

Interpolation inequalities and spectral estimates for magnetic operators

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Outline

- **Magnetic rings**
 - ▷ Results: a one-dimensional magnetic interpolation inequality
 - ▷ Consequences: Keller-Lieb-Thirring estimates, Bohm-Aharonov magnetic fields and a new Hardy inequality in \mathbb{R}^2
- **Interpolation inequalities and spectral estimates for magnetic operators in dimensions 2 and 3**
 - ▷ Theoretical results
 - ▷ Estimates, numerics and a conjecture

Preliminaries: a simple interpolation on the circle

On $(-\pi, \pi] \approx \mathbb{S}^1 \ni s$, let us consider the uniform probability measure $d\sigma = ds/(2\pi)$ and denote by $\|\psi\|_{L^p(\mathbb{S}^1)}$ the corresponding L^p norm. The inequality

$$\|\psi'\|_{L^2(\mathbb{S}^1)}^2 + \alpha \|\psi\|_{L^2(\mathbb{S}^1)}^2 \geq \mu_{0,p}(\alpha) \|\psi\|_{L^p(\mathbb{S}^1)}^2 \quad (1)$$

holds for some concave function $\alpha \mapsto \mu_{0,p}(\alpha)$ on $(0, +\infty)$

Lemma

- If $p > 2$ and $0 < \alpha \leq 1/(p-2)$, then $\mu_{0,p}(\alpha) = \alpha$
 - If $p = -2$ and $\alpha = 1/(p-2) = -1/4$, then $\mu_{0,p}(-1/4) = -1/4$
- In both cases, equality achieved only by constant functions

Case $p = -2$ (Exner, Harrell, Loss, 1998):

$$\|\psi'\|_{L^2(\mathbb{S}^1)}^2 + \frac{1}{4} \|\psi\|_{L^p(\mathbb{S}^1)}^2 \geq \frac{1}{4} \|\psi\|_{L^2(\mathbb{S}^1)}^2$$

Case $p > 2$: Bakry-Emery method applies to *Kolmogorov's inequality*

Carré du champ method

Let $\mathcal{F}[u] := \|u'\|_{L^2(\mathbb{S}^1)}^2 + \frac{1}{p-2} (\|u\|_{L^2(\mathbb{S}^1)}^2 - \|u\|_{L^p(\mathbb{S}^1)}^2)$ and consider a positive solution of the parabolic equation

$$\frac{\partial u}{\partial t} = u'' + (p-1) \frac{|u'|^2}{u}$$

If $p = -2$ (new application of the *carré du champ* method)

$$-\frac{d}{dt} \mathcal{F}[u(t, \cdot)] = \underbrace{\int_{-\pi}^{\pi} (|u''|^2 - |u'|^2) d\sigma}_{\geq 0 \text{ (Poincaré)}} + \int_{-\pi}^{\pi} \frac{|u'|^4}{u^2} d\sigma$$

If $p > 1$, $p \neq 2$, the method is well known (Bakry, Emery, 85)

Magnetic rings

▷ A magnetic interpolation inequality on \mathbb{S}^1 : with $p > 2$

$$\|\psi' + i a \psi\|_{L^2(\mathbb{S}^1)}^2 + \alpha \|\psi\|_{L^2(\mathbb{S}^1)}^2 \geq \mu_{a,p}(\alpha) \|\psi\|_{L^p(\mathbb{S}^1)}^2$$

▷ Consequences

- A Keller-Lieb-Thirring inequality
- A new Hardy inequality for Bohm-Aharonov magnetic fields in \mathbb{R}^2

Magnetic flux, a reduction

Assume that $a : \mathbb{R} \rightarrow \mathbb{R}$ is a 2π -periodic function such that its restriction to $(-\pi, \pi] \approx \mathbb{S}^1$ is in $L^1(\mathbb{S}^1)$ and define the space

$$X_a := \{ \psi \in C_{\text{per}}(\mathbb{R}) : \psi' + i a \psi \in L^2(\mathbb{S}^1) \}$$

• A standard change of gauge (see *e.g.* (Ilyin, Laptev, Loss, Zelik, 2016))

$$\psi(s) \mapsto e^{i \int_{-\pi}^s (a(\sigma) - \bar{a}) d\sigma} \psi(s)$$

where $\bar{a} := \int_{-\pi}^{\pi} a(s) d\sigma$ is the *magnetic flux*, reduces the problem to

a is a constant function

• For any $k \in \mathbb{Z}$, ψ by $s \mapsto e^{iks} \psi(s)$ shows that $\mu_{a,p}(\alpha) = \mu_{k+a,p}(\alpha)$

$$a \in [0, 1]$$

• $\mu_{a,p}(\alpha) = \mu_{1-a,p}(\alpha)$ because

$$|\psi' + i a \psi|^2 = |\chi' + i(1-a)\chi|^2 = |\bar{\psi}' - i a \bar{\psi}|^2 \text{ if } \chi(s) = e^{-is} \overline{\psi(s)}$$

$$a \in [0, 1/2]$$

Optimal interpolation

We want to characterize the *optimal constant* in the inequality

$$\|\psi' + i a \psi\|_{L^2(\mathbb{S}^1)}^2 + \alpha \|\psi\|_{L^2(\mathbb{S}^1)}^2 \geq \mu_{a,p}(\alpha) \|\psi\|_{L^p(\mathbb{S}^1)}^2$$

written for any $p > 2$, $\alpha \in (-a^2, +\infty)$, $\psi \in X_a$

$$\mu_{a,p}(\alpha) := \inf_{\psi \in X_a \setminus \{0\}} \frac{\int_{-\pi}^{\pi} (|\psi' + i a \psi|^2 + \alpha |\psi|^2) d\sigma}{\|\psi\|_{L^p(\mathbb{S}^1)}^2}$$

$p = -2 = 2d/(d-2)$ with $d = 1$ (Exner, Harrell, Loss, 1998)

$p = +\infty$ (Galunov, Olienik, 1995) (Ilyin, Laptev, Loss, Zelik, 2016)

$\lim_{\alpha \rightarrow -a^2} \mu_{a,p}(\alpha) = 0$ (JD, Esteban, Laptev, Loss, 2016)

Using a Fourier series $\psi(s) = \sum_{k \in \mathbb{Z}} \psi_k e^{iks}$, we obtain that

$$\|\psi' + i a \psi\|_{L^2(\mathbb{S}^1)}^2 = \sum_{k \in \mathbb{Z}} (a+k)^2 |\psi_k|^2 \geq a^2 \|\psi\|_{L^2(\mathbb{S}^1)}^2$$

$\psi \mapsto \|\psi' + i a \psi\|_{L^2(\mathbb{S}^1)}^2 + \alpha \|\psi\|_{L^2(\mathbb{S}^1)}^2$ is coercive for any $\alpha > -a^2$

An interpolation result for the magnetic ring

Theorem

For any $p > 2$, $a \in \mathbb{R}$, and $\alpha > -a^2$, $\mu_{a,p}(\alpha)$ is achieved and

- (i) if $a \in [0, 1/2]$ and $a^2(p+2) + \alpha(p-2) \leq 1$, then $\mu_{a,p}(\alpha) = a^2 + \alpha$ and equality in (1) is achieved only by the constant functions
- (ii) if $a \in [0, 1/2]$ and $a^2(p+2) + \alpha(p-2) > 1$, then $\mu_{a,p}(\alpha) < a^2 + \alpha$ and equality in (1) is not achieved by the constant functions

If $\alpha > -a^2$, $a \mapsto \mu_{a,p}(\alpha)$ is monotone increasing on $(0, 1/2)$

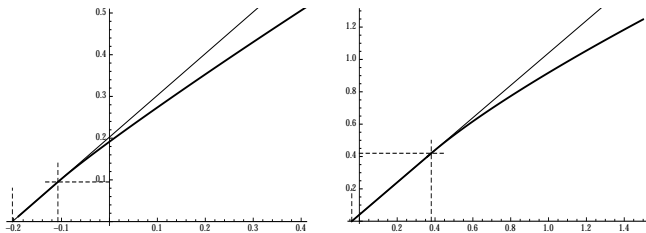


Figure: $\alpha \mapsto \mu_{a,p}(\alpha)$ with $p = 4$ and (left) $a = 0.45$ or (right) $a = 0.2$

Reformulations of the interpolation problem (1/3)

Any minimizer $\psi \in X_a$ of $\mu_{a,p}(\alpha)$ satisfies the Euler-Lagrange equation

$$(H_a + \alpha)\psi = |\psi|^{p-2}\psi, \quad H_a\psi = -\left(\frac{d}{ds} + ia\right)^2\psi$$

up to a multiplication by a constant and $v(s) = \psi(s)e^{ias}$ satisfies the condition

$$v(s + 2\pi) = e^{2i\pi a} v(s) \quad \forall s \in \mathbb{R} \quad (2)$$

Hence

$$\mu_{a,p}(\alpha) = \min_{v \in Y_a \setminus \{0\}} \mathbf{Q}_{p,\alpha}[v]$$

where $Y_a := \{v \in C(\mathbb{R}) : v' \in L^2(\mathbb{S}^1), (2) \text{ holds}\}$ and

$$\mathbf{Q}_{p,\alpha}[v] := \frac{\|v'\|_{L^2(\mathbb{S}^1)}^2 + \alpha \|v\|_{L^2(\mathbb{S}^1)}^2}{\|v\|_{L^p(\mathbb{S}^1)}^2}$$

Reformulations of the interpolation problem (2/3)

With $v = u e^{i\phi}$ the boundary condition becomes

$$u(\pi) = u(-\pi), \quad \phi(\pi) = 2\pi(a + k) + \phi(-\pi) \quad (3)$$

for some $k \in \mathbb{Z}$, and $\|v'\|_{L^2(\mathbb{S}^1)}^2 = \|u'\|_{L^2(\mathbb{S}^1)}^2 + \|u\phi'\|_{L^2(\mathbb{S}^1)}^2$

Hence

$$\mu_{a,p}(\alpha) = \min_{(u,\phi) \in Z_a \setminus \{0\}} \frac{\|u'\|_{L^2(\mathbb{S}^1)}^2 + \|u\phi'\|_{L^2(\mathbb{S}^1)}^2 + \alpha \|u\|_{L^2(\mathbb{S}^1)}^2}{\|u\|_{L^p(\mathbb{S}^1)}^2}$$

where $Z_a := \{(u, \phi) \in C(\mathbb{R})^2 : u', u\phi' \in L^2(\mathbb{S}^1), (3) \text{ holds}\}$

Reformulations of the interpolation problem (3/3)

We use the Euler-Lagrange equations

$$-u'' + |\phi'|^2 u + \alpha u = |u|^{p-2} u \quad \text{and} \quad (\phi' u^2)' = 0$$

Integrating the second equation, and *assuming that u never vanishes*, we find a constant L such that $\phi' = L/u^2$. Taking (3) into account, we deduce from

$$L \int_{-\pi}^{\pi} \frac{ds}{u^2} = \int_{-\pi}^{\pi} \phi' ds = 2\pi (a + k)$$

that

$$\|u \phi'\|_{L^2(\mathbb{S}^1)}^2 = L^2 \int_{-\pi}^{\pi} \frac{d\sigma}{u^2} = \frac{(a + k)^2}{\|u^{-1}\|_{L^2(\mathbb{S}^1)}^2}$$

Hence

$$\phi(s) - \phi(0) = \frac{a + k}{\|u^{-1}\|_{L^2(\mathbb{S}^1)}^2} \int_{-\pi}^s \frac{ds}{u^2}$$

Let us define

$$Q_{a,p,\alpha}[u] := \frac{\|u'\|_{L^2(\mathbb{S}^1)}^2 + a^2 \|u^{-1}\|_{L^2(\mathbb{S}^1)}^{-2} + \alpha \|u\|_{L^2(\mathbb{S}^1)}^2}{\|u\|_{L^p(\mathbb{S}^1)}^2}$$

Lemma

For any $a \in (0, 1/2)$, $p > 2$, $\alpha > -a^2$,

$$\mu_{a,p}(\alpha) = \min_{u \in H^1(\mathbb{S}^1) \setminus \{0\}} Q_{a,p,\alpha}[u]$$

is achieved by a function $u > 0$

Proofs

• The existence proof is done on the original formulation of the problem using the diamagnetic inequality

• $\psi(s) e^{ias} = v_1(s) + i v_2(s)$, solves

$$-v_j'' + \alpha v_j = (v_1^2 + v_2^2)^{\frac{p}{2}-1} v_j, \quad j = 1, 2$$

and the Wronskian $w = (v_1 v_2' - v_1' v_2)$ is constant so that $\psi(s) = 0$ is incompatible with the twisted boundary condition

• if $a^2(p+2) + \alpha(p-2) \leq 1$, then $\mu_{a,p}(\alpha) = a^2 + \alpha$ because

$$\begin{aligned} \|u'\|_{L^2(\mathbb{S}^1)}^2 + a^2 \|u^{-1}\|_{L^2(\mathbb{S}^1)}^{-2} + \alpha \|u\|_{L^2(\mathbb{S}^1)}^2 &= (1-4a^2) \|u'\|_{L^2(\mathbb{S}^1)}^2 + \alpha \|u\|_{L^2(\mathbb{S}^1)}^2 \\ &\quad + 4a^2 \left(\|u'\|_{L^2(\mathbb{S}^1)}^2 + \frac{1}{4} \|u^{-1}\|_{L^2(\mathbb{S}^1)}^2 \right) \end{aligned}$$

if $a^2(p+2) + \alpha(p-2) > 1$, the test function $u_\varepsilon := 1 + \varepsilon w_1$

$$\mathcal{Q}_{a,p,\alpha}[u_\varepsilon] = a^2 + \alpha + (1 - a^2(p+2) - \alpha(p-2)) \varepsilon^2 + o(\varepsilon^2)$$

proves the linear instability of the constants and $\mu_{a,p}(\alpha) < a^2 + \alpha$

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|_{L^2(\mathbb{S}^1)}^2 + a^2 \|u^{-1}\|_{L^2(\mathbb{S}^1)}^{-2} + \alpha \|u\|_{L^2(\mathbb{S}^1)}^2}{\|u\|_{L^p(\mathbb{S}^1)}^2},$$

$$\mu_{a,p}(\alpha) = \min_{u \in H^1(\mathbb{S}^1) \setminus \{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

$$\mathcal{Q}_{p,\alpha}[u] = \mathcal{Q}_{a=0,p,\alpha}[u], \quad \nu_p(\alpha) := \inf_{v \in H_0^1(\mathbb{S}^1) \setminus \{0\}} \mathcal{Q}_{p,\alpha}[v]$$

Proposition

$\forall p > 2, \alpha > -a^2$, we have $\mu_{a,p}(\alpha) < \mu_{1/2,p}(\alpha) \leq \nu_p(\alpha) = \mu_{1/2,p}(\alpha)$

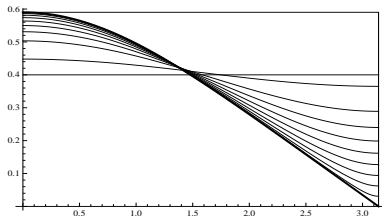


Figure: $p = 4, \alpha = 0, a = 0.40, 0.41, \dots, 0.49$; $u'' + u^{p-1} = 0$

A Keller-Lieb-Thirring inequality

Magnetic Schrödinger operator $H_a - \varphi = -\left(\frac{d}{ds} + i a\right)^2 \psi - \varphi$

• The function $\alpha \mapsto \mu_{a,p}(\alpha)$ is monotone increasing, concave, and therefore has an inverse, denoted by $\alpha_{a,p} : \mathbb{R}^+ \rightarrow (-a^2, +\infty)$, which is monotone increasing, and convex

Corollary

Let $p > 2$, $a \in [0, 1/2]$, $q = p/(p-2)$ and assume that φ is a non-negative function in $L^q(\mathbb{S}^1)$. Then

$$\lambda_1(H_a - \varphi) \geq -\alpha_{a,p}(\|\varphi\|_{L^q(\mathbb{S}^1)})$$

and $\alpha_{a,p}(\mu) = \mu - a^2$ iff $4a^2 + \mu(p-2) \leq 1$ (optimal φ is constant)

Equality is achieved

Bohm-Aharonov magnetic fields

On the two-dimensional Euclidean space \mathbb{R}^2 , let us introduce the polar coordinates $(r, \vartheta) \in [0, +\infty) \times \mathbb{S}^1$ of $\mathbf{x} \in \mathbb{R}^2$ and consider a magnetic potential \mathbf{a} in a transversal (Poincaré) gauge, or Poincaré gauge

$$(\mathbf{a}, \mathbf{e}_r) = 0 \quad \text{and} \quad (\mathbf{a}, \mathbf{e}_\vartheta) = a_\vartheta(r, \vartheta)$$

Magnetic Schrödinger energy

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{a})\Psi|^2 d\mathbf{x} = \int_0^{+\infty} \int_{-\pi}^{\pi} \left(|\partial_r \Psi|^2 + \frac{1}{r^2} |\partial_\vartheta \Psi + i r a_\vartheta \Psi|^2 \right) r d\vartheta dr$$

Bohm-Aharonov magnetic fields: $a_\vartheta(r, \vartheta) = a/r$ for some constant $a \in \mathbb{R}$ (a is the *magnetic flux*), with magnetic field $b = \text{curl } \mathbf{a}$

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{a})\Psi|^2 d\mathbf{x} \geq \tau \int_{\mathbb{R}^2} \frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} |\Psi|^2 d\mathbf{x} \quad \forall \varphi \in L^q(\mathbb{S}^1), \quad q \in (1, +\infty)$$

$$\implies \tau = \tau(a, \|\varphi\|_{L^q(\mathbb{S}^1)}) \quad ?$$

Hardy inequalities

(Hoffmann-Ostenhof, Laptev, 2015) proved Hardy's inequality

$$\int_{\mathbb{R}^d} |\nabla \Psi|^2 \, d\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} |\Psi|^2 \, d\mathbf{x}$$

where the constant τ depends on the value of $\|\varphi\|_{L^q(\mathbb{S}^{d-1})}$ and $d \geq 3$

Bohm-Aharonov vector potential in dimension $d = 2$

$$\mathbf{a}(\mathbf{x}) = a \left(\frac{x_2}{|\mathbf{x}|^2}, \frac{-x_1}{|\mathbf{x}|^2} \right), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \quad a \in \mathbb{R}$$

and recall the inequality (Laptev, Weidl, 1999)

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{a}) \Psi|^2 \, d\mathbf{x} \geq \min_{k \in \mathbb{Z}} (a - k)^2 \int_{\mathbb{R}^2} \frac{|\Psi|^2}{|\mathbf{x}|^2} \, d\mathbf{x}$$

A new Hardy inequality

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{a}) \Psi|^2 dx \geq \tau \int_{\mathbb{R}^2} \frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} |\Psi|^2 dx \quad \forall \varphi \in L^q(\mathbb{S}^1), \quad q \in (1, +\infty)$$

Corollary

Let $p > 2$, $a \in [0, 1/2]$, $q = p/(p-2)$ and assume that φ is a non-negative function in $L^q(\mathbb{S}^1)$. Then the inequality holds with $\tau > 0$ given by

$$\alpha_{a,p}(\tau \|\varphi\|_{L^q(\mathbb{S}^1)}) = 0$$

Moreover, $\tau = a^2 / \|\varphi\|_{L^q(\mathbb{S}^1)}$ if $4a^2 + \|\varphi\|_{L^q(\mathbb{S}^1)}(p-2) \leq 1$

For any $a \in (0, 1/2)$, by taking φ constant, small enough in order that $4a^2 + \|\varphi\|_{L^q(\mathbb{S}^1)}(p-2) \leq 1$, we recover the inequality

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{a}) \Psi|^2 dx \geq a^2 \int_{\mathbb{R}^2} \frac{|\Psi|^2}{|\mathbf{x}|^2} dx$$

Proofs (Keller-Lieb-Thirring inequality)

Hölder's inequality

$$\|\psi' + i a \psi\|_{L^2(\mathbb{S}^1)}^2 - \int_{-\pi}^{\pi} \varphi |\psi|^2 d\sigma \geq \|\psi' + i a \psi\|_{L^2(\mathbb{S}^1)}^2 - \mu \|\psi\|_{L^p(\mathbb{S}^1)}^2$$

where $\mu = \|\varphi\|_{L^q(\mathbb{S}^1)}$ and $\frac{1}{q} + \frac{2}{p} = 1$: choose $\mu_{a,p}(\alpha) = \mu$

$$\|\psi' + i a \psi\|_{L^2(\mathbb{S}^1)}^2 - \mu \|\psi\|_{L^p(\mathbb{S}^1)}^2 \geq -\alpha \|\psi\|_{L^2(\mathbb{S}^1)}^2$$

Proofs (Hardy inequality)

Let $\tau \geq 0$, $\mathbf{x} = (r, \vartheta) \in \mathbb{R}^2$ be polar coordinates in \mathbb{R}^2

$$\begin{aligned} & \int_{\mathbb{R}^2} \left(|(i\nabla + \mathbf{a})\Psi|^2 - \tau \frac{\varphi}{|x|^2} |\Psi|^2 \right) d\mathbf{x} \\ &= \int_0^\infty \int_{\mathbb{S}^1} \left(\underbrace{r |\partial_r \Psi|^2}_{\geq 0} + \frac{1}{r} |\partial_\vartheta \Psi + i a \Psi|^2 - \tau \frac{\varphi}{r} |\Psi|^2 \right) d\vartheta dr \\ &\geq \lambda_1 (H_a - \tau \varphi) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r} |\Psi|^2 d\vartheta dr \\ &\geq -\alpha_{a,p}(\tau \|\varphi\|_{L^q(\mathbb{S}^1)}) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r} |\Psi|^2 d\vartheta \end{aligned}$$

- If $\tau = 0$, then $\alpha_{a,p}(\tau \|\varphi\|_{L^q(\mathbb{S}^1)}) = \alpha_{a,p}(0) = -a^2$
- $\alpha_{a,p}(\tau \|\varphi\|_{L^q(\mathbb{S}^1)}) > 0$ for τ large
- $\implies \exists! \tau > 0$ such that $\alpha_{a,p}(\tau \|\varphi\|_{L^q(\mathbb{S}^1)}) = 0$

Comments

- ▷ The region $a^2(p+2) + \alpha(p-2) < 1$ is exactly the set where the constant functions are linearly stable critical points
- ▷ The proof of the *rigidity result* is based
 - neither on the *carré du champ* method, at least directly
 - nor on a Fourier representation of the operator as it was the case in earlier proofs ($p = +\infty$, or $p > 2$ and $\alpha = 0$)
- ▷ Magnetic rings: see (Bonnaillie-Noël, Hérau, Raymond, 2017)
- ▷ Deducing *Hardy's inequality* applied with *Bohm-Aharonov* magnetic fields from a *Keller-Lieb-Thirring inequality* is an extension of (Hoffmann-Ostenhof, Laptev, 2015) to the magnetic case
- ▷ Our results are not limited to the semi-classical regime

Magnetic interpolation inequalities in the Euclidean space

- ▷ Three interpolation inequalities and their dual forms
- ▷ Estimates in dimension $d = 2$ for constant magnetic fields
 - Lower estimates
 - Upper estimates and numerical results
 - A linear stability result (numerical) and an open question
- Warning: assumptions are not repeated
- Estimates are given only in the case $p > 2$ but similar estimates hold in the other cases

Magnetic Laplacian and spectral gap

In dimensions $d = 2$ and $d = 3$: the *magnetic Laplacian* is

$$-\Delta_{\mathbf{A}} \psi = -\Delta \psi - 2i \mathbf{A} \cdot \nabla \psi + |\mathbf{A}|^2 \psi - i (\operatorname{div} \mathbf{A}) \psi$$

where the magnetic potential (resp. field) is \mathbf{A} (resp. $\mathbf{B} = \operatorname{curl} \mathbf{A}$) and

$$H_{\mathbf{A}}^1(\mathbb{R}^d) := \{ \psi \in L^2(\mathbb{R}^d) : \nabla_{\mathbf{A}} \psi \in L^2(\mathbb{R}^d) \}, \quad \nabla_{\mathbf{A}} := \nabla + i \mathbf{A}$$

Spectral gap inequality

$$\|\nabla_{\mathbf{A}} \psi\|_2^2 \geq \Lambda[\mathbf{B}] \|\psi\|_2^2 \quad \forall \psi \in H_{\mathbf{A}}^1(\mathbb{R}^d) \quad (4)$$

- Λ depends only on $\mathbf{B} = \operatorname{curl} \mathbf{A}$
- Assumption: equality in (4) holds for some $\psi \in H_{\mathbf{A}}^1(\mathbb{R}^d)$
- If \mathbf{B} is a constant magnetic field, $\Lambda[\mathbf{B}] = |\mathbf{B}|$
- If $d = 2$, $\operatorname{spec}(-\Delta_{\mathbf{A}}) = \{(2j+1)|\mathbf{B}| : j \in \mathbb{N}\}$ is generated by the *Landau levels*. The *Lowest Landau Level* corresponds to $j = 0$

Magnetic interpolation inequalities

$$\|\nabla_{\mathbf{A}} \psi\|_2^2 + \alpha \|\psi\|_2^2 \geq \mu_{\mathbf{B}}(\alpha) \|\psi\|_p^2 \quad \forall \psi \in H_{\mathbf{A}}^1(\mathbb{R}^d) \quad (5)$$

for any $\alpha \in (-\Lambda[\mathbf{B}], +\infty)$ and any $p \in (2, 2^*)$,

$$\|\nabla_{\mathbf{A}} \psi\|_2^2 + \beta \|\psi\|_p^2 \geq \nu_{\mathbf{B}}(\beta) \|\psi\|_2^2 \quad \forall \psi \in H_{\mathbf{A}}^1(\mathbb{R}^d) \quad (6)$$

for any $\beta \in (0, +\infty)$ and any $p \in (1, 2)$

$$\|\nabla_{\mathbf{A}} \psi\|_2^2 \geq \gamma \int_{\mathbb{R}^d} |\psi|^2 \log \left(\frac{|\psi|^2}{\|\psi\|_2^2} \right) dx + \xi_{\mathbf{B}}(\gamma) \|\psi\|_2^2 \quad \forall \psi \in H_{\mathbf{A}}^1(\mathbb{R}^d) \quad (7)$$

(limit case corresponding to $p = 2$) for any $\gamma \in (0, +\infty)$

$$C_p := \begin{cases} \min_{u \in H^1(\mathbb{R}^d) \setminus \{0\}} \frac{\|\nabla u\|_2^2 + \|u\|_2^2}{\|u\|_p^2} & \text{if } p \in (2, 2^*) \\ \min_{u \in H^1(\mathbb{R}^d) \setminus \{0\}} \frac{\|\nabla u\|_2^2 + \|u\|_p^2}{\|u\|_2^2} & \text{if } p \in (1, 2) \end{cases}$$

$$\mu_{\mathbf{0}}(1) = C_p \text{ if } p \in (2, 2^*), \quad \nu_{\mathbf{0}}(1) = C_p \text{ if } p \in (1, 2)$$

$$\xi_{\mathbf{0}}(\gamma) = \gamma \log(\pi e^2 / \gamma) \text{ if } p = 2$$

Technical assumptions

$\mathbf{A} \in L_{\text{loc}}^\alpha(\mathbb{R}^d)$, $\alpha > 2$ if $d = 2$ or $\alpha = 3$ if $d = 3$ and

$$\lim_{\sigma \rightarrow +\infty} \sigma^{d-2} \int_{\mathbb{R}^d} |\mathbf{A}(x)|^2 e^{-\sigma|x|} dx = 0 \quad \text{if } p \in (2, 2^*)$$

$$\lim_{\sigma \rightarrow +\infty} \frac{\sigma^{\frac{d}{2}-1}}{\log \sigma} \int_{\mathbb{R}^d} |\mathbf{A}(x)|^2 e^{-\sigma|x|^2} dx = 0 \quad \text{if } p = 2$$

$$\lim_{\sigma \rightarrow +\infty} \sigma^{d-2} \int_{|x| < 1/\sigma} |\mathbf{A}(x)|^2 dx \quad \text{if } p \in (1, 2)$$

A statement

Theorem

$p \in (2, 2^*)$: $\mu_{\mathbf{B}}$ is monotone increasing on $(-\Lambda[\mathbf{B}], +\infty)$, concave and

$$\lim_{\alpha \rightarrow (-\Lambda[\mathbf{B}])_+} \mu_{\mathbf{B}}(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \rightarrow +\infty} \mu_{\mathbf{B}}(\alpha) \alpha^{\frac{d-2}{2} - \frac{d}{p}} = C_p$$

$p \in (1, 2)$: $\nu_{\mathbf{B}}$ is monotone increasing on $(0, +\infty)$, concave and

$$\lim_{\beta \rightarrow 0_+} \nu_{\mathbf{B}}(\beta) = \Lambda[\mathbf{B}] \quad \text{and} \quad \lim_{\beta \rightarrow +\infty} \nu_{\mathbf{B}}(\beta) \beta^{-\frac{2p}{2p+d(2-p)}} = C_p$$

$\xi_{\mathbf{B}}$ is continuous on $(0, +\infty)$, concave, $\xi_{\mathbf{B}}(0) = \Lambda[\mathbf{B}]$ and

$$\xi_{\mathbf{B}}(\gamma) = \frac{d}{2} \gamma \log\left(\frac{\pi e^2}{\gamma}\right) (1 + o(1)) \quad \text{as} \quad \gamma \rightarrow +\infty$$

Constant magnetic fields: equality is achieved

Nonconstant magnetic fields: only partial answers are known

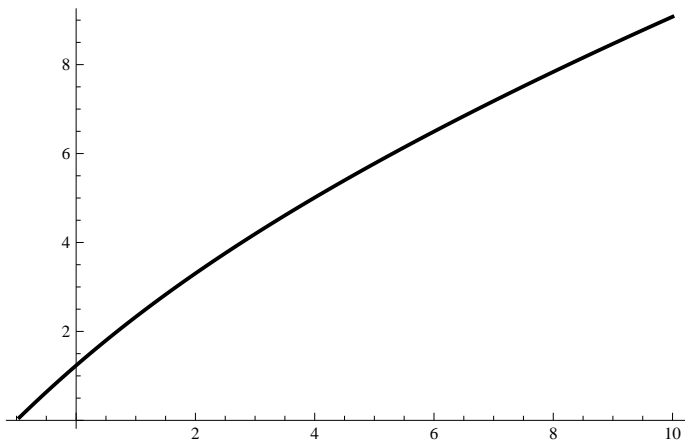


Figure: Case $d = 2$, $p = 3$, $B = 1$: plot of $\alpha \mapsto (2\pi)^{\frac{2}{p}-1} \mu_{\mathbf{B}}(\alpha)$

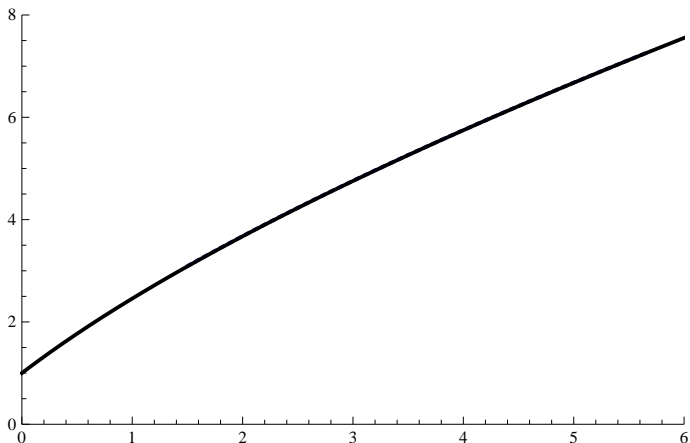


Figure: Case $d = 2$, $p = 1.4$, $B = 1$: plot of $\beta \mapsto \nu_{\mathbf{B}}(\beta)$. The horizontal axis is measured in units of $(2\pi)^{1-\frac{2}{p}} \beta$

Magnetic Keller-Lieb-Thirring inequalities

$\lambda_{\mathbf{A},V}$ is the principal eigenvalue of $-\Delta_{\mathbf{A}} + V$

$\alpha_{\mathbf{B}} : (0, +\infty) \rightarrow (-\Lambda, +\infty)$ is the inverse function of $\alpha \mapsto \mu_{\mathbf{B}}(\alpha)$

Corollary

- (i) For any $q = p/(p-2) \in (d/2, +\infty)$ and any potential $0 \geq V \in L^q(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},V} \geq -\alpha_{\mathbf{B}}(\|V\|_q)$$

$$\lim_{\mu \rightarrow 0^+} \alpha_{\mathbf{B}}(\mu) = \Lambda \text{ and } \lim_{\mu \rightarrow +\infty} \alpha_{\mathbf{B}}(\mu) \mu^{\frac{2(q+1)}{d-2-2q}} = -C_p^{\frac{2(q+1)}{d-2-2q}}$$

- (ii) For any $q = p/(2-p) \in (1, +\infty)$ and any $0 < W^{-1} \in L^q(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},W} \geq \nu_{\mathbf{B}}(\|W^{-1}\|_q^{-1})$$

- (iii) For any $\gamma > 0$ and any $W \geq 0$ s.t. $e^{-W/\gamma} \in L^1(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},W} \geq \xi_{\mathbf{B}}(\gamma) - \gamma \log \left(\int_{\mathbb{R}^d} e^{-W/\gamma} dx \right)$$

A more general duality result

Proposition

Let $d = 2$ or 3 . Let $\phi \in L^1_{\text{loc}}(\mathbb{R}^d)$ be an arbitrary potential

(i) If $q > d/2$, $p = \frac{2q}{q-1}$, we have

$$\lambda_{\mathbf{A},\phi} \geq -(\alpha_{\mathbf{B}} (\|(\lambda - \phi)\|_{q,+}) - \lambda)$$

(ii) If $q \in (1, +\infty)$, $p = \frac{2q}{q+1}$, we have

$$\lambda_{\mathbf{A},\phi} \geq \lambda + \nu_{\mathbf{B}} (\|(\phi - \lambda)^{-1}\|_q^{-1})$$

These estimates hold for any $\lambda \in \mathbb{R}$ such that all above norms are well defined, with the additional condition that $\phi \geq \lambda$ a.e. in Case (ii)

Preliminaries: interpolation without magnetic field

Assume that $p > 2$ and let C_p denote the best constant in

$$\|\nabla u\|_2^2 + \|u\|_2^2 \geq C_p \|u\|_p^2 \quad \forall u \in H^1(\mathbb{R}^d)$$

By scaling, if we test the inequality by $u(\cdot/\lambda)$, we find that

$$\|\nabla u\|_2^2 + \lambda^2 \|u\|_2^2 \geq C_p \lambda^{2-d(1-\frac{2}{p})} \|u\|_p^2 \quad \forall u \in H^1(\mathbb{R}^d) \quad \forall \lambda > 0$$

An optimization on $\lambda > 0$ shows that the best constant in the scale-invariant inequality

$$\|\nabla u\|_2^{d(1-\frac{2}{p})} \|u\|_2^{2-d(1-\frac{2}{p})} \geq S_p \|u\|_p^2 \quad \forall u \in H^1(\mathbb{R}^d)$$

is given by

$$S_p = \frac{1}{2^p} (2p - d(p-2))^{1-d\frac{p-2}{2p}} (d(p-2))^{\frac{d(p-2)}{2p}} C_p$$

... and with magnetic field

Proposition

Let $d = 2$ or 3 . For any $p \in (2, +\infty)$, any $\alpha > -\Lambda = -\Lambda[\mathbf{B}] < 0$

$$\mu_{\mathbf{B}}(\alpha) \geq \mu_{\text{interp}}(\alpha) := \begin{cases} S_p (\alpha + \Lambda) \Lambda^{-d \frac{p-2}{2p}} & \text{if } \alpha \in \left[-\Lambda, \frac{\Lambda(2p-d(p-2))}{d(p-2)} \right] \\ C_p \alpha^{1-d \frac{p-2}{2p}} & \text{if } \alpha \geq \frac{\Lambda(2p-d(p-2))}{d(p-2)} \end{cases}$$

Let $t \in [0, 1]$. From the diamagnetic inequality $\|\nabla|\psi|\|_2 \leq \|\nabla_{\mathbf{A}}\psi\|_2$ and from the inequality with $\lambda = \frac{\alpha + \Lambda t}{1-t}$, we deduce that

$$\begin{aligned} \|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha \|\psi\|_2^2 &\geq t (\|\nabla_{\mathbf{A}}\psi\|_2^2 - \Lambda \|\psi\|_2^2) \\ &\quad + (1-t) \left(\|\nabla|\psi|\|_2 + \frac{\alpha + \Lambda t}{1-t} \|\psi\|_2 \right)^2 \\ &\geq C_p (1-t)^{\frac{d(p-2)}{2p}} (\alpha + t\Lambda)^{1-d \frac{p-2}{2p}} \|\psi\|_2^2 \end{aligned}$$

and optimize on $t \in [\max\{0, -\alpha/\Lambda\}, 1]$

Lower estimates: $d = 2$, constant magnetic field

Assume that $\mathbf{B} = (0, B)$ is constant, $d = 2$ and choose

$$\mathbf{A}_1 = \frac{B}{2}x_2, \quad \mathbf{A}_2 = -\frac{B}{2}x_1 \quad \forall x = (x_1, x_2) \in \mathbb{R}^2$$

Proposition

(Loss, Thaller, 1997) Consider a constant magnetic field with field strength B in two dimensions. For every $c \in [0, 1]$, we have

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 dx \geq (1 - c^2) \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + cB \int_{\mathbb{R}^2} \psi^2 dx$$

and equality holds with $\psi = u e^{iS}$ and $u > 0$ if and only if

$$(-\partial_2 u^2, \partial_1 u^2) = \frac{2u^2}{c} (\mathbf{A} + \nabla S)$$

... a computation ($d = 2$, constant magnetic field)

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 dx &= \int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} |\mathbf{A} + \nabla S|^2 u^2 dx \\ &= (1 - c^2) \int_{\mathbb{R}^2} |\nabla u|^2 dx + \underbrace{\int_{\mathbb{R}^2} (c^2 |\nabla u|^2 + |\mathbf{A} + \nabla S|^2 u^2) dx}_{\geq \int_{\mathbb{R}^2} 2c |\nabla u| |\mathbf{A} + \nabla S| u dx} \end{aligned}$$

with equality only if $c |\nabla u| = |\mathbf{A} + \nabla S| u$

$$2 |\nabla u| |\mathbf{A} + \nabla S| u = |\nabla u^2| |\mathbf{A} + \nabla S| \geq (\nabla u^2)^\perp \cdot (\mathbf{A} + \nabla S)$$

where $(\nabla u^2)^\perp := (-\partial_2 u^2, \partial_1 u^2)$

Equality case: $(-\partial_2 u^2, \partial_1 u^2) = \gamma (\mathbf{A} + \nabla S)$ for $\gamma = 2u^2/c$

Integration by parts yields

$$\int_{\mathbb{R}^2} (c^2 |\nabla u|^2 + |\mathbf{A} + \nabla S|^2 u^2) dx \geq B c \int_{\mathbb{R}^2} u^2 dx$$

Lower estimate ($d = 2$, constant magnetic field): a result

Proposition

Consider a constant magnetic field with field strength B in two dimensions. Given any $p \in (2, +\infty)$, and any $\alpha > -B$, we have

$$\mu_{\mathbf{B}}(\alpha) \geq C_p (1 - c^2)^{1 - \frac{2}{p}} (\alpha + cB)^{\frac{2}{p}} =: \mu_{\text{LT}}(\alpha)$$

with

$$c = c(p, \eta) = \frac{\sqrt{\eta^2 + p - 1} - \eta}{p - 1} = \frac{1}{\eta + \sqrt{\eta^2 + p - 1}} \in (0, 1)$$

and $\eta = \alpha(p - 2)/(2B)$

Upper estimate (1): $d = 2$, constant magnetic field

For every integer $k \in \mathbb{N}$ we introduce the special symmetry class

$$\psi(x) = \left(\frac{x_2 + i x_1}{|x|} \right)^k v(|x|) \quad \forall x = (x_1, x_2) \in \mathbb{R}^2 \quad (\mathcal{C}_k)$$

(Esteban, Lions, 1989): if $\psi \in \mathcal{C}_k$, then

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 dx = \int_0^{+\infty} |v'|^2 r dr + \int_0^{+\infty} \left(\frac{k}{r} - \frac{B r}{2} \right)^2 |v|^2 r dr$$

and optimality is achieved in \mathcal{C}_k

Test function $v_\sigma(r) = e^{-r^2/(2\sigma)}$: an optimization on $\sigma > 0$ provides an explicit expression of $\mu_{\text{Gauss}}(\alpha)$ such that

Proposition

If $p > 2$, then

$$\mu_{\mathbf{B}}(\alpha) \leq \mu_{\text{Gauss}}(\alpha) \quad \forall \alpha > -\Lambda[\mathbf{B}]$$

This estimate is not optimal because v_σ does not solve the Euler-Lagrange equations

Upper estimate (2): $d = 2$, constant magnetic field

A more numerical point of view. The Euler-Lagrange equation in \mathcal{C}_0 is

$$-v'' - \frac{v'}{r} + \left(\frac{B^2}{4} r^2 + \alpha\right) v = \mu_{\text{EL}}(\alpha) \left(\int_0^{+\infty} |v|^p r dr\right)^{\frac{2}{p}-1} |v|^{p-2} v$$

We can restrict the problem to positive solutions such that

$$\mu_{\text{EL}}(\alpha) = \left(\int_0^{+\infty} |v|^p r dr\right)^{1-\frac{2}{p}}$$

and then we have to solve the reduced problem

$$-v'' - \frac{v'}{r} + \left(\frac{B^2}{4} r^2 + \alpha\right) v = |v|^{p-2} v$$

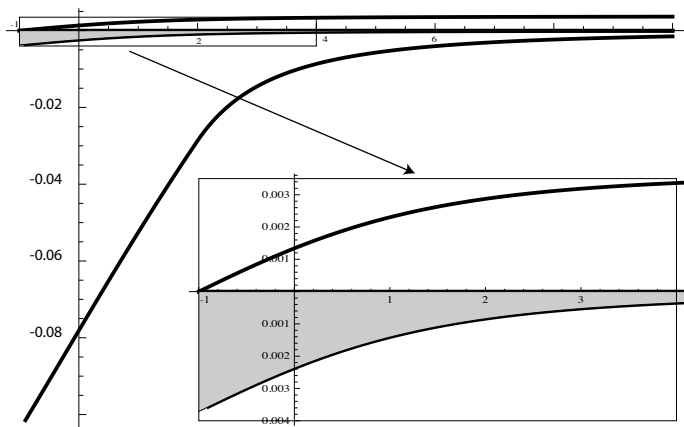


Figure: Case $d = 2$, $p = 3$, $B = 1$: comparison of the upper estimates $\alpha \mapsto \mu_{\text{Gauss}}(\alpha)$ and $\alpha \mapsto \mu_{\text{EL}}(\alpha)$ with the lower estimates $\alpha \mapsto \mu_{\text{interp}}(\alpha)$ and $\alpha \mapsto \mu_{\text{LT}}(\alpha)$

Plots represent the curves $\log_{10}(\mu_{\text{Gauss}}/\mu_{\text{EL}})$, $\log_{10}(\mu_{\text{LT}}/\mu_{\text{EL}})$ and $\log_{10}(\mu_{\text{interp}}/\mu_{\text{EL}})$ so that $\alpha \mapsto \mu_{\text{EL}}(\alpha)$ corresponds to a straight line at level 0. The exact value associated with μ_{B} lies in the grey area

Asymptotics (1): *Lowest Landau Level*

Proposition

Let $d = 2$ and consider a constant magnetic field with field strength B . If ψ_α is a minimizer for $\mu_{\mathbf{B}}(\alpha)$ such that $\|\psi_\alpha\|_p = 1$, then there exists a non trivial $\varphi_\alpha \in \text{LLL}$ such that

$$\lim_{\alpha \rightarrow (-B)_+} \|\psi_\alpha - \varphi_\alpha\|_{\mathbf{H}_{\mathbf{A}}^1(\mathbb{R}^2)} = 0$$

Let $\psi_\alpha \in \mathbf{H}_{\mathbf{A}}^1(\mathbb{R}^2)$ be an optimal function for (5) such that $\|\psi_\alpha\|_p = 1$ and let us decompose it as $\psi_\alpha = \varphi_\alpha + \chi_\alpha$, where $\varphi_\alpha \in \text{LLL}$ and χ_α is in the orthogonal of LLL

$$\mu_{\mathbf{B}}(\alpha) \geq (\alpha + B) \|\varphi_\alpha\|_2^2 + (\alpha + 3B) \|\chi_\alpha\|_2^2 \geq (\alpha + 3B) \|\chi_\alpha\|_2^2 \sim 2B \|\chi_\alpha\|_2^2$$

as $\alpha \rightarrow (-B)_+$ because $\|\nabla \chi_\alpha\|_2^2 \geq 3B \|\chi_\alpha\|_2^2$

Since $\lim_{\alpha \rightarrow (-B)_+} \mu_{\mathbf{B}}(\alpha) = 0$, $\lim_{\alpha \rightarrow (-B)_+} \|\chi_\alpha\|_2 = 0$ and

$$\mu_{\mathbf{B}}(\alpha) = (\alpha + B) \|\varphi_\alpha\|_2^2 + \|\nabla_{\mathbf{A}} \chi_\alpha\|_2^2 + \alpha \|\chi_\alpha\|_2^2 \geq \frac{2}{3} \|\nabla_{\mathbf{A}} \chi_\alpha\|_2^2$$

concludes the proof

Asymptotics (2): *semi-classical regime*

Let us consider the small magnetic field regime. We assume that the magnetic potential is given by

$$\mathbf{A}_1 = \frac{B}{2}x_2, \quad \mathbf{A}_2 = -\frac{B}{2}x_1 \quad \forall x = (x_1, x_2) \in \mathbb{R}^2$$

if $d = 2$. In dimension $d = 3$, we choose $\mathbf{A} = \frac{B}{2}(-x_2, x_1, 0)$ and observe that the constant magnetic field is $\mathbf{B} = (0, 0, B)$, while the spectral gap in (4) is $\Lambda[\mathbf{B}] = B$.

Proposition

Let $d = 2$ or 3 and consider a constant magnetic field \mathbf{B} of intensity B with magnetic potential \mathbf{A}

For any $p \in (2, 2^*)$ and any fixed α and $\mu > 0$, we have

$$\lim_{\varepsilon \rightarrow 0_+} \mu_{\varepsilon \mathbf{B}}(\alpha) = C_p \alpha^{\frac{d}{p} - \frac{d-2}{2}}$$

Consider any function $\psi \in H_{\mathbf{A}}^1(\mathbb{R}^d)$ and let $\psi(x) = \chi(\sqrt{\varepsilon}x)$, $\sqrt{\varepsilon} \mathbf{A}(x/\sqrt{\varepsilon}) = \mathbf{A}(x)$ with our conventions on \mathbf{A} .

Numerical stability of radial optimal functions

Let us denote by ψ_0 an optimal function in (\mathcal{C}_0) such that

$$-\psi_0'' - \frac{\psi_0'}{r} + \left(\frac{B^2}{4} r^2 + \alpha \right) \psi_0 = |\psi_0|^{p-2} \psi_0$$

and consider the test function

$$\psi_\varepsilon = \psi_0 + \varepsilon e^{i\theta} v$$

where $v = v(r)$ and $e^{i\theta} = (x_1 + i x_2)/r$

As $\varepsilon \rightarrow 0_+$, the leading order term is

$$2\pi \left[\int_{\mathbb{R}^2} |v'|^2 dx + \int_{\mathbb{R}^2} \left(\left(\frac{1}{r} - \frac{Br}{2} \right)^2 + \alpha \right) |v|^2 dx - \frac{p}{2} \int_0^{+\infty} |\psi_0|^{p-2} v^2 r dr \right] \varepsilon^2$$

and we have to solve the eigenvalue problem

$$-v'' - \frac{v'}{r} + \left(\left(\frac{1}{r} - \frac{Br}{2} \right)^2 + \alpha \right) v - \frac{p}{2} |\psi_0|^{p-2} v = \mu v$$

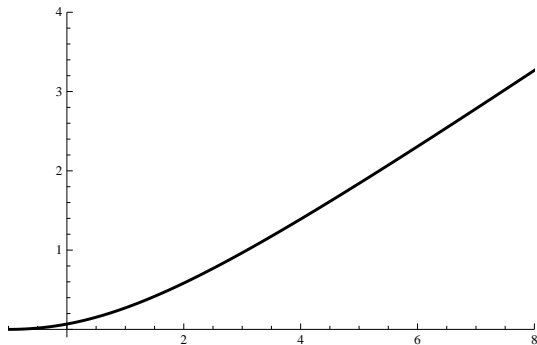


Figure: Case $p = 3$ and $B = 1$: plot of the eigenvalue μ as a function of α . A careful investigation shows that μ is always positive, including in the limiting case as $\alpha \rightarrow (-B)_+$, thus proving the numerical stability of the optimal function in \mathcal{C}_0 with respect to perturbations in \mathcal{C}_1 .

An open question of symmetry

• (Bonheure, Nys, Van Schaftingen, 2016) for a fixed $\alpha > 0$ and for \mathbf{B} small enough, the optimal functions are radially symmetric functions, *i.e.*, belong to \mathcal{C}_0

This regime is equivalent to the regime as $\alpha \rightarrow +\infty$ for a given \mathbf{B} , at least if the magnetic field is constant

• Numerically our upper and lower bounds are (in dimension $d = 2$, for a constant magnetic field) numerically extremely close

• The optimal function in \mathcal{C}_0 with respect to perturbations in \mathcal{C}_1

Prove that the optimality case is achieved among radial function if $d = 2$ and \mathbf{B} is a constant magnetic field

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Thank you for your attention !