Now consider the Neumann eigenvalue problem for the p-Laplacian

$$-\Delta_p u = \Lambda_p^p |u|^{p-2} u \quad \text{in } \Omega$$

$$|\nabla u|^{p-2} \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega,$$

with (first nontrivial eigenvalue) Λ_p^p minimizing

$$\mathcal{R}_p(v) := \frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p} \text{ on } W^{1,p}(\Omega) \cap \left\{ \int_\Omega |v|^{p-2} v = 0 \right\}. \tag{2}$$

What is known about Λ_p and what happens as $p \to 1$ or $p \to \infty$?

As $p \to 1$, formally we minimize $\int_\Omega |Dv|\, dx$
on functions satisfying $||v||_{L^1(\Omega)} = 1$ and $\int_\Omega \mathrm{sign} v\, dx = 0$.
Minimizers can be chosen piecewise constant: $c > 0$ on Ω^+ and $-c$ on Ω^-.

This constitutes a geometric partioning problem:
Divide Ω into two disjoint subsets Ω^+ and Ω^- of equal volume
such that their relative perimeter $P(\partial\Omega^+; \Omega)$ becomes minimal.

Problems of this nature are relevant in developing fast algorithms
for parallel computing (Gajewski, Gärtner: Domain separation
by means of sign changing eigenfunctions of p-Laplacians. 1998-2001)

For $p \to \infty$ we have $\Lambda_p \to \Lambda_\infty := 2/diam(\Omega)$, while $\lambda_\infty = 1/R(\Omega)$.
The first nontrivial eigenfunctions u_p converge to a viscosity solution u_∞ of

$$\begin{cases} \min\{|\nabla u| - \Lambda u, -\Delta_\infty u\} = 0 & \text{in } \{u > 0\} \cap \Omega \\ \max\{-|\nabla u| - \Lambda u, -\Delta_\infty u\} = 0 & \text{in } \{u < 0\} \cap \Omega \\ -\Delta_\infty u = 0 & \text{in } \{u = 0\} \cap \Omega \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

While (3) was already derived in (Rossi, Saintier 14), we also prove that $\Lambda \geq \Lambda_\infty$ for any Λ having a nontrivial solution to (3) and any convex Ω. This shows that (for convex Ω) Λ_∞ is in fact the first positive eigenvalue of (3).

The fact that $\Lambda_\infty = 2/diam(\Omega)$ has interesting consequences.

Corollary 1: *If Ω^* denotes a ball of same volume as Ω, then the Szegö-Weinberger inequality $\Lambda_\infty(\Omega) \leq \Lambda_\infty(\Omega^*)$ holds.*

For general p we do not know that $\Lambda_p(\Omega) \leq \Lambda_p(\Omega^*)$, except
a) for $p = 2$ and general Ω or
b) for $p = \infty$ and convex Ω (Cor.1).
c) For $p = 1$ and convex plane Ω this was an old conjecture of Polya
(Esposito, Ferone, Kawohl, Nitsch, Trombetti 12).

Corollary 2: *For convex Ω we have $\Lambda_\infty(\Omega) \leq \lambda_\infty(\Omega)$. Moreover, equality holds only if Ω is a ball.*

A stronger statement is known for finite p, namely $\Lambda_p(\Omega) < \lambda_p(\Omega)$
a) for $p = 2$ & general Ω (Faber-Krahn & Cor. 1) or $p = 1$ & convex plane Ω
b) for $p \in (1, \infty)$ and convex Ω (Brasco, Nitsch, Trombetti 14)

Corollary 3: *For convex Ω no Neumann eigenfunction associated to Λ_∞ can have a closed nodal line inside Ω.*

Else there is a nodal domain $\Omega' \subset\subset \Omega$ so that
$\Lambda_\infty(\Omega) = \lambda_\infty(\Omega') = 1/R(\Omega') > 1/R(\Omega) = \lambda_\infty(\Omega)$ contradicting Cor. 2.

Lemma 1: *For Ω simply connected and Lipschitz $\Lambda_p \to \Lambda_\infty := 2/diam(\Omega)$, where diam denotes intrinsic diameter.*

For $x, y \in \Omega$ the intrinsic distance $d_\Omega(x, y)$ is the length of the geodesic in Ω connecting x to y and $diam(\Omega) = \sup_{x,y\in\Omega} d_\Omega(x, y)$.

Proof in 2 steps.
Step 1: $\limsup_{p\to\infty} \Lambda_p \leq \Lambda_\infty$.
Pick $x_0 \in \Omega$ and adjust $c_p \in \mathbb{R}$ so that $w(x) = d_\Omega(x, x_0) - c_p$ is admissible test function for $\mathcal{R}_p$. Then

$$\Lambda_p \leq \mathcal{R}_p(w) = \left(\frac{1}{|\Omega|} \int_\Omega |d_\Omega(x, x_0) - c_p|^p \right)^{-1/p} .$$

Since $0 \leq c_p \leq diam(\Omega)$, up to subseqence $c_p \to c_\infty$ and

$$\liminf_{p\to\infty} \left(\frac{1}{|\Omega|} \int_\Omega |d_\Omega(x, x_0) - c_p|^p \right)^{1/p} = sup_{x\in\Omega}|d_\Omega(x, x_0) - c_\infty| \geq diam(\Omega)/2$$

Step 2: $\liminf_{p\to\infty} \Lambda_p \geq \Lambda_\infty$.

Step 2: $\liminf_{p\to\infty} \Lambda_p \geq \Lambda_\infty$.
For $p > m > n$ the family of eigenfunctions u_p is uniformly bounded in $W^{1,m}(\Omega)$ and equicont. so it converges in C^0 and weakly in $W^{1,m}$ to a limit u_∞.

$$\frac{||\nabla u_\infty||_m}{||u_\infty||_m} \leq \liminf_{p\to\infty} \frac{||\nabla u_p||_m}{||u_p||_m} \leq \liminf_{p\to\infty} \frac{||\nabla u_p||_p}{||u_p||_m} = \liminf_{p\to\infty} \Lambda_p \frac{||u_p||_p}{||u_p||_m}$$

Sending $m \to \infty$ gives $\liminf_{p\to\infty} \Lambda_p \geq \frac{||\nabla u_\infty||_\infty}{||u_\infty||_\infty}$
so it remains to estimate $||u_\infty||_\infty$ in terms of $||\nabla u_\infty||_\infty$.

Condition $\int_\Omega |u_p|^{p-2} u_p = 0$ implies $\sup u_\infty = -\inf u_\infty$

$$||u_\infty||_\infty = \frac{1}{2}(\sup u_\infty - \inf u_\infty) \leq \frac{1}{2} diam(\Omega) ||\nabla u_\infty||_\infty.$$

Thus

$$\liminf_{p\to\infty} \Lambda_p \geq \frac{||\nabla u_\infty||_\infty}{||u_\infty||_\infty} \geq \frac{2}{diam(\Omega)} = \Lambda_\infty$$

□

Observation:

Suppose $||u_\infty||_\infty$ is scaled to 1, $u(x) = \inf u$ and $u(y) = \sup u$.
Then u_∞ increases with constant slope Λ_∞ along the geodesic from x to y.

In particular in the one-dimensional case that $\Omega = (-1, 1)$ we have $\Lambda_\infty = 1$, and the function $u(x) = x$ is a viscosity solution of (3).

$$\begin{cases} \min\{|\nabla u| - \Lambda u, -\Delta_\infty u\} = 0 & \text{in } \{u > 0\} \cap \Omega \\ \max\{-|\nabla u| - \Lambda u, -\Delta_\infty u\} = 0 & \text{in } \{u < 0\} \cap \Omega \\ -\Delta_\infty u = 0 & \text{in } \{u = 0\} \cap \Omega \\ \partial u / \partial \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

While it solves the differential equation $-\Delta_\infty u = u'' = 0$ in the classical (and thus in the viscosity) sense, it clearly violates the boundary condition $u'(\pm 1) = 0$ in the classical sense.

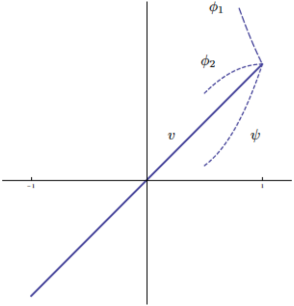
$u(x) = x$ clearly violates the boundary condition $u'(\pm 1) = 0$ in the classical sense. To check the Neumann condition in the right end point $x = 1$ in the viscosity sense, one must must verify

$$\min\{\min\{|\phi'| - \Lambda\phi, -|\phi'|^2\phi''\}\ , \phi'\}(1) \leq 0 \tag{4}$$

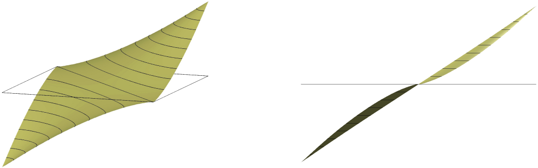
for any C^2 test function ϕ touching u in $x = 1$ from above, and

$$\max\{\min\{|\psi'| - \Lambda\psi, -|\psi'|^2\psi''\}\ ,\ \psi'\}(1) \geq 0 \tag{5}$$

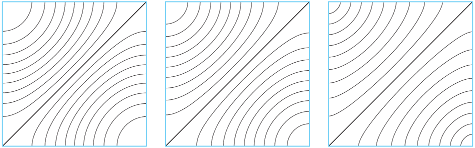
for any smooth test function ψ touching u from below.



Moreover, on a rectangle u_∞ has constant (and maximal) slope along a diagonal.



Numerical simulation of u_{15} and side view in diagonal direction.



Level plots of u_p for $p = 2$, $p = 4$ and $p = 8$

For a square and $p = 2$ (the linear case) Λ_2 is a double eigenvalue and there are solutions with diagonal and horizontal or vertical nodal lines.

There are one-dimensional solutions also for $p = \infty$ but they are associated to higher eigenvalues than Λ_∞.
Take $\Omega = (-1, 1)^2$ and $u(x) = x_1$. Then $u \in C^2(\Omega)$, $-\Delta_\infty u = 0$ in Ω.
Now PDE in (3) is satisfied if also $1 = |\nabla u| \geq \Lambda u$ on $\{u > 0\}$, that implies

$$\Lambda \leq 1.$$

The Neumann bc. is satisfied in classical sense on horizontal parts of $\partial\Omega$. However, for Neumann bc. to hold in the viscosity sense on the right part, we must verify

$$\min\{\min\{|\nabla\phi| - \Lambda\phi, -\Delta_\infty\phi\}\ ,\ \partial\phi/\partial\nu\}(x_0) \leq 0$$

for any C^2 test function ϕ touching u in $x_0 \in \partial\Omega$ from above, and

$$\max\{\min\{|\nabla\psi| - \Lambda\psi, -\Delta_\infty\psi\}\ ,\ \partial\psi/\partial\nu\}(x_0) \geq 0$$

for any smooth test function ψ touching u from below.

Recall $|\nabla u| = \partial u/\partial\nu = 1$ at $x_1 = 1$. Therefore only the very first constraint for ψ is active and implies

$$\Lambda \geq 1.$$

This shows that $u(x) = x_1$ is a viscosity solution to (3) with eigenvalue $\Lambda = 1$, but

$$\Lambda = 1 > \frac{1}{\sqrt{2}} = \frac{2}{diam(\Omega)} = \Lambda_\infty.$$

Eigenvalue problem for the normalized p-Laplacian

Also the eigenvalue problem

$$-\Delta_p^N u = \lambda_p u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{6}$$

has been studied for $p \in (1, \infty)$ by Birindelli/Demengel, Rossi et al and for $p = \infty$ by Juutinen. Now one has the estimate

$$\left(\frac{\pi}{2\rho(\Omega)}\right)^2 \le \lambda_\infty(\Omega) \le \left(\frac{\pi}{2R(\Omega)}\right)^2,$$

in which $\rho(\Omega)$ and $R(\Omega)$ denote the outer and inradius of Ω, and which becomes sharp for a ball of radius R.
For $p \ge 2$ the first eigenfunction is again unique modulo scaling. At least for starshaped domains Ω there is convergence of the first eigenvalue λ_p to λ_∞ as $p \to \infty$, but the limit $p \to 1$ appears to be open.

Open Problem

Let $\Omega = B_R(0)$ be a ball and u a radially symmetric eigenfunction. One should also expect many nonradial eigenfunctions to exist, but for $p \neq 2$ we are not aware of any results in this direction, not even in two dimensions. For radial functions and with the Ansatz $u(x) = v(|x|)$ the eigenvalue problem (6) transforms into

$$(p-1)v''(r) + \frac{n-1}{r}v'(r) + p\lambda v(r) = 0 \quad \text{in } (0,R), \qquad v'(0) = 0 = v(R). \quad (7)$$

The formal limit $p \to 1$ in (7) is

$$\frac{n-1}{r}v'(r) + \lambda v(r) = 0 \quad \text{in } (0,R), \qquad v'(0) = 0 = v(R). \quad (8)$$

with the viscosity solution $v(r) = v(0)\, e^{-\frac{\lambda}{2(n-1)}r^2}$, which violates the boundary condition in the classical sense at $r = R$, but since it solves (8) there in the classical sense, it still satisfies the boundary condition $v(R) = 0$ in the viscosity sense. It is therefore reasonable to expect that also for more general domains the first eigenfunctions in (32) converge (as $p \to 1$) to a viscosity solution of

$$(n-1)Hv_\nu + \lambda v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \quad (9)$$

$$(p-1)v''(r)+\frac{n-1}{r}v'(r)+p\lambda v(r)=0 \quad \text{in } (0,R), \qquad v'(0)=0=v(R). \quad (7)$$

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$$(n-1)Hv_\nu+\lambda v=0 \quad \text{in } \Omega, \qquad v=0 \quad \text{on } \partial\Omega, \qquad (9)$$

and that they develop a boundary layer.

For $p > 1$ problem

$$(p-1)v''(r) + \frac{n-1}{r}v'(r) + p\lambda v(r) = 0 \quad \text{in } (0,R), \qquad v'(0) = 0 = v(R). \quad (7)$$

is a Bessel type equation, and this observation was used by Kawohl/Krömer/Kurtz to explicitly derive a countable and complete orthonormal system of eigenfunctions to (7) in a suitably weighted L^2-space. These form a countable, but presumably incomplete sequence of eigenfunctions to (6)

$$-\Delta_p^N u = \lambda_p u \quad \text{in } B_R(0), \qquad u = 0 \quad \text{on } \partial B_R(0).$$

Other open issues concerning p-Laplacian:

- Uniqueness of first Dirichlet eigenfunction for $p = 1$ or ∞ and convex Ω?
  There is nonuniqueness for nonconvex Ω.

- Nodal pattern on disc for any $p \neq 2$?
  Not radial for both second Dirichlet and Neumann eigenf., but diameter?

- Hot spot conjecture true for general $p \in (1, \infty)$ and convex Ω?
  Known for $p = 1$ and ∞. Open even for $p = 2$, see Polymath thread of T.Tao.