INTRODUCTION TO LORENTZIAN POLYNOMIALS BY PETTER BRÄNDÉN

These lectures were given at the Institute Mittag-Leffler during the semester on Algebraic and Enumerative Combinatorics by Petter Brändén based on his paper with June Huh with the same title [6], available here `https://arxiv.org/abs/1902.03719`. See also the related work by Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant [2], and Gil Kalai's blog post on the "Mihail-Vazirani conjecture for matroids" for a nice summary and lots of links to references. These notes were typeset by Sara Billey with additional comments by Vasu Tewari.

1. Lecture 1

We consider two approaches to log-concavity and the related property of real-rootedness for polynomials.

(1) **Hodge theory**. Examples include Huh's earlier work, Richard Stanley's work on the g-theorem, and Gromov's work on convex sets and Kahler manifolds.
(2) **Stability**. These idea come from statistical physics and control theory originally and concern zeros of multivariate polynomials. See earlier work by Borcea-Brändén on stable polynomials [4] Also, David Wagner's work on matroid inequalities.

The join of these two approaches gives a framework for understanding log-concavity and similar inequalities. Plus, in a sense it completely characterizes matroids using either an analytic or algebraic framework. It also gives a framework for analyzing algorithms in computer science. For example, there are applications to mixing times of Markov processes. Our original motivation came from Mason's conjecture.

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Conjecture 1.1. *Let I_k be the number of independent sets of size k in a matroid on a ground set with n elements. Then, for all $0 < k < n$, we have the inequality*

$$\frac{I_k^2}{\binom{n}{k}^2} \geq \frac{I_{k-1}}{\binom{n}{k-1}} \cdot \frac{I_{k+1}}{\binom{n}{k+1}}. \tag{1.1}$$

We have equality for the uniform matroid of rank n on an n element ground set, denoted $U_{n,n}$.

Newton's inequalities on the coefficients of a polynomial look very similar to Mason's inequalities. To state these inequalities, let $P(x,y) = \prod_{i=1}^{n}(x+\alpha_i y)$ be a bivariate homogeneous polynomial with all $\alpha_i \in \mathbb{R}$. We can write the coefficients in the expansion of $P(x,y)$ using the elementary symmetric functions as

$$P(x,y) = \prod_{i=1}^{n}(x+\alpha_i y) = \sum_{k=0}^{n} e_k(\alpha_1,\ldots,\alpha_n)x^{n-k}y^k.$$

Briefly, let e_k denote $e_k(\alpha_1,\ldots,\alpha_n)$. Then, *Newton's Inequalities* say that for all $0 < k < n$, we have

$$\frac{e_k^2}{\binom{n}{k}^2} \geq \frac{e_{k-1}}{\binom{n}{k-1}} \cdot \frac{e_{k+1}}{\binom{n}{k+1}}. \tag{1.2}$$

Furthermore, the partial derivatives $\partial_x P(x,y)$ and $\partial_y P(x,y)$ both factor into a product of linear factors with real coefficients of the same form, so we can continue to apply partial derivatives until we get a quadratic polynomial.

Homework 1.2. Show

$$\partial_y^{k-1}\partial_x^{n-k-1}P(x,y) = n!\left(\frac{e_{k-1}x^2}{\binom{n}{k-1}} + \frac{2e_k^2 xy}{\binom{n}{k}} + \frac{e_{k+1}y^2}{\binom{n}{k+1}}.\right).$$

Since $\partial_y^{k-1}\partial_x^{n-k-1}P(x,y)$ at $y=1$ has only real roots, the discriminant of this quadratic is nonnegative which implies (1.2).

The *Hessian* of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is $H_f = (\partial_i\partial_j f)_{i,j=1}^n$. If we let $\tilde{e}_k = e_k/\binom{n}{k}$, then the Hessian of $Q =: \partial_y^{k-1}\partial_x^{n-k-1}P(x,y)$ is

$$H_Q = 2n!\begin{pmatrix} \tilde{e}_{k-1} & \tilde{e}_k \\ \tilde{e}_k & \tilde{e}_{k+1} \end{pmatrix}. \tag{1.3}$$

Thus, H_Q is a symmetric matrix with nonnegative real entries, so it has two real eigenvalues. If H_Q is not identically zero, then it has at least one positive eigenvalue since $H_Q\binom{1}{1} > 0$. The eigenvalues of such a 2×2 matrix can only come in three types $(+,+)$, $(+,0)$, or $(+,-)$. If $\det H_Q \leq 0$, then H_Q has at most one positive eigenvalue. We summarize these observations with a remark.

Remark 1.3. If $\alpha_i \geq 0$ for all i, then H_Q has signature $(+,-)$, $(+,0)$, or $(0,0)$.

More generally, for any real symmetric matrix A with nonzero eigenvalues, we say A has *Lorentz signature* $(+,-,-,\dots,-)$ if it has one positive eigenvalue and the rest are all negative. Equivalently, if and only if the quadratic form $q = x^T Ax$ may be written as

$$q = \ell_1^2 - \ell_2^2 - \ell_3^2 - \cdots - \ell_n^2 \tag{1.4}$$

where $\ell_i = x^T v_i$ for each i, $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$, and $x = (x_1, \dots, x_n)^T$ is the column vector of coordinates on $\mathbb{R}^n$.

Definition 1.4. The set of *strictly Lorentzian polynomials*, $\mathring{L}_n^d$, is defined as follows. Let H_n^d be the homogeneous polynomials of degree d in $\mathbb{R}[x_1, x_2, \dots, x_n]$. For $d = 2$, the strictly Lorentzian polynomials $\mathring{L}_n^d$ are the set of all $f \in H_n^2$ such that f has all positive coefficients and H_f has Lorentz signature. For $d > 2$,

$$\mathring{L}_n^d = \{f \in H_n^d : \ f \text{ has positive coefficients and } \ \partial_{i_1}\partial_{i_2}\cdots\partial_{i_{d-2}}f \in \mathring{L}_n^2\}$$

Definition 1.5. For $d \geq 2$, the *Lorentzian polynomials* L_n^d is the closure of $\mathring{L}_n^d$ in the product topology on H_n^d.

Question: What happens if you drop the condition that coefficients are all positive? Do you run into problems?

Answer: Yes. Consider $x^3 + y^3$.

Example 1.6. Let $f = \sum_{k=0}^{d} a_k x^k y^{d-k}$ be a homogeneous polynomial of degree $d \geq 2$, with all $a_k > 0$. Under what conditions is $f \in \mathring{L}_n^d$? By definition, $f \in \mathring{L}_n^d$ if and only if every possible way to successively differentiate f down to a quadratic $Q = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_{d-2}}f \in \mathring{L}_n^2$. This is equivalent to requiring $\det H_Q < 0$ since we have assumed $a_k > 0$ for all k by Remark 1.3. As we saw, the condition $\det H_Q < 0$ is equivalent to saying the coefficients $a_1, \dots, a_n$ satisfy the strict version of Newton's inequalities (1.2). In particular, if f is a Lorentzian polynomial, then

f has *no internal zeros*, which means the set of k's for which $a_k \neq 0$ form an interval in the integers.

There are many examples of Lorentzian polynomials.

(1) Minkowski polynomials of convex bodies $C_1, C_2, \ldots, C_n \subset \mathbb{R}^n$. Each convex body can be rescaled to get another convex body. We can add two convex bodies using the Minkowski sum. Putting these two operations together we miraculously get a polynomial function
$$f(x_1, x_2, \ldots, x_n) = \mathrm{vol}(x_1 C_1 + \cdots + x_n C_n).$$
If we expand
$$f(x_1, x_2, \ldots, x_n) = \sum_{j_1, \ldots, j_n = 1}^{n} V(C_{j_1}, \ldots, C_{j_n}) x_{j_1} x_{j_2} \cdots x_{j_n},$$
then the coefficients $V(C_{j_1}, \ldots, C_{j_n})$ are called *mixed volumes*. These functions V satisfy many interesting properties including being symmetric in their inputs and multilinear. They also satisfy the *monotonicity inequality*
$$V(C_1, \ldots, C_n) \leq V(C_1', \ldots, C_n)$$
for $C_1 \subset C_1'$, and the *Alexandrov-Fenchel inequality*
$$V(C_1, \ldots, C_n) \geq \sqrt{V(C_1, C_1, C_3 \ldots, C_n) V(C_2, C_2, C_3 \ldots, C_n)}.$$
By looking carefully at Alexanderov's proof of this inequality, one can see such polynomials are Lorentzian. In fact, Alexanderov's proof anticipates the definition of Lorentzian polynomials. It might be a good research direction to find a more elementary proof of the Alexandrov-Fenchel inequality.

(2) Volume polynomials from the theory of nef divisors of a projective variety. See [6].

(3) Several families of polynomials from matroid theory are Lorentzian. We will see more of that later.

(4) Normalized Schur polynomials. These are defined as follows. Recall the Schur polynomials $s_\lambda = \sum_{\mu \vdash |\lambda|} K_{\lambda,\mu} m_\mu$. For the normalized version, we rescale each term according to $\mu! = \mu_1! \mu_2! \cdots$, so the *normalized Schur polynomials* are
$$ns_\lambda = \sum_{\mu \vdash |\lambda|} K_{\lambda,\mu} m_\mu / \mu!.$$

In recent work, June Huh, Jacob Matherne, Karola Mészáros, and Avery St. Dizier show the normalized Schur functions are Lorentzian [8] . They use this machinery to show the Kostka numbers $K_{\lambda,\mu}$ satisfy an inequality similar to log-concavity for any $1 \leq i, j \leq \ell(\mu)$, namely $K_{\lambda,\mu}^2 \geq K_{\lambda,\mu+e_i-e_j} K_{\lambda,\mu-e_i+e_j}$.

(5) Stable polynomials. This is a rich topic with a long history going back to Schur, Pólya, and Laguerre, so we will spend some time on the details.

Definition 1.7. A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is *stable* if $f(\alpha_1, \ldots, \alpha_n) \neq 0$ for all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with $\mathrm{Im}(\alpha_j) > 0\ \forall j$.

For example, when $n = 1$, a polynomial $f(x_1) \in \mathbb{R}[x_1]$ is stable if and only if f is real-rooted. The variety of a stable polynomial avoids the Cartesian product of the half planes in $\mathbb{C}$ with positive imaginary part. The most important examples of multivariate stable polynomials come from determinants, but these are just a small subset of all stable polynomials.

Lemma 1.8. *If A_0 is a real symmetric matrix, $A_1, A_2, \ldots, A_n$ are positive semidefinite (PSD), then the polynomial defined by*

$$f(x_1, \ldots, x_n) = \det(A_0 + A_1 x_1 + \cdots + A_n x_n)$$

is stable.

Proof. Set $x_j = \alpha_j + i\beta_j \in \mathbb{C}$ with $\beta_j > 0$ and $\alpha_j, \beta_j \in \mathbb{R}$ for each $1 \leq j \leq n$. Let $A = A_0 + A_1\alpha_1 + \cdots + A_n\alpha_n$ and $B = A_1\beta_1 + \cdots + A_n\beta_n$. Then, by definition

$$f(x_1, \ldots, x_n) = \det(A + iB).$$

We may assume A_1 is PD since stability is a closed property, B is PD since it is the sum of at least one PD matrix and other PSD matrices. Since B is positive definite, there exists another positive definite matrix P such that $B = P^2$. Therefore, by the multiplicative properties of determinant

$$f(x_1, \ldots, x_n) = \det(A + iB) = \det(B) \det\left(P^{-1}AP^{-1} + iI\right),$$

where I stands for the identity matrix. Note, $P^{-1}AP^{-1}$ is symmetric since A is symmetric and P is PD, so $-i$ is not one of its eigenvalues. Therefore, $f(x_1, \ldots, x_n) \neq 0$ since B is PD. $\square$

Definition 1.9. A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is *strictly stable* if the following two conditions hold

- All coefficients of f are positive.
- If for any $\beta \in \mathbb{R}^n_{>0}$ and $\alpha \in \mathbb{R}^n$ not parallel to β, the polynomial $f(t\beta + \alpha) \in \mathbb{R}[t]$ has only real and simple zeros.

1.1. **Discrete Convexity.** What does a typical Lorentzian polynomial look like? Two conditions completely characterize the Lorentzian polynomials. One is based on Hessians of derivatives of the polynomial and the other is a generalization of the basis exchange axiom for matroids.

Definition 1.10. A collection $J \subset \mathbb{N}^n$ is *M-convex* or *matroid-convex* if for all $\alpha, \beta \in J$ and all i such that $\alpha_i > \beta_i$ there exists a j such that $\beta_j > \alpha_j$ with $\alpha - e_i + e_j \in J$.

For example, $\{(0,0), (-1,1), (-2,2)\}$ is M-convex, but $\{(0,0), (-2,2)\}$ is not. If $J \subset \{0,1\}^n$, then J is M-convex if and only if J is the set of bases of a matroid.

Definition 1.11. The *support* of a multivariate polynomial $f = \sum a_\alpha x^\alpha$, where $x^\alpha = \prod x_i^{\alpha_i}$, is

$$\mathrm{supp}(f) = \{\alpha \in \mathbb{N}^n : \ a_\alpha \neq 0\}.$$

Theorem 1.12. [,] *Let $f \in H_n^d$ be a homogeneous polynomial with nonnegative coefficients. Then f is Lorentzian if and only if*

(1) *The support of f is M-convex.*
(2) *The Hessian of $\partial_{i_1}\partial_{i_2}\cdots\partial_{i_{d-2}} f$ has at most one positive eigenvalue for all $1 \leq i_1, i_2, \ldots, i_{d-2} \leq n$.*

Both directions of this theorem are nontrivial. To show that if f is Lorentzian then the second property holds trivially, but the first uses the formal Hodge-Riemann relations. In the other direction, to show a polynomial satisfying the two conditions is Lorentzian requires studying linear operators preserving the two properties.

Questions:

(1) Ivan: If $d = 1$, is every polynomial with nonnegative coefficients Lorentzian? Answer: Yes!
(2) Vasu: Do integer points in permutahedra form M-convex sets? Yes.

2. Lecture 2

We reviewed the definition of Lorentzian polynomials L_n^d and Theorem 1.12 characterizing these polynomials in terms of M-convex sets. As an example of using this theorem, we show the generating polynomials of matroids are Lorentzian.

Definition 2.1. Let $\mathcal{B}$ be the set of bases of a matroid, or equivalently an M-convex set in $\{0,1\}^n$. Then the *bases generating polynomial* is

$$f_B = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i.$$

Corollary 2.2. *Let $\mathcal{B}$ be the set of all bases of a matroid. Then, f_B is Lorentzian.*

Proof. By definition, f_B is homogeneous with nonnegative coefficients, hence we only need to show properties (1) and (2) hold from Theorem 1.12. Property (1) follows directly from the basis exchange axiom for matroids. To prove (2) holds, note any partial derivative is the generating polynomial of the contraction of the matroid $\partial_i f_{\mathcal{B}} = f_{\mathcal{B}/i}$. So, it suffices to prove (2) holds in the case when the rank of the matroid is 2.

The bases for a rank 2 matroids are quite simple. Partition the ground set into parallel classes and loops which never appear in a basis. Then every basis is obtained by selecting 2 elements from different parallel classes. If $S_1, S_2, \ldots, S_k$ are the distinct parallel classes, define the linear forms $\sigma_i = \sum_{j \in S_i} x_j$ for each i. Then,

$$f_{\mathcal{B}} = e_2(\sigma_1, \ldots, \sigma_k) = \frac{1}{2}\left[(\sigma_1 + \cdots + \sigma_k)^2 - \sigma_1^2 - \cdots - \sigma_k^2\right].$$

Recall from (1.4) that any quadratic form q with nonnegative coefficients is Lorentzian if and only if we can write it as $q = \ell_1^2 - \ell_2^2 - \cdots - \ell_k^2$ for some $1 \leq k \leq n$ and linear forms ℓ_i. Hence, $f_{\mathcal{B}}$ is Lorentzian, and the proof is complete. $\square$

More generally, we can define a *generating polynomial* for any finite subset $J \subset \mathbb{N}^n$ by

$$f_J := \sum_{\alpha \in J} \frac{x^\alpha}{\alpha!} := \sum_{\alpha \in J} \frac{x_1^{\alpha_1} \ldots x_n^{\alpha_n}}{\alpha_1! \ldots \alpha_n!}.$$

The property of a subset being M-convex is completely characterized by the Lorentzian property. We refer the reader to the proof in the paper for the following theorem.

Theorem 2.3. *If $J \subset \mathbb{N}^n$ is finite, then f_J is Lorentzian if and only if J is M-convex.*

Next we consider the *projectified space of Lorentzian polynomials*, $PL_n^d := \mathbb{P}(L_n^d)$ as a topological space. Let

$$PL_J = \{f \in PL_n^d : \operatorname{supp}(f) = J\}.$$

By Theorem 2.3, $f_J \in PL_J$ provided J is M-convex and $\sum_{i \in J} i = d$, in which case, PL_J is nonempty. Since every Lorentzian polynomial has M-convex support, we can partition

$$PL_n^d = \bigsqcup PL_J$$

as the disjoint union over all M-convex subsets $J \subset \mathbb{N}^n$ such that $\sum_{i \in J} i = d$.

Theorem 2.4. *For all d, n, the spaces PL_n^d,* interior(PL_n^d)*, and PL_J are all connected and contractible.*

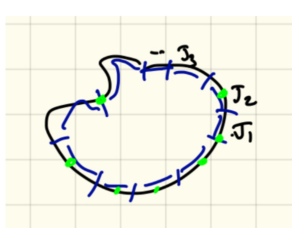
In fact, there is an explicit homotopy from any $f = \sum_{\alpha \in J} a_\alpha x^\alpha / \alpha! \in PL_J$ to $f_J = \sum_{\alpha \in J} x^\alpha / \alpha! \in PL_J$ given by

$$t \in [0,1] \longrightarrow f_t := \sum_{\alpha \in J} a_\alpha^{1-t} x^\alpha / \alpha! \in PL_J.$$

Note, $f_0 = f$ and $f_1 = f_J$.

Homework 2.5. Show f_t is Lorentzian for all $t \in [0,1]$. Hint: Clearly the support being M-convex comes for free, so we just need to differentiate to degree 2. Then use results on spectra of matrices from linear algebra. See [3, Cor. 2.10].

So PL_n^d is some nonconvex ball with stratified boundary indexed by say $J_1, J_2, \ldots, J_k$. Each J_i is represented by its generating polynomial.



Question (Ivan): For which J is f_J in the relative interior of PL_J?

Petter's Answer Yes, someone should look into that. For example, we know the *simplex polynomial*

$$f_{\Delta_n^d} = \sum_{\alpha_1+\cdots+\alpha_n=d} x^\alpha/\alpha! = \frac{1}{n!}\left(x_1+\cdots+x_n\right)^d$$

is on the boundary, and its support is all weak compositions of d into n parts.

2.1. **Linear Preservers.** Observe that differentiation is a linear operator that preserves the space of Lorentzian polynomials by definition. What other linear operators preserve the Lorentzian property? Let's call these "*linear preservers.*"

Definition 2.6. Let $\kappa = (\kappa_1, \dots, \kappa_k) \in \mathbb{N}^n$, and define

$$P_\kappa = \{f \in \mathbb{R}[x_1, \dots, x_n] : \deg_i(f) \le \kappa_i\}.$$

Here $\deg_i(f) \le \kappa_i$ means the exponent of x_i in any monomial in the support of f is at most κ_i.

Consider all linear operators of the form $T : P_\kappa \longrightarrow \mathbb{R}[x_1, \dots, x_n]$. When does such a T preserve the Lorentzian property? One simple conditions is that it should preserve homogeneity since Lorentzian polynomials are homogeneous by definition. Surprisingly, it suffices to consider the *symbol polynomial* of T, namely

$$G_T = T\left[(x_1+y_1)^{\kappa_1}(x_2+y_2)^{\kappa_2}\dots(x_n+y_n)^{\kappa_n}\right] = \sum_{\alpha\le\kappa}\binom{\kappa}{\alpha}T(x^\alpha)y^{\kappa-\alpha}$$

where a binomial coefficient for weak compositions κ, α both of length n is defined by

$$\binom{\kappa}{\alpha} = \prod_{i=1}^{n}\binom{\kappa_i}{\alpha_i}.$$

Theorem 2.7. *If G_T is Lorentzian, then T preserves the Lorentzian property.*

This construction was first proved by Borcea-Brändén [4] for stable polynomials where the converse holds also. There is no known converse for Lorentzian polynomials because it doesn't extend to the nonhomogeneous case.

Remark 2.8. If G_T is homogeneous and stable, then T preserves the Lorentzian property by a theorem Borcea-Brändén [4]. Which one? And, should we just call it a theorem instead of a remark?

Example 2.9. Consider the linear operator T which makes a nonnegative change of variables encoded by an $n \times n$ matrix $A = (a_{i,j})$ with nonnegative entries. By the usual matrix action on polynomials,

$$T(f) = f(Ax) = f\left(\sum_j a_{1,j}x_j,\ \sum_j a_{2,j}x_j,\ \dots,\ \sum_j a_{n,j}x_j\right).$$

In this case, if $T : P_\kappa \longrightarrow \mathbb{R}[x_1, \dots, x_n]$, then we claim T preserves the Lorentzian property. Observe,

$$G_T = T\left[(x_1+y_1)^{\kappa_1}(x_2+y_2)^{\kappa_2}\dots(x_n+y_n)^{\kappa_n}\right] = \prod_{i=1}^{n}\left(y_i + \sum_j a_{i,j}x_j\right)^{\kappa_i}$$

is homogeneous and stable since it does not vanish on the intersection of the positive imaginary halfplanes in $\mathbb{C}^n$. Hence G_T is Lorentzian. So, the claim holds by Theorem 2.7.

Question(Liam): What is the obstruction to nonhomogeneous polynomials or polynomials with coefficients of different signs being Lorentzian?

Petter's Answer: It is subtle question. For non-homogeneous polynomials with nonnegative coefficients there is the notion of complete log-concavity [2]. For polynomials with coefficients of different signs there is no "good" notion known, except for bivariate polynomials.

2.2. **Proof sketch of Mason's Conjecture.** Next, we will show how Mason's conjecture follows from the theory of Lorentzian polynomials.

Lemma 2.10. *Let $f \in \mathbb{R}[x_0, x_1, \dots, x_n]$ be Lorentzian of degree d, and consider the expansion*

$$f = \sum_{k=0}^{d} a_k(x_1, \dots, x_n)x_0^{d-k},$$

so that $a_k(x_1, \dots, x_n)$ is homogeneous of degree k for each k. Then

$$\frac{a_k(x)}{\binom{d}{k}^2} \geq \frac{a_{k-1}(x)}{\binom{d}{k}}\frac{a_{k+1}(x)}{\binom{d}{k}}.$$

for all $x \in \mathbb{R}^n_{\geq 0}$ and $1 \leq k \leq n-1$.

Proof. Fix $y \in \mathbb{R}^n_{\geq 0}$. By Example 2.9, the polynomial $g(s,t)$ obtained from f by the change of variables $x_0 = s$ and $x_i = y_i t$, for $1 \leq i \leq n$ is Lorentzian. Since

$$g(s,t) = \sum_{k=0}^{d} a_k(y_1, \dots, y_n) t^k s^{d-k},$$

the proof now follows from Example 1.6. □

Let M be a matroid on a ground set $E = [n]$. A subset I of E is independent if it is contained in some basis. Similar to the bases generating polynomial of M, let

$$I_k(x_1, \dots, x_n) = \sum \prod_{i \in I} x_i,$$

where the sum is over all independent sets I of size k in E. Mason's conjecture is the special case when $x = (1, \dots, 1)$ of the next theorem. Theorem 2.11 also generalizes Newton's inequalities. Indeed Newton's inequalities correspond to the uniform matroids $U_{n,n}$.

Theorem 2.11. *For all $1 \leq k \leq n$, and $x \in \mathbb{R}^n_{\geq 0}$,*

$$\frac{I_k(x)}{\binom{d}{k}^2} \geq \frac{I_{k-1}(x)}{\binom{d}{k}} \frac{I_{k+1}(x)}{\binom{d}{k}}.$$

Proof. We prove that the polynomial $I_M = \sum_{k=0}^{n} I_k(x_1, \dots, x_n) x_0^{n-k}$ is Lorentzian. The lemma then follows from Lemma 2.10.

Property (1): The support condition follows easily since the polynomials come from a matroid. Property (2): For $1 \leq i \leq n$, the derivative $\partial_i I_M = I_{M/i}$, where M/i is M contracted by i. Hence the proof reduces to the case when M has rank two. We leave it as a linear algebra exercise to prove that the Hessian of I_M for rank two matroids have Lorentz signature.

□

Question (Christos): Were the related inequalities known before for all k?

Petter's Answer: No. Weaker versions of Mason's conjecture were known.

2.3. **Formal Hodge-Riemann Relations.** The classical Hodge–Riemann relations have been extended to the case of matroids by Adiprasito-Huh-Katz [1, Thm. 1.4]. Roughly they imply that a certain bilinear form coming from multiplication by a 1-form is positive definite on the kernel. The Hodge-Riemann relations can also be phrased as saying a certain volume polynomial of a Cartier divisor is Lorentzian. Here we define a formal Hodge-Riemann relations. The proof again reduces to rank two matroids. We refer the reader to the paper for complete details.

Theorem 2.12. *If $f \in \mathring{L}_n^d$ and $x \in \mathbb{R}^n_{>0}$, then the Hessian $H_f(x) := (\partial_i \partial_j f(x))_{i,j=1}^n$ has Lorentzian signature $(+, -, \dots, -)$.*

Lemma 2.13. *(Lefschetz-like lemma) Let $f \in H_n^d$ and $x \in \mathbb{R}^n_{>0}$. Suppose $\partial_i H_f(x)$ has exactly one positive eigenvalue for all $1 \leq i \leq n$, then*

$$\ker H_f(x) = \bigcap_{i=1}^n \ker H_{\partial_i f}(x).$$

Proof. Exercise in linear algebra. □

Remark 2.14. It was this lemma that lead us to come up with the notion of Lorentzian polynomials.

2.4. **Tropical Geometry to Convexity.** Looking at the support of a polynomial is on par with tropicalization. Here we describe the notion of discrete concave functions. See Murota, 2003.

There is an alternative and more symmetrical definition of M-convex sets. This statement is stronger so we state it as a proposition. It does require proof.

Proposition 2.15. *A subset $J \subset \mathbb{N}^n$ is M-convex if and only if for all $\alpha, \beta \in J$ with $\alpha_i > \beta_i$, there exists a j such that $\beta_j > \alpha_j$ and both*

(1) $\alpha - e_i + e_j \in J$, *and*
(2) $\beta - e_j + e_i \in J$.

Definition 2.16. A function $g : \mathbb{N}^n \longrightarrow \mathbb{R} \cup \{-\infty\}$ is *M-concave* if for all $\alpha, \beta \in \mathbb{N}^n$ and i such that $\alpha_i > \beta_i$, there exists a j such that $\beta_j > \alpha_j$ and

$$g(\alpha) + g(\beta) \leq g(\alpha - e_i + e_j) + g(\beta - e_j + e_i).$$

Observe that $\mathrm{supp}(g) := \{\alpha : g(\alpha) \neq -\infty\}$ is an M-convex set.

If $\mathrm{supp}(g) \subset \{0,1\}^n$, then g is a valuated matroid as defined by Dress and Wenzel [7], see also [9]. For rank 2, the valuated matroids are exactly the tree distances of phylogenetic trees which we define next.

Definition 2.17. A *phylogenetic tree* is a connected graph with no cycles and no vertices of degree two.

Theorem 2.18. *(Fundamental Theorem of Phylogenetics)*[9] *The set of all rank 2 valuated matroids coincides with the set of all tree distances.*

Let $T = (V, E)$ be a phylogenetic tree with edge weights $w_e \in \mathbb{R}$ for each $e \in E$. The *tree distances* are given by $d : \binom{V}{2} \longrightarrow \mathbb{R}$ where $d(v_i, v_j)$ is the sum of the edge weights along the unique path between vertices v_i and v_j.

The following theorem characterizes M-concave functions in terms of Lorentzian polynomials.

Theorem 2.19. *Let $g : \mathbf{N} \longrightarrow \mathbb{R} \cup \{-\infty\}$ be a discrete function with finite support. Then the following are equivalent.*

1. *For all $q \geq 1$, the polynomial $\sum_{\alpha \in J} q^{g(\alpha)} x^\alpha / \alpha!$ is Lorentzian.*
2. *for all q sufficiently large, $\sum_{\alpha \in J} q^{g(\alpha)} x^\alpha / \alpha!$ is Lorentzian.*
3. *g is M-concave.*
4. *g is a tropical Lorentzian polynomial (which has not been defined yet in these notes).*

The proof of Theorem 2.19 is a surprising tour through the mathematics. The support condition is simple. Handling the rank 2 case is another characterization of the matrices related to phylogenetic trees and the metric properties of tree distance.

2.5. **Open problems.** We conclude with several open problems related to the theory of Lorentzian polynomials.

1. One really challenging problem is a conjecture of Rota and Welsh. Let W_k be the number of flats of rank k in a matroid M

on a ground set of size n. The numbers W_k are called *Whitney numbers*. Then, for $1 \le k \le n-1$

$$\frac{W_k^2}{\binom{n}{k}^2} \ge \frac{W_{k-1}}{\binom{n}{k-1}} \cdot \frac{W_{k+1}}{\binom{n}{k+1}}. \tag{2.1}$$

(2) Let N be the *normalization* map on polynomials in $\mathbb{R}[x_1, \dots, x_n]$ defined by $N(x^\alpha) = x^\alpha/\alpha!$ and extend by linearity. Then a beautiful conjecture of Huh-Matherne-Mészáros-St.Dizier says that all normalized Schubert polynomials $N(\mathfrak{S}_\pi)$ are Lorentzian. The four authors prove that $\mathrm{N}(x_1^{n-1} \cdots x_n^{n-1} \mathfrak{S}_\pi(x_1^{-1}, \dots, x_n^{-1}))$ is Lorentzian. Fortunately, inverting the alphabet in Schurs does not really change them much, which is what allows them to conclude that normalized Schur polynomials are Lorentzian. As far as the original conjecture is concerned, they establish that the normalized Schubert $\mathfrak{S}_\pi$ is Lorentzian if π avoids the patterns 1432 and 1423. Their proof utilizes the fact that Petter mentioned saying Lorentzian-ness is preserved under a linear change of variables with nonnegative coefficients.

(3) Lots of conjectured unimodal or log concave families of polynomials are yet to be "Lorentzianized."

(4) Let P be a finite poset and $e_k(P)$ be the number of order preserving surjections $\sigma : P \longrightarrow \{1, 2, \dots, k\}$. Is the sequence $(e_k(P) : k \ge 1)$ always log-concave? Note, this polynomial is not necessarily real-rooted.

This sequence is related to the Neggers-Stanley conjecture which Petter and Stembridge found a counter example to [5]. This conjecture asserted that the univariate polynomial counting the linear extensions of a partially ordered set by their number of descents has real zeros only.

Questions:

(1) Francesco: Normalized Schur polynomials are Lorentzian, so what about a normalized version of the fundamental quasisymmetric polynomials. Petter said the support condition fails. Vasu Tewari added that it appears not to be true. Consider the normalized fundamental quasisymmetric $\mathrm{N}F_{21}(x_1, x_2, x_3) = \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2x_3 + \frac{1}{2}x_1^2x_3 + x_1x_2x_3$. Applying ∂_2 to this gives the quadratic form $\frac{1}{2}x_1^2 + x_2x_3 + x_1x_3$ which may be rewritten as $\frac{1}{2}(x_1+x_3)^2 - \frac{1}{2}(x_2-x_3)^2 + \frac{1}{2}x_2^2$.

(2) Sara: How exactly do you turn a log-concavity questions for a family of univariate polynomials into a multivariate family which can be attacked by the theory of Lorentzian polynomials? Petter said that can be tricky. One approach might be to follow along the lines Postnikov laid out to find a volume polynomial with the right combinatorial properties.
(3) Ben: What about other signatures?

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