On complete monotonicity of inverse powers of some stable polynomials

Khazhgali Kozhasov (TU Braunschweig)
joint work with Mateusz Michalek and Bernd Sturmfels

Institut Mittag-Leffler

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Outline

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- Motivation: discrete wave equation

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They observed that Taylor coefficients $a_{k,\ell,m}$ of the function

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satisfy the difference equation $(*)$.

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In 1930 Lewy wrote to G. Szegö asking him to prove positivity of Taylor coefficients ($a_{k,\ell,m} > 0$) in general.

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He expresses coefficients as some integrals of products of Bessel functions which are shown to be positive (nonnegative).

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Let $G = (V, E)$ be a series-parallel graph. Then for any $\alpha \geq 1/2$ and any positive vector $x \in \mathbb{R}^E_{>0}$, the function

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If G is the n-cycle, $T_G(x) = \sum_{i=1}^n \prod_{j \neq i} x_j$ and Szegö's result follows from Scott and Sokals' theorem with $x = (1, \ldots, 1)$.

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Status: open in general!

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What about complete monotonicity of $E_{d,n}^{-\alpha}$ for other d?

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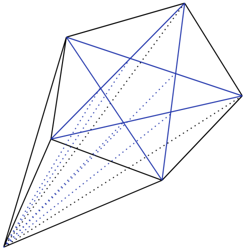

Example: pentagonal cone and its chamber complex

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If $\alpha_1 = \dots = \alpha_d = 1$, the function $q(z)$ measures the volume of the $(d-n)$-dimensional polytope $L^{-1}(z) \cap \mathbb{R}^d_{\geq 0}$.

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Thank you!