Operator Theory and Analytic Function Spaces

Dynamics of the Gauss maps and the Hilbert transform

Alfonso Montes-Rodríguez
University of Sevilla, Spain

A pair (Γ, Λ), where $\Gamma \subset \mathbb{R}^2$ is a locally rectifiable curve and $\Lambda \subset \mathbb{R}^2$ is a *Heisenberg uniqueness pair* if an absolutely continuous finite complex-valued Borel measure supported on Γ whose Fourier transform vanishes on Λ necessarily is the zero measure. Here, absolute continuity is with respect to arc length measure. Recently, it was shown by Hedenmalm and Montes that if Γ is the hyperbola $x_1 x_2 = M^2/(4\pi^2)$ (where $M > 0$ is the mass), and Λ is the lattice-cross $(\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, where α, β are positive reals, then (Γ, Λ) is a Heisenberg uniqueness pair if and only if $\alpha\beta M^2 \leq 4\pi^2$. The Fourier transform of a measure supported on a hyperbola solves the one-dimensional Klein-Gordon equation, so the theorem supplies very thin uniqueness sets for a class of solutions to this equation. By rescaling, we may assume that the mass equals $M = 2\pi$, and then the above-mentioned theorem is equivalent to the following assertion: *the functions*

$$\mathrm{e}^{\mathrm{i}\pi\alpha m t}, \quad \mathrm{e}^{-\mathrm{i}\pi\beta n/t}, \qquad m, n \in \mathbb{Z},$$

span a weak-star dense subspace of $L^\infty(\mathbb{R})$ if and only if $0 < \alpha\beta \leq 1$. The proof involved ideas from Ergodic Theory. To be more specific, in the critical regime $\alpha\beta = 1$, the crucial fact was that the Gauss-type map $t \mapsto -1/t$ modulo $2\mathbb{Z}$ on $[-1, 1]$ has an ergodic absolutely continuous invariant measure with infinite total mass. However, the case of the semi-axis $\mathbb{R}_+$ as well as the holomorphic counterpart remained open. We completely solve these two problems. Both results can be stated in terms of Heisenberg uniqueness, but here, we prefer the concrete formulation. As for the semi-axis, *we can show that the restriction to* $\mathbb{R}_+$ *of the functions*

$$\mathrm{e}^{\mathrm{i}\pi\alpha m t}, \quad \mathrm{e}^{-\mathrm{i}\pi\beta n/t}, \qquad m, n \in \mathbb{Z},$$

span a weak-star dense subspace of $L^\infty(\mathbb{R}_+)$ *if and only if* $0 < \alpha\beta \leq 4$. In the critical regime $\alpha\beta = 4$, the weak-star span misses the mark by one dimension only. The proof of this statement is based on the ergodic properties of the standard Gauss map $t \mapsto 1/t \mod \mathbb{Z}$ on the interval $[0, 1]$. In particular, we find that for $1 < \alpha\beta < 4$, there exist nontrivial functions $f \in L^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi\alpha m t} f(t)\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi\beta n/t} f(t)\mathrm{d}t = 0, \qquad m, n \in \mathbb{Z},$$

and that each such function is uniquely determined by its restriction to any of the semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$. This is an instance of *dynamical unique continuation.*

As for the holomorphic counterpart, *we show that the functions*

$$\mathrm{e}^{\mathrm{i}\pi\alpha m t}, \quad \mathrm{e}^{-\mathrm{i}\pi\beta n/t}, \qquad m, n \in \mathbb{Z}_+ \cup \{0\},$$

span a weak-star dense subspace of $H^\infty_+(\mathbb{R})$ *if and only if* $0 < \alpha\beta \le 1$. Here, $H^\infty_+(\mathbb{R})$ is the subspace of $L^\infty(\mathbb{R})$ which consists of those functions whose Poisson extensions to the upper half-plane are holomorphic. In the critical regime $\alpha\beta = 1$, the proof relies on the nonexistence of a certain invariant distribution in the predual of real H^∞ for the above-mentioned Gauss-type map on the interval $](1, 1)$, which is a new result of dynamical flavor. To attain it, we need to handle in a subtle way series of powers of transfer operators, a rather intractable problem where even the recent advances by Melbourne and Terhesiu do not apply. More specifically, our approach – which is obtained by combining ideas from Ergodic Theory with ideas from Harmonic Analysis – involves a splitting of the Hilbert kernel, as induced by the transfer operator. The careful analysis of this splitting involves detors to the Hurwitz zeta function as well as to the theory of totally positive matrices.

Joint work with Håkan Hedenmalm, KTH.