Number theory

Orbits of rotations and beyond

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The orbit of a rotation of the unit circle has one of two behaviours as a dynamical system, depending on whether the angle of rotation is a rational or an irrational multiple of 2π. In the rational case, any orbit is periodic; in the irrational case, any orbit is dense and in fact uniformly distributed. A quantitative form of the uniform distribution of the irrational orbits can be studied via the discrepancy of the sequence $\{n\alpha\}$ which in turn depends heavily on the continued fraction expansion of α.

The study of orbits of rotations leads naturally to the notion of twisted Diophantine approximation – a form of inhomogeneous Diophantine approximation, where one fixes the homogeneous parameter and studies the possible approximation properties of the inhomogeneous parameter as it varies.

With Bugeaud, Harrap and Velani, we proved that the natural analogue of badly approximable elements exist in abundance. To be precise, we proved that for any $\alpha \in \mathbb{R}$, the set

$$\left\{ x \in [0,1) : \|n\alpha - x\| \geq \frac{K(x)}{n} \text{ for some } K(x) > 0 \text{ for all } n \in \mathbb{N} \right\}$$

has maximal Hausdorff dimension and that this property is stable under intersection with sufficiently nice fractals. In recent work in progress with Tseng, this result is extended to the case when the rotation is replaced by an Interval Exchange Transformation (IET). The added difficulty of the IET setting comes from some highly geometric and dynamical obstacles to a sufficiently nice behaviour of the appropriate analogue of continued fractions.