Asymptotically optimal eigenvalues of the Robin Laplacian

James Kennedy

Group of Mathematical Physics
University of Lisbon

Joint work with Pedro Freitas

Eigenvalues and Inequalities
Institut Mittag-Leffler

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Grupo de Física Matemática da Universidade de Lisboa

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Eigenvalues of the Laplacian

$\Omega \subset \mathbb{R}^2$ a bounded, sufficiently smooth domain. Consider

$$-\Delta u = \lambda u \qquad \text{in } \Omega$$

with one of the following boundary conditions on $\partial\Omega$:

$$\begin{aligned} u &= 0 && \text{(Dirichlet), or} \\ \frac{\partial u}{\partial \nu} &= 0 && \text{(Neumann), or} \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{(Robin),} \end{aligned}$$

where ν is the outward-pointing unit normal to $\partial\Omega$ and $\alpha > 0$ is a constant.

Dirichlet:	$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \ldots \to \infty$
Neumann:	$0 = \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \to \infty$

Motivation I: Pólya's conjecture

Conjecture (Pólya)

For all Ω and all $k \geq 1$, we have

$$\mu_{k+1}(\Omega) \leq \frac{4\pi}{|\Omega|} k \leq \lambda_k(\Omega).$$

Supporting evidence:

Theorem (Weyl asymptotics)

For any fixed Ω, as $k \to \infty$

$$\lambda_k(\Omega) = \frac{4\pi}{|\Omega|} k + 2\pi \frac{|\partial\Omega|}{|\Omega|^{3/2}} k^{1/2} + o(k^{1/2})$$

$$\mu_k(\Omega) = \frac{4\pi}{|\Omega|} k - 2\pi \frac{|\partial\Omega|}{|\Omega|^{3/2}} k^{1/2} + o(k^{1/2})$$

Motivation II: shape optimisation

Consider

$$\lambda_k^* := \inf\{\lambda_k(\Omega) : \Omega \subset \mathbb{R}^2 \text{ “nice”}, |\Omega| = 1\}$$
$$= \inf\{\lambda_k(\Omega) \cdot |\Omega| : \Omega \subset \mathbb{R}^2 \text{ “nice”}\}$$

Examples:

1. $k = 1$: Minimum achieved when Ω is a disk (Theorem of Faber–Krahn)
2. $k = 2$: Minimum when Ω disjoint union of two equal disks (Theorem of Hong–Krahn–Szego)

Consider likewise

$$\mu_k^* := \sup\{\mu_k(\Omega) \cdot |\Omega| : \Omega \subset \mathbb{R}^2 \text{ nice}\}.$$

Pólya's conjecture may be formulated as

$$\frac{\mu_{k+1}^*}{k} \leq 4\pi \leq \frac{\lambda_k^*}{k} \qquad \text{for all } k \geq 1.$$

Motivation II: shape optimisation

Theorem (Colbois–El Soufi, 2014)

The sequence (λ_k^*) is subadditive, while (μ_k^*) is superadditive.

Lemma (Fekete)

Suppose (a_k) is subadditive. Then

$$\lim_{k\to\infty} \frac{a_k}{k} \text{ exists and equals } \inf_k \frac{a_k}{k}.$$

Corollary

$$\lim_{k\to\infty} \frac{\lambda_k^*}{k} \text{ exists and equals } \inf_k \frac{\lambda_k^*}{k}.$$

Pólya's conjecture is equivalent to $\lim_{k\to\infty} \frac{\lambda_k^*}{k} \geq 4\pi$.
(actually "$=$" due to Weyl. Analogous for Neumann.)

Motivation I: Pólya's conjecture

Moral

It is "useful" to study the asymptotic behaviour of the optimal values λ_k^*, μ_k^*.

Let Ω_k^* be such that $\lambda_k(\Omega_k^*) = \lambda_k^*$ and let $B =$ disk of unit area.

"Conjecture"

$\Omega_k^* \to B$ as $k \to \infty$ (in Hausdorff distance??). In particular,

$$\frac{\lambda_k^*}{\lambda_k(B)} \longrightarrow 1 \qquad \text{as } k \to \infty,$$

which would imply Pólya.

Supporting evidence: Weyl

$$\lambda_k(\Omega) = \frac{4\pi}{|\Omega|}k + 2\pi\frac{|\partial\Omega|}{|\Omega|^{3/2}}k^{1/2} + o(k^{1/2}).$$

Isoperimetric inequality $\implies$ second term is smallest when $\Omega = B$.

Model problems

Idea

Try to prove this for a restricted class of domains.

Example: rectangles

$$R_a := \text{rectangle of side lengths } a, a^{-1},\ a \geq 1,$$
$$a_k^* := \arg\min\{\lambda_k(R_a) : a \geq 1\}$$

Theorem (Antunes–Freitas, 2013)

$a_k^* \to 1$ (corresp. to the square) as $k \to \infty$.

(Related to a lattice point counting problem.)

Model problems

Subsequent works: (among others)

perimeter restriction (more general domains)	Bucur–Freitas (2013) Antunes–Freitas (2016)
Neumann rectangles	van den Berg–Bucur–Gittins (2016)
hyperrectangles	van den Berg–Gittins (2017) Gittins–Larson (2017)
lattice triangles	Marshall–Steinerberger (2017)
other lattice point problems ($\implies$ other operators)	Ariturk–Laugesen (2017) Laugesen–Liu (2016/18)

The Robin Laplacian

Recall

$$-\Delta u = \lambda u \qquad \text{in } \Omega \subset \mathbb{R}^2 \text{ nice}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \qquad \text{on } \partial\Omega,\ \alpha > 0.$$

The associated form is

$$a_\alpha(u, v) = \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} \alpha uv, \qquad u, v \in H^1(\Omega);$$

$$\alpha = 0 \quad \Longrightarrow \quad \text{Neumann}$$

$$\alpha = \infty \quad \Longrightarrow \quad \text{Dirichlet}$$

Eigenvalues

$$0 < \lambda_1(\Omega, \alpha) \le \lambda_2(\Omega, \alpha) \le \ldots \to \infty$$

- $\alpha \mapsto \lambda_k(\Omega, \alpha)$ is (piecewise) smooth and monotonically increasing, for fixed Ω
- $\alpha \to 0 \implies \lambda_k(\Omega, \alpha) \to \mu_k(\Omega)$,
  $\alpha \to \infty \implies \lambda_k(\Omega, \alpha) \to \lambda_k(\Omega)$

The Robin Laplacian

Question

Is there a version of Pólya's conjecture for $\lambda_k(\Omega, \alpha)$?

1. Eigenvalue *minimisation* problem like Dirichlet:

$$\lambda_k^*(\alpha) := \inf\{\lambda_k(\Omega, \alpha) : \Omega \subset \mathbb{R}^2 \text{ nice}, \ |\Omega| = 1\}$$

   Example: $\lambda_1^*(\alpha)$ achieved by the disk (Theorem of Bossel–Daners)

2. Weyl asymptotics like Neumann:

$$\lambda_k(\Omega, \alpha) = \frac{4\pi}{|\Omega|} k - 2\pi \frac{|\partial\Omega|}{|\Omega|^{3/2}} k^{1/2} + o(k^{1/2})$$

3. Robin Laplacian lacks "nice" properties of Dirichlet Laplacian (domain monotonicity, homothetic scalings)

A simple estimate

$$\lambda_1(U,\alpha) = \inf_{0\neq u\in H^1(U)} \frac{\int_U |\nabla u|^2 + \int_{\partial U} \alpha |u|^2}{\int_U |u|^2} \leq \frac{|\partial U|}{|U|}\alpha.$$

Now suppose $|\Omega| = 1$ and take

$\Omega_k :=$ disjoint union of k equal copies U of Ω of area $1/k$ each.

Then

$$\lambda_k(\Omega_k,\alpha) = \lambda_1\left(U,\alpha\right) \leq \frac{k^{-1/2}|\partial\Omega|}{k^{-1}|\Omega|}\alpha = |\partial\Omega|\alpha k^{1/2}$$

Example

$B =$ disk of unit area, $B_k =$ disjoint union of k equal disks of total area 1. Then

$$\lambda_k(B_k,\alpha) \leq |\partial B|\alpha k^{1/2} = 2\pi^{1/2}\alpha k^{1/2}.$$

Shape optimisation for λ_k

B_k is "individually asymptotically minimising" for λ_k:

Theorem (K., 2010)

Fix $k \geq 3$ and Ω. There exists $\alpha_{k,\Omega} > 0$ such that

$$\lambda_k(\Omega, \alpha) \geq \lambda_k(B_k, \alpha) \qquad \text{for all } \alpha \in (0, \alpha_{k,\Omega}).$$

Conjecture (cf. Antunes–Freitas–K., 2013)

Fix $k \geq 3$. There exists $\alpha_k > 0$ such that for any Ω

$$\lambda_k(\Omega, \alpha) \geq \lambda_k(B_k, \alpha) \qquad \text{for all } \alpha \in (0, \alpha_k).$$

We should have $\alpha_k \to \infty$ as $k \to \infty$.

A Pólya conjecture for the Robin Laplacian?

For *fixed* α, this means we expect

$$\begin{aligned} \lambda_k^*(\alpha) &= \lambda_k(B_k, \alpha) \quad \text{for } k \text{ suff. large} \\ &\sim 2\pi^{1/2}\alpha k^{1/2}, \end{aligned}$$

although we also expect $\lambda_k^*(\alpha) \to \lambda_k^*$ as $\alpha \to \infty$, for fixed k (with convergence of the optimal domains to their Dirichlet counterparts).

Conjecture ("modified Pólya conjecture")

Fix $\alpha > 0$ and $|\Omega| > 0$. Then there exists $C = C(\alpha, |\Omega|) > 0$ such that

$$\lambda_k(\Omega, \alpha) \geq Ck^{1/2} \qquad \text{for all } \Omega \text{ and } k \geq 1.$$

(How does C depend on α and $|\Omega|$?)

The problem on rectangles and unions of rectangles

Look at the modified Pólya conjecture on two classes of domains:

- $\mathcal{R} = \{\text{rectangles } \Omega \subset \mathbb{R}^2 : |\Omega| = 1\}$ (proxy for convex?)
- $\mathcal{U}$ = set of finite disjoint unions of rectangles of total area 1

Theorem (Freitas–K., 2018)

There exist constants $C_r, C_u > 0$ depending on $\alpha > 0$ such that

$$\lambda_k(\Omega, \alpha) \geq C_u k^{1/2} \qquad \text{for all } \Omega \in \mathcal{U},$$

$$\lambda_k(\Omega, \alpha) \geq C_r k^{2/3} \qquad \text{for all } \Omega \in \mathcal{R}.$$

The powers of k are optimal.

In particular, we cannot recover a "classical" Pólya bound by restricting to convex sets.

Remark

The proof does not make significant use of lattice point counting arguments, but rather how the problem scales.

The problem on rectangles and unions of rectangles

Further questions:

- How do the optimal constants C_u, C_r depend on α?
- Is S_k = disjoint union of k equal squares optimal for λ_k for α small?

Theorem (Freitas–K., 2018)

There exists a constant $C_1 > 0$ such that for all $\Omega \in \mathcal{U}$

$$\lambda_k(\Omega, \alpha) \geq \lambda_k(S_k, \alpha) \qquad \text{whenever } \alpha \leq C_1 k^{1/2}.$$

There exists another constant $C_2 > 0$ such that, for any fixed $\Omega \in \mathcal{U}$,

$$\lambda_k(\Omega, \alpha) \leq \lambda_k(S_k, \alpha) \qquad \text{whenever } \alpha \geq C_2 k^{1/2} + o(k^{1/2});$$

in particular, S_k does not minimise λ_k if $\alpha \geq C_2 k^{1/2}$.

$\lambda_k(S_k, \alpha) \sim 4\alpha k^{1/2}$ as $k \to \infty \implies 4\alpha$ is the "asymptotically correct" value of C_u.

Further consequences and open problems

Denote by

$$\sigma_k^*(\alpha) := \inf \left\{ \sum_{j=1}^{k} \lambda_k(\Omega, \alpha) : \Omega \in \mathcal{U} \right\}.$$

Corollary (Freitas–K., 2018)

Fix $\alpha > 0$. Then

$$\frac{8}{3}\alpha \leq \liminf_{k\to\infty} \frac{\sigma_k^*(\alpha)}{k^{3/2}} \leq \limsup_{k\to\infty} \frac{\sigma_k^*(\alpha)}{k^{3/2}} \leq 4\alpha.$$

Questions and conjectures:

1. Show that B_k actually achieves $\lambda_k^*(\alpha)$ for $\alpha \leq C_1 k^{1/2}$, and obtain the modified Pólya conjecture
2. Which domain actually achieves $\sigma_k^*(\alpha)$? Is it S_k? What about the infimum over general domains?
3. Higher dimensions

Thank you for your attention!