh}(\mathcal{P}))$, we want all $\mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_w)) \in \langle \mathcal{E} \rangle$.

(4.1) Let $\Delta : \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ be the diagonal imbedding, $p_1, p_2 : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ the projections, and $\mathcal{K}^\bullet \to \Delta_* \mathcal{O}_{\mathcal{P}}$ Kapranov's resolution. Then for each coherent sheaf $\mathcal{F}$ on $\mathcal{P}$ one has

(1) $$\mathcal{F} \simeq \mathbf{R}p_{1*}((p_2^* \mathcal{F}) \otimes^{\mathbf{L}}_{\mathcal{P}\times\mathcal{P}} \mathcal{K}^\bullet)$$

with each $\mathbf{R}p_{1*}((p_2^* \mathcal{F}) \otimes^{\mathbf{L}}_{\mathcal{P}\times\mathcal{P}} \mathcal{K}^i)$, $i \in \mathbb{Z}$, admitting a filtration with the subquotients

(2) $$\begin{aligned}\mathbf{R}p_{1*}\{(p_2^* \mathcal{F}) &\otimes^{\mathbf{L}}_{\mathcal{P}\times\mathcal{P}} (\mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_w)) \boxtimes \mathcal{L}_{\mathcal{P}}(\nabla^P(\tilde{\lambda}_w))\} \\ &\simeq \mathcal{L}_{\mathcal{P}}(\nabla^P(\lambda_w)) \otimes_{\mathbb{C}} \mathbf{R}\Gamma(\mathcal{P}, \mathcal{F} \otimes_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}(\nabla^P(\tilde{\lambda}_w))), \quad w \in W^P \text{ with } |\lambda_w| = i,\end{aligned}$$

where $|\lambda_w| = \sum_{i=1}^n \lambda_{w,i}$ if $\lambda_w = (\lambda_{w,1}, \dots, \lambda_{w,n}) = \lambda_{w,1}\varepsilon_1 + \cdots + \lambda_{w,n}\varepsilon_n$, and $\tilde{\lambda}_w$ is defined as follows. We have $\lambda_{i,w} = 0$ $\forall i \in [1, n-2]$. Letting $((\lambda^t_w)_1, \dots, (\lambda^t_w)_{n-2})$ denote the transpose of $(\lambda_{w,n-1}, \lambda_{w,n})$, we set $\tilde{\lambda}_w = (-(\lambda^t_w)_{n-2}, \dots, -(\lambda^t_w)_1, 0, 0)$.

Recall that $\mathcal{E}(w) = \mathcal{L}_{\mathcal{P}}(\nabla^P(\mu_w))$, $w \in W^P$, and $\{\mu_w | w \in W^P\} = \{\mu_w | w \in W^P_{\mathcal{K}}\} \sqcup \{\mu_w | w \in W^P \setminus W^P_{\mathcal{K}}\}$, $\{\lambda_w | w \in W^P\} = \{\lambda_w | w \in W^P_{\mathcal{K}}\} \sqcup \{\lambda_w | w \in W^P \setminus W^P_{\mathcal{K}}\}$ with $\{\mu_w | w \in W^P_{\mathcal{K}}\} = \{\lambda_w | w \in W^P_{\mathcal{K}}\}$. Thus we will be done if we show $\forall x \in W^P \setminus W^P_{\mathcal{K}}$, $\exists x' \in W^P \setminus W^P_{\mathcal{K}}$: $\forall y \in (W^P \setminus W^P_{\mathcal{K}}) \setminus \{x\}$,

(3) $$\mathbf{R}\Gamma(\mathcal{P}, \mathcal{E}(x') \otimes_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}(\nabla^P(\tilde{\lambda}_y))) = 0,$$

(4) $$\mathbf{R}\Gamma(\mathcal{P}, \mathcal{E}(x') \otimes_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}(\nabla^P(\tilde{\lambda}_x))) \neq 0.$$

We have $\{\mu_w | w \in W^P \setminus W^P_{\mathcal{K}}\} = \{\mu_{(b,b+1)} = \varpi_{b-1} - (n-b)\varpi_{n-2} | b \in [1, n-2]\}$ while $\{\lambda_w | w \in W^P \setminus W^P_{\mathcal{K}}\} = \{\lambda_{(1,c)} = (0, \dots, 0, c-2, 0) | c \in [3, n]\}$, and hence

$$\tilde{\lambda}_{(1,c)} = (\overbrace{0, \dots, 0}^{n-c}, \overbrace{-1, \dots, -1}^{c-2}, 0, 0) = \varpi_{n-c} - \varpi_{n-2}.$$

We will show

Proposition: $\forall b \in [1, n-2]$, $\forall c \in [3, n]$,

$$\mathrm{H}^\bullet(\mathcal{P}, \mathcal{E}(b, b+1) \otimes_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}(\nabla^P(\tilde{\lambda}_{(1,c)}))) = \delta_{b+c, n+1}\mathbb{C}.$$

(4.2) We have

(1) $$\begin{aligned}\mathrm{H}^\bullet(\mathcal{P}, \mathcal{E}(b, b+1) &\otimes_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}(\nabla^P(\tilde{\lambda}_{(1,c)}))) \\ &= \mathrm{H}^\bullet(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{b-1} - (n-b)\varpi_{n-2}) \otimes \nabla^P(\varpi_{n-c} - \varpi_{n-2}))) \\ &\simeq \mathrm{H}^\bullet(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{b-1}) \otimes \nabla^P(\varpi_{n-c}) \otimes (-(n-b+1)\varpi_{n-2}))).\end{aligned}$$

Let $w_b = (s_{b+1}s_b)(s_{b+2}s_{b+1}) \dots (s_{n-1}s_{n-2})$.

To begin with, consider the case $b = 1$. Thus

$$\begin{aligned}\mathrm{H}^\bullet(\mathcal{P}, \mathcal{E}(1,2) \otimes_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}(\nabla^P(\tilde{\lambda}_{(1,c)}))) &\simeq \mathrm{H}^\bullet(\mathcal{P}, \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{n-c}) \otimes (-n\varpi_{n-2}))) \\ &\simeq \mathrm{H}^\bullet(\mathcal{B}, \mathcal{L}(\varpi_{n-c} - n\varpi_{n-2})).\end{aligned}$$

One checks that $\forall c \in [3, n]$,

(2) $$w_1 \bullet (\varpi_{n-c} - n\varpi_{n-2}) = -\varpi_2 + \varpi_{n+2-c}.$$

It follows from Bott's theorem that $\mathrm{H}^\bullet(\mathcal{P}, \mathcal{E}(1,2) \otimes_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}(\nabla^P(\tilde{\lambda}_{(1,c)}))) = \delta_{c,n}\mathbb{C}$.

(4.3) Assume now $b \in [2, n-2]$, and let $c \in [3, n]$. By the Littlewood-Richardson rule $\nabla^P(\varpi_{b-1}) \otimes \nabla^P(\varpi_{n-c})$ is a direct sum of the following summands

$$\begin{cases} \nabla^P(\varpi_{b-k} + \varpi_{n+k-1-c}), \ k \in [1, c-1], & \text{if } c \leq b+1 \leq n+2-c, \\ \nabla^P(\varpi_{b-k} + \varpi_{n+k-1-c}), \ k \in [1, b], & \text{if } c > b+1 \leq n+2-c, \\ \nabla^P(\varpi_{b-1+k} + \varpi_{n-k-c}), \ k \in [0, n-1-b], & \text{if } c \leq b+1 > n+2-c, \\ \nabla^P(\varpi_{b-1+k} + \varpi_{n-k-c}), \ k \in [0, n-c], & \text{if } c > b+1 > n+2-c. \end{cases}$$

Thus $\mathcal{E}(b, b+1) \otimes_{\mathcal{P}} \mathcal{L}_{\mathcal{P}}(\nabla^P(\tilde{\lambda}_{(1,c)}))$ in (4.2.1) is a direct sum of

$$\begin{cases} \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{b-k} + \varpi_{n+k-1-c} - (n-b+1)\varpi_{n-2})), \ k \in [1, c-1], & \text{if } c \leq b+1 \leq n+2-c, \\ \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{b-k} + \varpi_{n+k-1-c} - (n-b+1)\varpi_{n-2})), \ k \in [1, b], & \text{if } c > b+1 \leq n+2-c, \\ \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{b-1+k} + \varpi_{n-k-c} - (n-b+1)\varpi_{n-2})), \ k \in [0, n-1-b], & \text{if } c \leq b+1 > n+2-c, \\ \mathcal{L}_{\mathcal{P}}(\nabla^P(\varpi_{b-1+k} + \varpi_{n-k-c} - (n-b+1)\varpi_{n-2})), \ k \in [0, n-c], & \text{if } c > b+1 > n+2-c. \end{cases}$$

One checks

Lemma: *Let $b \in [2, n-2]$ and $c \in [3, n]$.*

(i) If $c \leq b+1 \leq n-c+2$, then $\forall k \in [1, c-1]$,

$$w_b \bullet (\varpi_{b-k} + \varpi_{n+k-1-c} - (n-b+1)\varpi_{n-2}) = \varpi_{b-k} - \varpi_{b-1} - \varpi_{b+1} + \varpi_{n+k+1-c} \in -\rho + \Lambda^+.$$

In particular, $w_b \bullet (\varpi_{b-k} + \varpi_{n+k-1-c} - (n-b+1)\varpi_{n-2}) \in \Lambda^+$ iff $k = 1$ and $b + c = n + 1$, in which case $w_b \bullet (\varpi_{b-k} + \varpi_{n+k-1-c} - (n-b+1)\varpi_{n-2}) = 0$.

(ii) If $c > b+1 \leq n-c+2$, then $\forall k \in [1, b]$,

$$w_b \bullet (\varpi_{b-k} + \varpi_{n+k-1-c} - (n-b+1)\varpi_{n-2}) = \varpi_{b-k} - \varpi_{b-1} - \varpi_{b+1} + \varpi_{n+k+1-c} \in -\rho + \Lambda^+.$$

In particular, $w_b \bullet (\varpi_{b-k} + \varpi_{n+k-1-c} - (n-b+1)\varpi_{n-2}) \in \Lambda^+$ iff $k = 1$ and $b + c = n + 1$, in which case $w_b \bullet (\varpi_{b-k} + \varpi_{n+k-1-c} - (n-b+1)\varpi_{n-2}) = 0$.

(iii) If $c \leq b+1 > n-c+2$, then $\forall k \in [0, n-b-1]$,

$$w_b \bullet (\varpi_{b-1+k} + \varpi_{n-k-c} - (n-b+1)\varpi_{n-2}) = \varpi_{n-c-k} - \varpi_{b-1} - \varpi_{b+1} + \varpi_{b+k+1} \in (-\rho + \Lambda^+) \setminus \Lambda^+.$$

(iv) If $c > b+1 > n-c+2$, then $\forall k \in [0, n-c]$,

$$w_b \bullet (\varpi_{b-1+k} + \varpi_{n-k-c} - (n-b+1)\varpi_{n-2}) = \varpi_{n-c-k} - \varpi_{b-1} - \varpi_{b+1} + \varpi_{b+k+1} \in (-\rho + \Lambda^+) \setminus \Lambda^+.$$

(4.4) Together with (4.2) Proposition 4.1 follows from (4.3) by Bott's theorem. This completes the proof of Theorem 2.2.

Going over to positive characteristic, as the vanishing of cohomology groups in §§3 and 4 have been shown using the fact that the relevant weights can be sent by elements of W to weights in $(-\rho+\Lambda^+)\setminus\Lambda^+$, if $\mathrm{ch}\,\Bbbk = p \gg 0$ such that those weights sent into $(-\rho+\Lambda^+)\setminus\Lambda^+$ all lie in the closure of the bottom alcove for $W \ltimes p\mathbb{Z}R$, then together with (1.3) the theorem holds also over $\Bbbk$ [J, II.5.5]. Finally, if $\Lambda_1 = \{\lambda \in \Lambda | \langle \lambda, \alpha^\vee \rangle \in [0, p[\ \forall \alpha \in R^{\mathrm{s}}\}$, each $\lambda \in \Lambda$ admits an expression $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in \Lambda_1$ and $\lambda^1 \in \Lambda$. Let $L(\lambda^0)$ denote a simple $G_\Bbbk$-module of highest weight λ^0. If M is a $P_\Bbbk$-module, let $M^{[1]}$ denote the Frobenius twist of M. We remark for $p \geq h$ the Coxeter number of G that $\forall w \in W^P$, $\mu_w = w^{-1} \bullet (w \bullet 0)^1$, and that $L((w \bullet \lambda)^0) \otimes \nabla^P_\Bbbk(w^{-1} \bullet (w \bullet 0)^1)^{[1]}$ is a subquotient of $G_1 P_\Bbbk$-Verma module of highest weight 0 [AbK], [K14], and hence each $\mathcal{L}_{\mathcal{P}_\Bbbk}(\nabla^P(w^{-1} \bullet (w \bullet 0)^1))$ is a subquotient of the Frobenius direct image $F_*\mathcal{O}_{\mathcal{P}_\Bbbk}$ of the structure sheaf $\mathcal{O}_{\mathcal{P}_\Bbbk}$ of $\mathcal{P}_\Bbbk$.

References

[AbK] Abe, N. and Kaneda, M., *On the structure of parabolically induced G_1T-Verma modules*, JIM Jussieu **14** Issue 01 (2015), 185-220

[Be] Beilinson, A. A., *Coherent sheaves on $\mathbb{P}_n$ and problems of linear algebra*, Func. Anal. Appl. **12** (1979), 214-216.

[BLV] Buchweitz, R-O., Leuschke, G., Van den Bergh, M., *On the derived category of Grassmannians in arbitrary characteristic*, Comp. Math. doi:10.1112/S0010437X14008070

[FH] Fulton, W. and Harris, J., Representation Theory, Springer-Verlag, New York 1999

[H] Hartshorne, R., Algebraic Geometry, Springer-Verlag, New York 1977

[J] Jantzen, J. C., Representations of algebraic groups, Orlando 1987 (Academic Press)

[K08] Kaneda M., *Kapranov's tilting sheaf on the Grassmannian in positive characteristic*, Alg. Repr. Th. (2008), 347-354

[K09] Kaneda M., *The structure of Humphreys-Verm modules for projective spaces*, J. Alg. **322** (2009), 237-244

[K14] Kaneda, M., Exceptional collections of sheaves on quadrics in positive characteristic, São Paulo Journal of Mathematical Sciences **8** (2014), 117-156

[Kap83] Kapranov, M. M., *The derived category of coherent sheaves on Grassmannians*, Functional Anal. i Prilozhen, **17** (1983), 78-79

[Kap88] Kapranov, M. M., *On the derived category of coherent sheaves on some homogeneous spaces*, Inv. Math., **92** (1988), 479-508