Limiting absorption principle and spectral asymptotics for waveguides with singular perturbations

André Froehly

Leibniz Universität Hannover

Eigenvalues and Inequalities, 14 May - 18 May 2018

Motivation: The free Laplacian

Consider the free Laplacian $A_0 := -\Delta$ on $\mathbb{R}$. For $\lambda \notin [0,\infty)$ the resolvent is given by

$$(A_0 - \lambda)^{-1} f(x) = \frac{i}{2\sqrt{\lambda}} \int_{\mathbb{R}} e^{i\sqrt{\lambda}|x-y|} f(y)\, \mathrm{d}y, \qquad f \in L_2(\mathbb{R}),$$

where we assume that $\Im(\sqrt{\lambda}) > 0$.

Limiting absorption principle

For $\lambda \in [0,\infty)$ the following limits exist

$$(A_0 - (\lambda \pm i0))^{-1} : L_2(\mathbb{R}; (1+x^2)\,\mathrm{d}x) \to L_2(\mathbb{R}; (1+x^2)^{-1}\,\mathrm{d}x)$$

Observation: The resolvent kernel

$$\frac{i}{2\sqrt{\lambda}} e^{i\sqrt{\lambda}|x-y|}$$

is well-defined as a multi-valued holomorphic function of $\lambda \in \mathbb{C} \setminus \{0\}$.

Motivation: The free Laplacian

We set $\mathbb{H}_+ := \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ and

$$L_{2,\beta}(\mathbb{R}) := \left\{ f \in L_{2,\mathrm{loc}}(\mathbb{R}) : f(x)e^{-\beta|x|} \in L_2(\mathbb{R}) \right\}.$$

Lemma

Let $\beta > 0$. The function

$$\mathbb{H}_+ \ni \omega \mapsto (A_0 - \omega^2)^{-1} \in \mathscr{L}(L_{2,-\beta}(\mathbb{R}); L_{2,\beta}(\mathbb{R}))$$

has a meromorphic continuation to $\{\omega \in \mathbb{C} : \Im(\omega) > -\beta\}$.

The case $0 \neq V \in C_c^\infty(\mathbb{R})$**:** We consider the Schrödinger operator $-\Delta + V$ and its resolvent $R_V(\omega) := (-\Delta + V - \omega^2)^{-1}$. We have

$$R_V(\omega) = R_0(\omega)(I + VR_0(\omega))^{-1}, \qquad \Im(\omega) \gg 1.$$

Note that:

1. $VR_0(\omega) : L_{2,-\beta}(\mathbb{R}) \to L_{2,-\beta}(\mathbb{R})$ is compact;
2. $\|VR_0(\omega)\| \to 0$ as $\Im(\omega) \to \infty$.

Motivation: The free Laplacian

Meromorphic Fredholm theorem

The operator $I + VR_0(\omega) : L_{2,-\beta}(\mathbb{R}) \to L_{2,-\beta}(\mathbb{R})$ is invertible for almost every $\omega \in \mathbb{C}$ with $\Im(\omega) > -\beta$.

Consequence: $R_V(\omega)$ has also a meromorphic continuation.

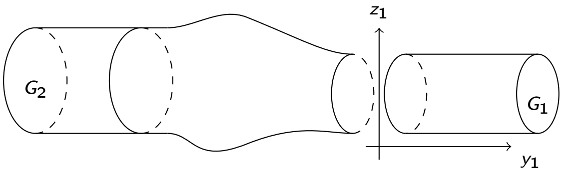
Notions:

- A pole of $R_V(\cdot)$ is called resonance.
- We call $u \in \operatorname{ran} R_0(\omega)$ an outgoing function.

1. u is outgoing if and only if $u(x) = c_\pm e^{i\omega|x|} + \textit{exponential decay}$ for $x \to \pm\infty$;
2. If ω is not a resonance then $u = R_V(\omega)f$ is the unique solution of $(-\Delta u + V - \omega^2)u = f$;
3. λ is a resonance if and only if there exists a non-trivial outgoing solution of $(-\Delta u + V - \lambda^2)u = 0$.

Statement of the problem

Let $\Omega \subseteq \mathbb{R}^d$ be a $C^{1,1}$-domain with cylindrical ends. On the cylindrical ends we use coordinates $(y_i, z_i) \in [1,\infty) \times G_i$. Let $\mathbf{n}$ be the outer normal.



We consider a symmetric, strongly elliptic differential operator

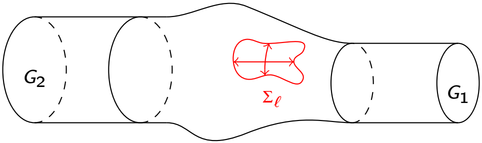
$$a(x,\nabla_x) := -\sum_{k,k=1}^{d} \partial_j (a_{jk}(x)\partial_k) + c(x),$$

such that we have $a_{kj}(y_i,z_i) = a^0_{jk}(z_i) \in C^{1,1}$ and $c(y,z_i) = c^0(z_i) \in C^{1,1}$ for $y_i \geq 1$. The conormal derivative is given by

$$\mathscr{B}u := \sum_{j,k=1}^{d} \mathbf{n} \cdot (a_{jk})_{jk} \nabla u_{\partial\Omega}, \qquad u \text{ smooth.}$$

Statement of the problem

We choose a family of Lipschitz domains on the boundary $\Sigma_\ell \subseteq \partial\Omega$ shrinking to a point.



We define the self-adjoint operator A_{Σ_ℓ} by $A_{\Sigma_\ell} u := a(x,\nabla_x)u$ and

$$u \in D(A_{\Sigma_\ell}) := \left\{ u \in H^1(\Omega) : a(x,\nabla_x)u \in L_2(\Omega) \wedge \mathscr{B}u|_{\Sigma_\ell} = 0 \wedge u|_{\partial\Omega\setminus\Sigma_\ell} = 0 \right\}.$$

1st aim: Find a meromorphic continuation of the resolvent.
2nd aim: Find the asymptotics of the resonances of A_{Σ_ℓ} as $\ell \to 0$.

Note that: The operator is defined by its form

$$u \mapsto \langle (a_{jk})_{jk} \nabla u, \nabla u \rangle + \langle cu, u \rangle,$$

where $u \in H^1(\Omega)$ shall satisfies $\operatorname{supp}(u|_{\partial\Omega}) \subseteq \overline{\Sigma_\ell}$.

The asymptotic formula for the Laplacian

Quantum waveguides:

Let $\Omega := \mathbb{R} \times (0,\pi)$, $\Sigma_\ell := [-\ell,\ell] \times \{\pi\}$ and $a(x,\nabla_x) = -\Delta$.

Theorem (Popov, 1999)

We have $\sigma_{\text{ess}}(A_{\Sigma_\ell}) = [1,\infty)$ and $\sigma_d(A_{\Sigma_\ell}) = \{\lambda(\ell)\}$ for sufficiently small $\ell > 0$. Moreover,

$$1 - \lambda(\ell) = \ell^4 + \mathcal{O}(\ell^5) \qquad \text{as } \ell \to 0.$$

More contributions concerning asymptotics for singular waveguides by Borisov, Exner, Gadyl'shin, Popov and by many more.

An operator with more involved dispersion curves

Elasticity operator:

Let $\Omega := \mathbb{R} \times (0,\pi)$, $\Sigma_\ell := [-\ell,\ell] \times \{\frac{\pi}{2}\}$.

Theorem (Weidl and Hänel/Froehly, 2017)

The elasticity operator $a(x,\nabla_x) = -\Delta - \mathrm{grad}\,\mathrm{div}$ *and*

$$D(A_{\Sigma_\ell}) := \left\{ u \in H^1(\Omega \setminus \Sigma_\ell; \mathbb{C}^2) : \mathscr{B}u = 0 \text{ on } \partial\Omega \setminus \Sigma_\ell \right\}$$

has two embedded eigenvalues $\lambda_i(\ell)$ *which satisfy*

$$\Lambda - \lambda_1(\ell) = \nu_1 \ell^4 + \mathscr{O}(\ell^5), \qquad \Lambda - \lambda_2(\ell) = \nu_2 \ell^8 + \mathscr{O}(\ell^9).$$

Here ν_1, ν_2 *are constants and* Λ *is a spectral threshold.*

The essential spectrum of A_{Σ_ℓ}

Assumption: Consider only one cylindrical end $[1,\infty) \times G$ with stabilising coefficient $a^0_{jk}(z)$, $b^0(z)$. We put

$$a^0(z, \nabla_{y,z}) := \sum_{j,k=1}^{d} \partial_j (a^0_{jk}(z) \partial_k) + c^0(z)$$

acting on functions on the cylinder $\mathbb{R} \times G$. We consider its self-adjoint realisation A^0 with Dirichlet b.c. on all of $\mathbb{R} \times \partial G$. Then

$$D(A^0) = H^2(\mathbb{R} \times G) \cap H^1_0(\mathbb{R} \times G), \qquad A^0 u = a^0(z, \nabla_{y,z}) u.$$

Then:

Lemma

We have $\sigma_{\mathrm{ess}}(A_{\Sigma_\ell}) = \sigma_{\mathrm{ess}}(A^0) = \sigma(A^0)$.

Proof: Use Weyl sequences.

What is the structure of $\sigma(A_{\Sigma_\ell})$ resp. $\sigma(A^0)$?

The dispersion curves of A^0

Let $\widehat{f},\widehat{g}$ denote the Fourier transform in the horizontal direction of functions $f,g \in L^2(\mathbb{R}\times G)$. Then we have

$$\langle (A^0-\omega^2)^{-1}f,g\rangle = \int_{\mathbb{R}} \langle (A(\xi)-\omega^2)^{-1}\widehat{f}(\xi),\widehat{g}(\xi)\rangle \; \mathrm{d}\xi,$$

where $A(\xi)$, $\xi\in\mathbb{C}$, is a m-sectorial operator in $L_2(G)$,

$$D(A(\xi)) = H^2(G)\cap H_0^1(G), \qquad A(\xi)u = a^0(z,i\xi,\nabla_z)u.$$

The eigenvalues of $A(\xi)$ depend holomorphically on ξ.

Lemma (cf. Theorem VIII.3.9. in [Kato]

There exist real-analytic families $\mu_k:\mathbb{R}\to\mathbb{R}$, $P_k:\mathbb{R}\to\mathscr{L}(L_2(G))$ such that

$$A(\xi) = \sum_{k=1}^{\infty} \mu_k(\xi)P_k(\xi), \qquad \xi\in\mathbb{R}.$$

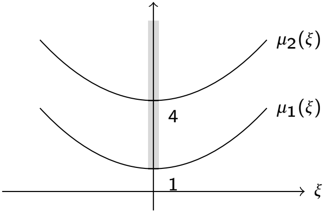
Here $P_k(\xi)$ are mutually orthogonal projections.

Consequence: We have $\lambda\in\sigma(A^0)$ if and only if $\lambda=\mu_k(\xi)$ for some $k\in\mathbb{N}$ and $\xi\in\mathbb{R}$.

An example and spectral thresholds

Example: Consider $a^0(z, \nabla_{y,z}) = -\partial_y^2 - \partial_z^2$ on $\mathbb{R} \times G := \mathbb{R} \times (0, \pi)$. Then:

- We have $A(\xi) = -\partial_z^2 + \xi^2$ with Dirichlet b.c. on $\{0, \pi\}$.
- For $k \geq 1$ we have $\mu_k(\xi) = k^2 + \xi^2$ and $P_k(\xi) = P_k$ is the projection onto the space spanned by $z \mapsto \sin(kz)$.



Observation: $\mu_k(\xi) \geq c\xi^2$ as $\xi \to \infty$.

Definition

A value $\omega \in \mathbb{R}$ is called spectral threshold if and only if there exists $k \in \mathbb{N}$ and $\xi \in \mathbb{R}$ such that:

$$\mu_k(\xi) = \omega^2 \quad \text{and} \quad \mu_k'(\xi) = 0.$$

The meromorphic continuation of the resolvent

Let $\Lambda_1 < \Lambda_2 \dots$ be the spectral thresholds of A^0. Then $\Lambda_n \to \infty$ as $n \to \infty$. We denote by

$$R^0(\omega) = (A^0 - \omega^2)^{-1}, \qquad \omega \in \mathbb{H}_+,$$

the resolvent and for $\beta \in \mathbb{R}$ we define

$$L_{2,\beta}(\mathbb{R} \times G) := \left\{ f \in L_{2,\mathrm{loc}}(\mathbb{R} \times G) : f(y,z) e^{-\beta|y|} \in L_2(\mathbb{R} \times G) \right\}.$$

Theorem

Let $\beta > 0$. There exists a open neighbourhood $U \supseteq \overline{\mathbb{H}_+}$ such that

$$\mathbb{H}_+ \ni \omega \mapsto R^0(\omega) \in \mathscr{L}(L_{2,-\beta}(\mathbb{R} \times G); L_{2,\beta}(\mathbb{R} \times G))$$

has a meromorphic continuation to multiple-valued function on $U \setminus \{\pm\Lambda_k : k \in \mathbb{N}\}$ with algebraic branching points at $\pm\Lambda_k$.

Note that: An analogous result holds true for the operator A_{Σ_ℓ}.

Proof of the Theorem

Let $f, g \in L_{2,-\beta}(\mathbb{R} \times G)$. Then

$$\langle R^0(\omega) f, g\rangle = \int_{\mathbb{R}} \langle (A(\xi) - \omega^2)^{-1} \widehat{f}(\xi)\,,\,\widehat{g}(\xi)\rangle\ \mathrm{d}\xi,$$

Then $\xi \mapsto \widehat{f}(\xi), \xi \mapsto \widehat{g}(\xi)$ have analytic extensions to $\{\xi \in \mathbb{C} : |\Im(\xi)| < \beta\}$.
Define

$$\Xi_\beta(\omega) := \left\{\xi \in \mathbb{C} : |\Im(\xi)| < \beta \wedge (A(\xi) - \omega^2)^{-1} \text{ does not exist}\right\}.$$

Then Ξ_β is finite and for suitable $\beta_1 \in (0, \beta)$ we have

$$\langle R^0(\omega) f, g\rangle = \int_{\mathbb{R} + i\beta_1} \langle (A(\xi) - \omega^2)^{-1} \widehat{f}(\xi)\,,\,\widehat{g}(\xi)\rangle\ \mathrm{d}\xi + 2\pi i \mathrm{Res}_{\xi \in \Xi_{\beta_1}, \Im(\xi) > 0}[\dots]$$

Observation: As $A(\xi) = \sum_{k=1}^{\infty} \mu_k(\xi) P_k(\xi)$ we have

$$\Xi_{\beta_1}(\omega) = \left\{\xi \in \mathbb{C} : |\Im(\xi)| < \beta_1 \wedge \exists k : \mu_k(\xi) = \omega^2\right\} \quad (\beta_1 \text{ small}).$$

Proof of the Theorem

Let $f, g \in L_{2,-\beta}(\mathbb{R} \times G)$. Then

$$\langle R^0(\omega) f, g\rangle = \int_{\mathbb{R}} \langle (A(\xi) - \omega^2)^{-1} \widehat{f}(\xi)\,,\, \widehat{g}(\xi)\rangle \;\mathrm{d}\xi,$$

Then $\xi \mapsto \widehat{f}(\xi), \xi \mapsto \widehat{g}(\xi)$ have analytic extensions to $\{\xi \in \mathbb{C} : |\Im(\xi)| < \beta\}$. Define

$$\Xi_\beta(\omega) := \left\{\xi \in \mathbb{C} : |\Im(\xi)| < \beta \,\wedge\, (A(\xi) - \omega^2)^{-1} \text{ does not exist}\right\}.$$

Then Ξ_β is finite and for suitable $\beta_1 \in (0, \beta)$ we have

$$\langle R^0(\omega) f, g\rangle = \int_{\mathbb{R}+i\beta_1} \langle (A(\xi) - \omega^2)^{-1} \widehat{f}(\xi)\,,\, \widehat{g}(\xi)\rangle \;\mathrm{d}\xi + 2\pi i \mathrm{Res}_{\xi \in \Xi_{\beta_1}, \Im(\xi) > 0}[\dots]$$

Observation: As $(A(\xi) - \omega^2)^{-1} = \sum_{k=1}^{\infty} \frac{1}{\mu_k(\xi) - \omega^2} P_k(\xi)$ we have

$$\Xi_{\beta_1}(\omega) = \left\{\xi \in \mathbb{C} : |\Im(\xi)| < \beta_1 \,\wedge\, \exists k : \mu_k(\xi) = \omega^2\right\} \quad (\beta_1 \text{ small}).$$

Proof of the Theorem

Let $\omega_0 \in \mathbb{R}$ and we denote by $(\kappa_1, k_1) \ldots, (\kappa_n, k_n)$ all pairs such that $\mu_{k_\alpha}(\kappa_\alpha) = \omega_0^2$. We have

$$\mu_{k_\alpha}(\xi) = \omega_0^2 + G_\alpha(\xi)^{m_\alpha}, \qquad \xi \text{ near } \kappa_\alpha,$$

with $G'_\alpha(\kappa_\alpha) \neq 0$. For ω near ω_0 we obtain

$$\Xi_{\beta_1}(\omega) := \left\{ \xi_{\alpha,l}(\omega) := G_\alpha^{-1}\left(e^{\frac{2\pi i}{m_\alpha} l} (\omega^2 - \omega_0^2)^{1/m_\alpha} \right) \right\}.$$

From $(A(\xi) - \omega^2)^{-1} = \sum_{k=1}^{\infty} \frac{1}{\mu_k(\xi) - \omega^2} P_k(\xi)$ we have

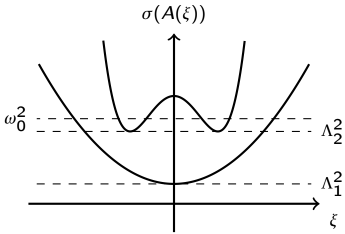
$$\begin{aligned} \langle K(\omega) f, g \rangle &:= 2\pi i \sum_{\Im(\xi_{\alpha,\ell}(\omega))>0} \mathrm{Res}_{\xi = \xi_{\alpha,l}(\omega)} \langle (A(\xi) - \omega^2)^{-1} \hat{f}(\xi)\,,\, \hat{g}(\xi) \rangle \\ &= 2\pi i \sum_{\Im(\xi_{\alpha,\ell}(\omega))>0} \frac{1}{\mu'_{k_\alpha}(\xi_{\alpha,l})} \langle P_k(\xi_{\alpha,l}) \hat{f}(\xi_{\alpha,l})\,,\, \hat{g}(\xi_{\alpha,l}) \rangle, \end{aligned}$$

and finally,

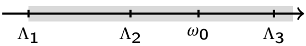
$$\langle R^0(\omega) f, g \rangle = \int_{\mathbb{R} + i\beta_1} \langle (A(\xi) - \omega^2)^{-1} \hat{f}(\xi)\,,\, \hat{g}(\xi) \rangle \; \mathrm{d}\xi + \langle K(\omega) f, g \rangle.$$

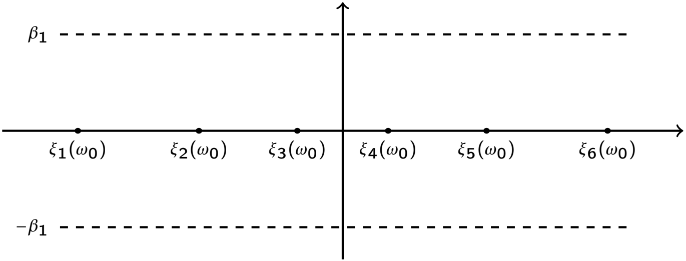
Example

Assume that we have the following dispersion curves:



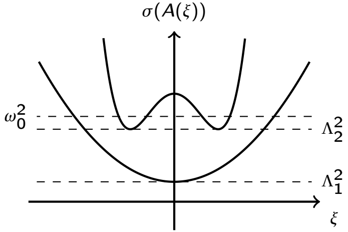
Then: $\Xi_\beta(\omega) = \{\xi_1(\omega), \dots, \xi_6(\omega)\}$.



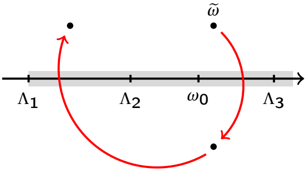


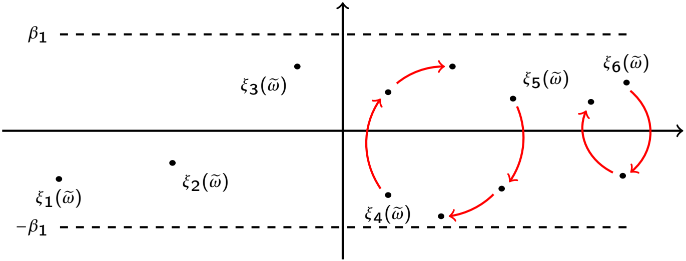
Example

Assume that we have the following dispersion curves:



Then: $\Xi_\beta(\omega) = \{\xi_1(\omega), \ldots, \xi_6(\omega)\}$.





Remark

As for the Laplacian on $\mathbb{R}$ one may show that R_{Σ_ℓ} has a meromorphic extension.

Let $\chi_i \in C^\infty(\mathbb{R})$ be chosen such that $\chi_i = 0$ on $(-\infty, R+1)$ and $\chi_i = 1$ on $(R+2, \infty)$, $i \in \{0,1\}$. Moreover, $\chi_1\chi_0 = \chi_1$. Then

$$\chi_1 R_{\Sigma_\ell}(\omega) = \chi_0 R^0(\omega)\chi_0\Big[[A(z,\nabla_{y,z}),\chi_1]R_{\Sigma_\ell}(\omega) + \chi_1\Big]. \tag{1}$$

Recall that we have $R^0(\omega) = L(\omega) + K(\omega)$, where

$$\langle L(\omega)f, g\rangle := \int_{\mathbb{R}+i\beta} \langle A(\xi) - \omega^2)^{-1}\widehat{f}(\xi)\,,\, \widehat{g}(\xi)\rangle \, \mathrm{d}\xi$$

$$\langle K(\omega)f, g\rangle := \sum_{\Im(\xi_{\alpha,l})>0} \frac{1}{\mu'_{k_\alpha}(\xi_{\alpha,l}(\omega))} \langle P_k(\xi_{\alpha,l}(\omega))\widehat{f}(\xi_{\alpha,l}(\omega))\,,\, \widehat{g}(\xi_{\alpha,l}(\omega))\rangle,$$

for $\omega \in \mathbb{H}_+$. Note that:

- $\chi_0 L(\omega)\chi_0 f \in H^1_{-\beta}(\Omega)$.
- $\mathrm{ran}(K(\omega)) = \mathrm{span}\left\{e^{i\xi_{\alpha,l}(\omega)y}\psi : \psi \in \mathrm{ran}(P_k(\xi_{\alpha,l}(\omega))\right\}$.

Incoming and outgoing solutions

Let $(\psi_{\alpha,\ell,j})_j$ be a basis of $\operatorname{ran}(P_k(\xi_{\alpha,l}(\omega))$. We put

$$U_{\alpha,\ell,j}(y,z) := \chi(y)e^{i\xi_{\alpha,l}(\omega)y}\psi_{\alpha,l,j}(z).$$

Definition

Let ω be fixed. A function $u \in H^1_\beta(\Omega)$ is called outgoing if and only if there exists $c_{\alpha,l,j} \in \mathbb{C}$ and a function $\tilde{u} \in H^1_{-\beta}(\Omega)$ such that

$$u(y,z) := \sum_{\alpha,l,j} c_{\alpha,l,i} U_{\alpha,\ell,j}(y,z) + \tilde{u}(y,z).$$

Lemma

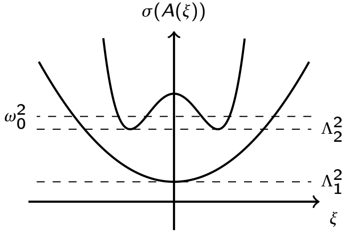
Let $f \in L_{2,-\beta}(\Omega)$. Then $u := R_{\Sigma_\ell}(\omega)f$ is the unique outgoing solution $u \in H^1_\beta(\Omega)$ of the boundary value problem

$$(a(x,\nabla_x) - \omega^2)u = f, \qquad \mathscr{B}u|_{\Sigma_\ell} = 0, \qquad u|_{\partial\Omega\setminus\Sigma_\ell} = 0.$$

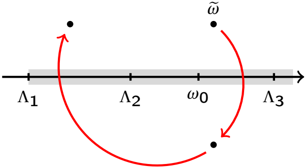
Remark: ω is a resonance of $R_V(\cdot)$ iff there exists a non-trivial outgoing solution of the above boundary value problem.

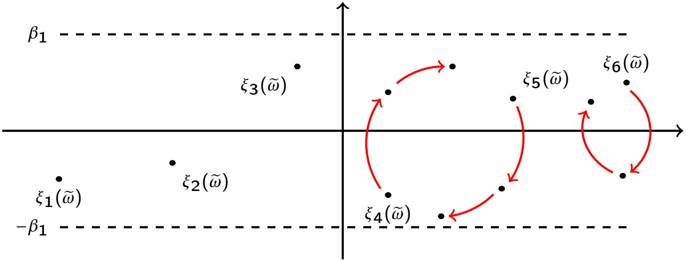
Example (revisited)

Assume that we have the following dispersion curves:



Then: $\Xi_\beta(\omega) = \{\xi_1(\omega), \ldots, \xi_6(\omega)\}$.





The Dirichlet-to-Neumann operator

Let $\omega \in \mathbb{C}$ such that $\Im(\omega) \geq 0$. We want to consider the boundary value problem

$$(a(x,\nabla_x) - \omega^2)u = 0, \qquad u|_{\partial\Omega} = g \qquad (*),$$

for suitable u and g. Recall that $A_\varnothing$ is the realisation of $a(x,\nabla_x)$ in $L_2(\Omega)$ with Dirichlet b.c. on $\partial\Omega$.

Lemma

Let $\omega^2 \notin \sigma(A_\varnothing)$ and $g \in H^{1/2}(\partial\Omega)$. Then there exists a unique solution $u := \mathbb{K}_\omega g \in H^1(\Omega)$ of $(*)$.

Then the Dirichlet-to-Neumann operator $\mathbb{D}_\omega : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is given by

$$\mathbb{D}_\omega g := \mathscr{B}\,\mathbb{K}_\omega g.$$

A perturbation formula

Let $\omega \in \mathbb{C}$ with $\Im(\omega) \geq 0$, $\omega^2 \notin \sigma(A_\varnothing)$. Assume that $0 \notin \sigma(A_\varnothing)$. Then:

Lemma

1. $\mathbb{K}_\omega^* = -\mathscr{B} R_\varnothing(\omega)$.
2. $\mathbb{K}_\omega = (I + \omega^2 R_\varnothing(\omega))\mathbb{K}_0$;
3. $\mathbb{D}_\omega = \mathbb{D}_0 - \omega^2 \mathbb{K}_0^* \Big(I + \omega^2 R_\varnothing(\omega)\Big)\mathbb{K}_0$.

Remark: For sufficiently small $\beta > 0$ we have:

- $R_\varnothing(0) : L_{2,-\beta}(\Omega) \to L_{2,-\beta}(\Omega)$;
- $\mathbb{K}_0 : L_{2,-\beta}(\partial\Omega) \to H^{1/2}_{-\beta}(\Omega)$.

Consequences: There exists $U \supseteq \overline{\mathbb{H}_+}$ such that the mappings $\omega \mapsto \mathbb{K}_\omega$ and $\omega \mapsto \mathbb{D}_\omega$ may be continued meromorphically on $U \setminus \{\pm\Lambda_k : k \in \mathbb{N}\}$.

The Dirichlet-to-Neumann operator (2)

Note that: We will be interested only in the boundary values on Σ_ℓ.
We define the spaces

$$H_{00}^{1/2}(\Sigma_\ell) := \{h \in H^{1/2}(\partial\Omega) : \operatorname{supp}(h) \subseteq \overline{\Sigma_\ell}\},$$
$$H^{-1/2}(\Sigma_\ell) := \{g \in \mathscr{D}'(\Sigma_\ell) : \exists G \in H^{-1/2}(\partial\Omega) \text{ s.t. } G|_{\Sigma_\ell} = g\}.$$

Then:

- We have $H_{00}^{1/2}(\Sigma_\ell) := [L_2(\Sigma_\ell), H_0^1(\Sigma_\ell)]_{1/2}$.
- We have a dual pairing between

$$\langle G|_\Sigma, h\rangle_{\Sigma_\ell} = \langle G, h\rangle_{\partial\Omega}, \quad G \in H^{-1/2}(\partial\Omega), h \in H_{00}^{1/2}(\Sigma_\ell).$$

We define the truncated Dirichlet-to-Neumann operator

$$\mathbb{D}_{\ell,\omega} : H_{00}^{1/2}(\Sigma_\ell) \to H^{-1/2}(\Sigma_\ell), \qquad \mathbb{D}_{\ell,\omega} g := r_{\Sigma_\ell} \mathbb{D}_\omega e_{\Sigma_\ell},$$

where e_{Σ_ℓ} is the extension by 0 and r_{Σ_ℓ} the restriction operator.

Theorem

ω is a resonance of A_{Σ_ℓ} if and only if $\ker(\mathbb{D}_{\ell,\omega})$ is non-trivial.

The window Σ_ℓ

Assumption: $\partial\Omega$ is smooth near Σ.
For simplicity let $\partial\Omega$ be flat near the window and define $\Sigma_\ell := \ell\Sigma$. We define the unitary operator $T_\ell : L_2(\Sigma) \to L_2(\Sigma_\ell)$,

$$(T_\ell g)(x) := \ell^{-\frac{d-1}{2}} g(\ell^{-1}x), \qquad g \in L_2(\Sigma^*), x \in \Sigma_\ell.$$

Let

$$Q_{\ell,\omega} : H_{00}^{1/2}(\Sigma^*) \to H^{-1/2}(\Sigma^*), \qquad Q_{\ell,\omega} = T_\ell^* \mathbb{D}_{\ell,\omega} T_\ell.$$

From the previous lemma we have

$$\mathbb{D}_\omega = \mathbb{D}_0 - \omega^2 \mathbb{K}_0^* \Big(I + \omega^2 R_\varnothing(\omega) \Big) \mathbb{K}_0,$$

and thus,

$$Q_{\ell,\omega} = Q_{\ell,0} - \omega^2 T_\ell^* r_{\Sigma_\ell} \mathbb{K}_0^* \Big(I + \omega^2 R_\varnothing(\omega) \Big) \mathbb{K}_0 e_{\Sigma_\ell} T_\ell.$$

Note that

$$\langle Q_{\ell,0} g, h \rangle_{\Sigma^*} = \langle \mathbb{D}_{\ell,0} T_\ell g, T_\ell h \rangle_{\Sigma_\ell} = \langle \mathbb{D}_0 T_\ell g, T_\ell h \rangle_{\partial\Omega}.$$

The window Σ_ℓ

Assumption: $\partial\Omega$ is smooth near Σ.
For simplicity let $\partial\Omega$ be flat near the window and define $\Sigma_\ell := \ell\Sigma$. We define the unitary operator $T_\ell : L_2(\Sigma) \to L_2(\Sigma_\ell)$,

$$(T_\ell g)(x) := \ell^{-\frac{d-1}{2}} g(\ell^{-1}x), \qquad g \in L_2(\Sigma^*),\, x \in \Sigma_\ell.$$

Let

$$Q_{\ell,\omega} : H^{1/2}_{00}(\Sigma^*) \to H^{-1/2}(\Sigma^*), \qquad Q_{\ell,\omega} = T_\ell^* \mathbb{D}_{\ell,\omega} T_\ell.$$

From the previous lemma we have

$$\mathbb{D}_{\ell,\omega} = \mathbb{D}_{\ell,0} - \omega^2 r_{\Sigma_\ell} \mathbb{K}_0^* \Big(I + \omega^2 R_\varnothing(\omega) \Big) \mathbb{K}_0 e_{\Sigma_\ell},$$

and thus,

$$Q_{\ell,\omega} = Q_{\ell,0} - \omega^2 T_\ell^* r_{\Sigma_\ell} \mathbb{K}_0^* \Big(I + \omega^2 R_\varnothing(\omega) \Big) \mathbb{K}_0 e_{\Sigma_\ell} T_\ell.$$

Note that

$$\langle Q_{\ell,0} g, h \rangle_{\Sigma^*} = \langle \mathbb{D}_{\ell,0} T_\ell g, T_\ell h \rangle_{\Sigma_\ell} = \langle \mathbb{D}_0 T_\ell g, T_\ell h \rangle_{\partial\Omega}.$$

A first asymptotic estimate for $Q_{\ell,0}$

Observation: $r_\Sigma \mathbb{D}_0 e_\Sigma : C_c^\infty(\Sigma) \to C^\infty(\Sigma)$ is Ψ-do. Let $p(t,\theta)$ be the full symbol of $\mathbb{D}_0$. Then

$$\langle Q_{\ell,0} g, h\rangle_{\Sigma_\ell} = \langle \mathbb{D}_0 T_\ell g, T_\ell h\rangle_{\Sigma^*}$$
$$= \frac{\ell^{1-d}}{(2\pi)^{d-1}} \int_{\Sigma_\ell^*} \int_{\mathbb{R}^{d-1}} \int_{\Sigma_\ell^*} e^{i(t-s)\theta} p(t,\theta) g(s/\ell) \overline{h(t/\ell)} \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}t$$

Let p_j be homogeneous of order $-j$ and $p(t,\theta) = \sum_{j=-1}^{\infty} p_j(t,\theta)$ the expansion into homogeneous symbols. Formally, we have

$$= \sum_{j=-1}^{\infty} \frac{\ell^{1-d}}{(2\pi)^{d-1}} \int_{\Sigma_\ell^*} \int_{\mathbb{R}^{d-1}} \int_{\Sigma_\ell^*} e^{i(t-s)\theta} p_j(t,\theta) g(s/\ell) \overline{h(t/\ell)} \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}t$$
$$= \sum_{j=-1}^{\infty} \frac{1}{(2\pi)^{d-1}} \int_{\Sigma^*} \int_{\mathbb{R}^{d-1}} \int_{\Sigma^*} e^{i(t-s)\theta} p_j(\ell t,\theta/\ell) g(s) \overline{h(t)} \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}t$$
$$= \sum_{j=-1}^{\infty} \frac{\ell^{-j}}{(2\pi)^{d-1}} \int_{\Sigma_\ell^*} \int_{\mathbb{R}^{d-1}} \int_{\Sigma_\ell^*} e^{i(t-s)\theta} p_j(\ell t,\theta) g(s) \overline{h(t)} \, \mathrm{d}s \, \mathrm{d}\theta \, \mathrm{d}t.$$

A decomposition formula

Define

$$\langle Q_0 g, h\rangle := \int_{\mathbb{R}^{d-1}} p_{-1}(0,\theta)\cdot(Fg)(\theta)\cdot(Fh)(\theta)\,\mathrm{d}\theta.$$

Lemma

$$Q_{\ell,0} = \frac{1}{\ell} Q_0 + \mathcal{O}(1).$$

Recall that

$$\ell Q_{\ell,\omega} = \ell Q_{\ell,0} - \ell\omega^2 T_\ell^* r_{\Sigma_\ell} \mathbb{K}_0^* \Big(I + \omega^2 R_\varnothing(\omega)\Big)\mathbb{K}_0 e_{\Sigma_\ell} T_\ell.$$

Assume that $\omega_0 \neq 0$ is a resonance of order 1 of $A_\varnothing$ and **not** a branching point, i.e., we have

$$1 = \dim \frac{1}{2\pi i}\int_{|\omega-\omega_0|=\varepsilon} R_\varnothing(\omega) 2\omega\ \mathrm{d}\omega, \qquad \varepsilon \text{ small enough.}$$

Properties: Let ℓ be sufficiently small. Then:

- $\omega \mapsto \ell Q(\ell,\omega)$ has a unique singularity near ω_0;
- $\|\ell Q(\ell,\omega) - Q_0\| = \ell \| Q(\ell,\omega) - \frac{1}{\ell} Q_0\| = \mathcal{O}(\ell)$ for $|\omega| = \varepsilon$, ε fixed.

A final asymptotic estimate

We have

$$R_{\varnothing}(\omega) = \frac{1}{2\omega_0} \cdot \frac{\Pi_0}{\omega_0 - \omega} + \mathscr{O}(1) \qquad \text{as } \omega \to \omega_0$$

for a rank-one operator Π_0 with integral kernel $\Pi_0(x,y) = u_0(x)u_0(y)$. Here u_0 is a sufficiently normalised outgoing solution of

$$(a(x,\nabla_x) - \omega_0^2)u_0 = 0, \qquad u|_{\partial\Omega} = 0.$$

Theorem

There exist $\ell_0 > 0$ such that for $\ell \in (0,\ell_0)$ the operator A_{Σ_ℓ} has exactly one resonance $\omega(\ell)$ near ω, which satisfies the asymptotic estimate

$$\omega_0 - \omega(\ell) = \nu \cdot (\mathscr{B}u_0(0))^2 \cdot \ell^d + \mathscr{O}(\ell^{d+1}), \qquad (2)$$

where $\nu := \frac{\langle Q_0^{-1}\mathbf{1},\mathbf{1}\rangle}{2\omega_0} > 0$. Here Q_0 is chosen as before and $\mathbf{1}$ is the constant function on the window.

Remark: For simple branching points of second order we replace $\omega_0 - \omega(\ell)$ by $\sqrt{\omega_0 - \omega(\ell)}$.

Thank you for your attention.