Generalized Permutohedra : Ehrhart Positivity and Minkowski Linear Functionals

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IML, Unimodality, Log Concavity and Beyond

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Some examples of valuations :

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2. Surface area.
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One reason for studying valuations is Hilbert's third problem.

Hilbert's third problem

Let $\mathcal{D}_2$ be the group of isometries of $\mathbb{R}^2$.

Definition

A dissection of a polytope $A \subset \mathbb{R}^2$ is a collection of polytopes $A_1, \ldots, A_k$ such that $A = \cup_{i=1}^k A_i$ and such that the interiors of the A_i are disjoint.

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Two polytopes $A, B \subset \mathbb{R}^2$ are called $\mathcal{D}_2$ equidissectable if there are dissections $A = \cup_{i=1}^k A_i$ and $B = \cup_{i=1}^n B_i$ and such that $A_i \sim_{\mathcal{D}_2} B_i$ for each i.

Caveat : What we are calling equidissectability is usually called *Scissors Congruence*.

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This was the first of Hilbert's 23 problems to be solved, actually in the same year!

Dehn Invariants

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be an additive map that is zero at π but is not identically zero. Such a f must be non-measurable. The associated Dehn invariant for a polytope $P \subset \mathbb{R}^3$ is

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Dehn showed the necessity of the following (settling Hilbert's third problem). The sufficiency was settled by Sydler in 1965.

Theorem (Dehn, Sydler)

Polytopes $P, Q \subset \mathbb{R}^3$ are equidissectable under $\mathcal{D}_3$ iff all their Dehn invariants are the same. It is easy to see that every Dehn invariant of the standard cube is zero while every Dehn invariant of the regular tetrahedron is non-zero.

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And of course, we have the Banach-Tarski paradox.

Theorem

Any two bounded sets $X, Y \subset \mathbb{R}^n$ with non-empty interior, for $n \geq 3$ are equidecomposable.

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The coefficients are called *Quermassintegrals* and are equal to the V_k upto a scaling.

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The study of valuations that take values in other semigroups is a beautiful and well developed area.

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Theorem (Ehrhart's theorem)

Let P be a lattice polytope : Then there is a polynomial Ehr_P of degree equal to the affine dimension of P such that

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If we write

$$\mathrm{Ehr}_p(x) = \sum_{k=0}^{d} E_k(P)x^k,$$

we see that E_d is the volume and $E_0 = 1$.

Some Ehrhart polynomials

Let B_d be the d dimensional box. Then

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This was introduced by John Reeve in 1902 to show that there is no analogue of Pick's theorem in higher dimensions. There are no lattice points in the Reeve tetrahedron apart from the vertices but the volume is $r/6$ which can be arbitrarily large.

$$\mathrm{Ehr}_{R_t} = \frac{rt^3}{6} + t^2 + \left(2 - \frac{r}{6}\right)t + 1.$$

The Betke-Kneser theorem

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Stanley in 1974 showed that the usual monomial basis is **not the best basis** to write the Ehrhart polynomial in.

Theorem (Stanley)

Let P be a lattice polynomial of dimension d. Then, there are natural numbers $h_0^*, \ldots, h_d^*$ such that

$$\mathrm{Ehr}(n) = h_0^* \binom{n+d}{d} + h_1^* \binom{n+d-1}{d} + \ldots + h_d^* \binom{n}{d}.$$

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Note that the elements of the h^* vector are not valuations: The h^* vector depends upon the dimension of the polytope.

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The most important valuation by far is the following.

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There is a unique integer valued valuation on polyconvex sets in $\mathbb{R}^d$ which assigns the value 1 to every (closed) polytope. This valuation is called the Euler Characteristic.

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Instead of considering valuations on convex sets, may we consider valuations on the larger class of *PolyConvex* sets.

Definition

A relatively open polyhedron is a polyhedral convex set that is open in the affine space that it lies in. A polyconvex set is a finite union of relatively open polyhedra.

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Let $P \in \mathcal{L}_d$. Then

$$\mathrm{Ehr}_P(-n) = (-1)^{\dim(P)}\,\mathrm{Ehr}_{P^o}(n),$$

where P^o is the relative interior of P.

This was conjectured by Ehrhart and proved by him in several cases. The full proof was given by Ian Macdonald in 1971.

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This theorem was generalized by McMullen to all translation invariant valuations.

Theorem (McMullen)

Let ϕ be a valuation on $\mathcal{L}_d$. Then

$$\varphi_P(-n) = \sum_{F \subset P} (-1)^{\dim(F)} \varphi_F(n).$$

Ehrhart Positivity

Recall the definition of the Ehrhart polynomial,

$$\mathrm{Ehr}_P(n) = |nP \cap \mathbb{Z}^d|,$$

Question

Which lattice polytopes have Ehrhart polynomials with non-negative coefficients?

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This shows that Ehrhart Positivity is **not a combinatorial property**, but a a geometric one.

Classes of Ehrhart Positive Polytopes

Given a vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d$, the associated Stanley-Pitman polytope is

$$PS_d(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^d_{\geq 0} \mid \sum_{j=1}^{i} x_i \leq \sum_{j=1}^{i} a_i,\ \ i \in [d]\}.$$

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Crosspolytopes and several derived polytopes are Ehrhart positive : This uses an interesting fact, namely that their Ehrhart polynomials have roots on S^1. That this implies positivity is easy to see.

Zonotopes

Zonotopes are Minkowski sums of line segments,

$$Z = \sum_{i=1}^{m} [0, v_i],$$

where the $v_i \in \mathbb{Z}^d$. In this case, the coefficients of the Ehrhart polynomial have meaning.

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The coefficient of t^k in Ehr_Z is

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where X is the collection of all linearly independent size k subsets and $h(X)$ is the gcd of all $k \times k$ minors of the matrix whose column vectors are the elements in X.

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Stanley's theorem specializes to: $E_Z(k)$ is the $\#$ forests on $[d+1]$ with exactly $d+1-k$ trees.

Matroid Polytopes

A matroid M is a finite set X and a collection of subsets T (called independent sets) which are

1. Downward closed.
2. For every $e \in T$ and $i \in X \setminus e$, there is a $j \in e$ such that $e \cup i \setminus \{j\} \in T$.

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Theorem (Fundamental Meta-theorem of Combinatorial Optimization)

Optimization problems where the associated matroid polytopes have compact facet description are tractable. If a problem is intractable, then the corresponding matroid polytope has complex facet structure.

Generalized Permutohedra

Conjecture (De Leora et. al. 2007)

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A generalized permutohedron is a polytope obtained by making parallel translates of facets of a regular permutohedron. Note that some vertices may vanish.

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Yet another definition is the following : A polytope P is a generalized permutohedron if there is another polytopes Q such that $P + Q = \lambda\Pi_d$. In other words, P is a weak Minkowski summand of Π_d.

Submodular functions

Another very useful characterization of generalized permutohedra uses submodular functions.

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A function f defined on subsets of $[d]$ is called submodular if for any $S \subset T \subset [n]$ and i not in T,

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be the **standard simplices** where $e_1, \ldots, e_d$ are the standard basis vectors in $\mathbb{R}^d$.

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Theorem (Jochemko, Ravichandran, 2019)

Let $\{y_I\}_{I \in [d]}$ be a vector of real numbers. Then the following are equivalent.

(i) *The signed Minkowski sum $\sum_{I \subseteq [d]} y_I \Delta_I$ defines a generalized permutahedron.*

(ii) *For all 2-element subsets $E \in \binom{[d]}{2}$ and all $T \subseteq [d]$ such that $E \subseteq T$*

$$\sum_{E \subseteq I \subseteq T} y_I \geq 0\,. \qquad (3.1)$$

Further, every generalized permutohedron is of the above form. In particular, the collection of all coefficients $\{y_I\}_{I \in [d]}$ such that $\sum_{I \subseteq [d]} y_I \Delta_I$ defines a generalized permutahedron is a polyhedral cone. The inequalities are facet-defining.

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This theorem uses the supermodular characterization of Permutohedra due to Postnikov together with a theorem of Schneider on facets of Minkowski sums of polytopes.

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With this in hand, we can calculate the linear term of the Ehrhart polynomial of any generalized Permutohedron.

The linear term of the n simplex is easily calculated to be

$$\mathcal{E}(\Delta_{i+1}) \ = \ 1 + \frac{1}{2} + \cdots + \frac{1}{i} =: h_i$$

Translation Invariant valuations on Generalized Permutohedra

For any 2-element subset $E \in \binom{[d]}{2}$ and any $T \subseteq [d]$ such that $E \subseteq T$ let v_E^T be the Minkowski linear functional defined by

$$v_E^T(\Delta_I) = \begin{cases} 1 & \text{if } E \subseteq I \subseteq T\,, \\ 0 & \text{otherwise.} \end{cases}$$

We characterize all positive, translation-invariant Minkowski linear functionals on $\mathcal{P}_d$.

Proposition

Let $\varphi\colon \mathcal{P}_d \to \mathbb{R}$ be a Minkowski linear functional. Then φ is positive and translation-invariant if and only if there are nonnegative real numbers c_E^T such that

$$\varphi = \sum_{E \in \binom{[d]}{2}} \sum_{T \supseteq E} c_E^T v_E^T\,.$$

In particular, the family of positive, translation-invariant Minkowski linear functionals is a polyhedral cone with rays v_E^T.

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The proof is not difficult: It essentially uses Conic Duality.

Symmetric Minkowski Linear Functionals

The Final step is to specialize the above theorem to the case when the functionals are also symmetric (under co-ordinate permutations).

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For all $1 \leq k \leq d-1$ let $f_k : \mathcal{P}_d \to \mathbb{R}$ be the symmetric, translation-invariant Minkowski linear functional defined by

$$(f_k)(\Delta_{i+1}) = \binom{i+1}{2}\binom{d-i-1}{k-i} \tag{3.2}$$

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Theorem

Let $\varphi \colon \mathcal{P}_d \to \mathbb{R}$ be a Minkowski linear functional. Then φ is positive, translation-invariant and symmetric if and only if there are real numbers $c_1, \ldots, c_{d-1} \geq 0$ such that

$$\varphi \;=\; \sum_{k=1}^{d-1} c_k f_k\,.$$

In particular, the family of all positive, Minkowski linear, translation- and symmetric functionals form a simplicial cone of dimension $d-1$.

Completing the proof

The final step is showing that the functional

$$\sum_{I \subset [d]} y_I \Delta_I \to \sum_{I \subset [d]} y_I h_{|I|},$$

satisfies the above condition. This uses some basic combinatorics with univariate polynomials.

Solid Angle Polynomials

Let $q \in \mathbb{R}^d$ be a point, $P \subseteq \mathbb{R}^d$ be a polytope and let $\mathcal{B}_\epsilon(q)$ denote the ball with radius ϵ centered at q. The **solid angle** of q with respect to P is defined by

$$\omega_q(P) \;=\; \lim_{\epsilon \to 0} \frac{(P \cap \mathcal{B}_\epsilon(q))}{\mathcal{B}_\epsilon}\,.$$

We note that the function $q \mapsto \omega_q(P)$ is constant on relative interiors of the faces of P. In particular, if $q \notin P$ then $\omega_q(P) = 0$, if q is in the interior of P then $\omega_q(P) = 1$ and if q lies inside the relative interior of a facet then $\omega_q(P) = \frac{1}{2}$. The **solid angle sum** of P is defined by

$$A(P) \;=\; \sum_{q \in \mathbb{Z}^d} \omega_q(P) \;\; \text{Fact: McMullen's theorem shows this is a polynomial}$$

Proposition

There is a 3-dimensional generalized permutahedron in $\mathbb{R}^4$ such that the linear term of its solid angle polynomial is negative.

Open Problems

Definition (Birkhoff Polytope)

The Birkhoff polytope is the set of all $n \times n$ matrices with non-negative entries so that each row and column sum is 1. This is a polytope of dimension $(n-1)^2$.

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What about general matching polytopes? We conjecture

Conjecture

General matching polytopes need not be Ehrhart positive.

Monotonicity

Coming to matroid polytopes, consider the following operations on a matroid $M = (E, T)$.

- **Deletion:** Given $e \in E$, the matroid $M \setminus e$ is the matroid on the ground set $E \setminus e$ with independent sets being independent sets in M not containing e.
- **Contraction:** Given $e \in E$, the matroid M/e is the matroid on the ground set $E \setminus e$ with independent sets being those sets such that appending e gives an independent set in M.

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Conjecture

The linear term of the Ehrhart polynomials on matroid polytopes is monotone with respect to deletion and contraction.

We have verified this in over a hundred special cases.

Final Slide

Thanks for Listening!