Eigenvalue inequalities for the Laplacian with mixed boundary conditions

Jonathan Rohleder

in collaboration with Vladimir Lotoreichik

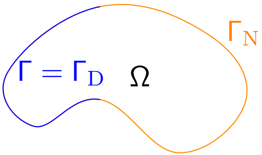
Eigenvalues and Inequalities, Institut Mittag-Leffler 2018

Mixed Laplacian eigenvalues

$\Omega \subset \mathbb{R}^d$, $d \geq 2$, bounded with sufficiently regular boundary $\partial\Omega$

Mixed Laplacian eigenvalues

$\Omega \subset \mathbb{R}^d$, $d \geq 2$, bounded with sufficiently regular boundary $\partial\Omega$

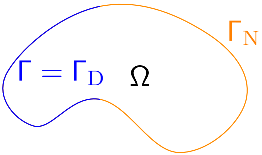


Mixed / Zaremba EV problem

$$
\begin{aligned}
-\Delta u &= \lambda u && \text{in } \Omega, \\
u &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\
\nu \cdot \nabla u &= 0 && \text{on } \Gamma_{\mathrm{N}}.
\end{aligned}
$$

Mixed Laplacian eigenvalues

$\Omega \subset \mathbb{R}^d$, $d \geq 2$, bounded with sufficiently regular boundary $\partial\Omega$



Mixed / Zaremba EV problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \nu \cdot \nabla u &= 0 && \text{on } \Gamma_N. \end{aligned}$$

Eigenvalues: $0 \leq \lambda_1^\Gamma < \lambda_2^\Gamma \leq \lambda_3^\Gamma \leq \cdots \to +\infty$

Comparison of Dirichlet and Neumann eigenvalues

Comparison of Dirichlet and Neumann eigenvalues

$\mu_k \leq \lambda_k$ (trivial)

Comparison of Dirichlet and Neumann eigenvalues

$\mu_k \leq \lambda_k$ (trivial)

$\mu_2 < \lambda_1$ for $d = 2$ (Pólya '52)

Comparison of Dirichlet and Neumann eigenvalues

$$\mu_k \leq \lambda_k \quad \text{(trivial)}$$

$$\mu_2 < \lambda_1 \quad \text{for } d = 2 \text{ (Pólya '52)}$$

$$\mu_{k+2} < \lambda_k \quad \text{for } d = 2,\ \Omega \text{ convex (Payne '55)}$$

Comparison of Dirichlet and Neumann eigenvalues

$$\mu_k \leq \lambda_k \quad \text{(trivial)}$$

$$\mu_2 < \lambda_1 \quad \text{for } d = 2 \text{ (Pólya '52)}$$

$$\mu_{k+2} < \lambda_k \quad \text{for } d = 2, \Omega \text{ convex (Payne '55)}$$

$$\mu_{k+d} \leq \lambda_k \quad \text{for } \Omega \text{ convex (Levine, Weinberger '86)}$$

History

Comparison of Dirichlet and Neumann eigenvalues

$$
\begin{aligned}
\mu_k &\le \lambda_k && \text{(trivial)}\\
\mu_2 &< \lambda_1 && \text{for } d = 2 \text{ (Pólya '52)}\\
\mu_{k+2} &< \lambda_k && \text{for } d = 2,\ \Omega \text{ convex (Payne '55)}\\
\mu_{k+d} &\le \lambda_k && \text{for } \Omega \text{ convex (Levine, Weinberger '86)}\\
\mu_{k+1} &\le \lambda_k && \text{for } C^1\text{-smooth } \Omega \text{ (Friedlander '91)}
\end{aligned}
$$

Comparison of Dirichlet and Neumann eigenvalues

$$\mu_k \le \lambda_k \quad \text{(trivial)}$$

$$\mu_2 < \lambda_1 \quad \text{for } d = 2 \text{ (Pólya '52)}$$

$$\mu_{k+2} < \lambda_k \quad \text{for } d = 2,\ \Omega \text{ convex (Payne '55)}$$

$$\mu_{k+d} \le \lambda_k \quad \text{for } \Omega \text{ convex (Levine, Weinberger '86)}$$

$$\mu_{k+1} \le \lambda_k \quad \text{for } C^1\text{-smooth } \Omega \text{ (Friedlander '91)}$$

$$\mu_{k+1} < \lambda_k \quad \text{for } \Omega \text{ Lipschitz (Filonov '04)}$$

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \text{span}\,\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \leq \lambda_k\}$,

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \text{span}\,\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \leq \lambda_k\}$, $\omega \in \mathbb{R}^n$ with $|\omega|^2 = \lambda_k$:

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \text{span}\,\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \leq \lambda_k\}$, $\omega \in \mathbb{R}^n$ with $|\omega|^2 = \lambda_k$:

$$\int_\Omega \left|\nabla(u + ce^{i\omega x})\right|^2 dx$$

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \operatorname{span}\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \leq \lambda_k\}$, $\omega \in \mathbb{R}^n$ with $|\omega|^2 = \lambda_k$:

$$\int_\Omega \left|\nabla(u + ce^{i\omega x})\right|^2 dx = \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\,\mathrm{Re} \int_\Omega \nabla u \cdot \nabla \overline{c} e^{-i\omega x} \; dx$$

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \operatorname{span}\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \leq \lambda_k\}$, $\omega \in \mathbb{R}^n$ with $|\omega|^2 = \lambda_k$:

$$\int_\Omega \left|\nabla(u + ce^{i\omega x})\right|^2 dx = \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\,\mathrm{Re} \int_\Omega u\left(-\Delta \bar{c} e^{-i\omega x}\right) dx$$

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \operatorname{span}\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \le \lambda_k\}$, $\omega \in \mathbb{R}^n$ with $|\omega|^2 = \lambda_k$:

$$\int_\Omega \left|\nabla(u + ce^{i\omega x})\right|^2 dx = \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\,\mathrm{Re} \int_\Omega u\left(-\Delta \bar{c} e^{-i\omega x}\right) dx$$

$$= \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\lambda_k \,\mathrm{Re} \int_\Omega u \bar{c} e^{-i\omega x} dx$$

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \operatorname{span}\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \leq \lambda_k\}$, $\omega \in \mathbb{R}^n$ with $|\omega|^2 = \lambda_k$:

$$\begin{aligned}
\int_\Omega \left|\nabla(u + ce^{i\omega x})\right|^2 dx &= \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\,\mathrm{Re} \int_\Omega u\left(-\Delta \bar{c} e^{-i\omega x}\right) dx \\
&= \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\lambda_k\,\mathrm{Re} \int_\Omega u\bar{c}e^{-i\omega x} dx \\
&\leq \lambda_k \int_\Omega \left(|u|^2 + |c|^2\right) dx + 2\lambda_k\,\mathrm{Re} \int_\Omega u\bar{c}e^{-i\omega x} dx
\end{aligned}$$

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \operatorname{span}\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \leq \lambda_k\}$, $\omega \in \mathbb{R}^n$ with $|\omega|^2 = \lambda_k$:

$$\begin{aligned}
\int_\Omega \left|\nabla(u + ce^{i\omega x})\right|^2 dx &= \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\operatorname{Re} \int_\Omega u\left(-\Delta \bar{c} e^{-i\omega x}\right) dx \\
&= \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\lambda_k \operatorname{Re} \int_\Omega u\bar{c}e^{-i\omega x} dx \\
&\leq \lambda_k \int_\Omega \left(|u|^2 + |c|^2\right) dx + 2\lambda_k \operatorname{Re} \int_\Omega u\bar{c}e^{-i\omega x} dx \\
&= \lambda_k \int_\Omega \left|u + ce^{i\omega x}\right|^2 dx
\end{aligned}$$

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \text{span}\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \le \lambda_k\}$, $\omega \in \mathbb{R}^n$ with $|\omega|^2 = \lambda_k$:

$$\begin{aligned}
\int_\Omega \left|\nabla(u + ce^{i\omega x})\right|^2 dx &= \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\,\mathrm{Re} \int_\Omega u\left(-\Delta \bar{c} e^{-i\omega x}\right) dx \\
&= \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\lambda_k\,\mathrm{Re} \int_\Omega u\bar{c}e^{-i\omega x} dx \\
&\le \lambda_k \int_\Omega \left(|u|^2 + |c|^2\right) dx + 2\lambda_k\,\mathrm{Re} \int_\Omega u\bar{c}e^{-i\omega x} dx \\
&= \lambda_k \int_\Omega \left|u + ce^{i\omega x}\right|^2 dx
\end{aligned}$$

$\Rightarrow \dfrac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega |w|^2 dx} \le \lambda_k$ $\forall w$ in a $(k+1)$-dim. subspace of $H^1(\Omega)$

Friedlander '91, Filonov '04

$$\mu_{k+1} < \lambda_k \quad \forall k \in \mathbb{N}$$

Proof: $u \in \operatorname{span}\{\ker(-\Delta_{\mathrm{D}} - \lambda) : \lambda \le \lambda_k\}$, $\omega \in \mathbb{R}^n$ with $|\omega|^2 = \lambda_k$:

$$\begin{aligned}
\int_\Omega \left|\nabla(u + ce^{i\omega x})\right|^2 dx &= \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\,\mathrm{Re} \int_\Omega u\left(-\Delta \bar{c} e^{-i\omega x}\right) dx \\
&= \int_\Omega \left(|\nabla u|^2 + |c\omega|^2\right) dx + 2\lambda_k\,\mathrm{Re} \int_\Omega u \bar{c} e^{-i\omega x} dx \\
&\le \lambda_k \int_\Omega \left(|u|^2 + |c|^2\right) dx + 2\lambda_k\,\mathrm{Re} \int_\Omega u \bar{c} e^{-i\omega x} dx \\
&= \lambda_k \int_\Omega \left|u + ce^{i\omega x}\right|^2 dx
\end{aligned}$$

$\Rightarrow \dfrac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega |w|^2 dx} \le \lambda_k$ $\forall w$ in a $(k+1)$-dim. subspace of $H^1(\Omega) \Rightarrow \mu_{k+1} \le \lambda_k$ $\square$

Our aim: Inequalities between N / D and mixed eigenvalues

Our aim: Inequalities between N / D and mixed eigenvalues

Trivial: $\mu_k \leq \lambda_k^\Gamma \leq \lambda_k$

Our aim: Inequalities between N / D and mixed eigenvalues

Trivial: $\mu_k \leq \lambda_k^\Gamma \leq \lambda_k$

What about $\mu_{k+?} \leq \lambda_k^\Gamma$?

Our aim: Inequalities between N / D and mixed eigenvalues

Trivial: $\mu_k \leq \lambda_k^\Gamma \leq \lambda_k$

What about $\mu_{k+?} \leq \lambda_k^\Gamma$?

What about $\lambda_{k+?}^\Gamma \leq \lambda_k$?

Tangential space at $x \in \partial\Omega$:

$$\mathcal{T}_x = \left\{ \tau \in \mathbb{R}^d : \tau \cdot \nu(x) = 0 \right\} \subset \mathbb{R}^d.$$

Tangential space at $x \in \partial\Omega$:

$$\mathcal{T}_x = \left\{ \tau \in \mathbb{R}^d : \tau \cdot \nu(x) = 0 \right\} \subset \mathbb{R}^d.$$

For $\Sigma \subset \partial\Omega$: $\mathcal{S}(\Sigma) := \bigcap_{x \in \Sigma} \mathcal{T}_x$.

Mixed and Neumann eigenvalues

Tangential space at $x \in \partial\Omega$:

$$\mathcal{T}_x = \left\{ \tau \in \mathbb{R}^d : \tau \cdot \nu(x) = 0 \right\} \subset \mathbb{R}^d.$$

For $\Sigma \subset \partial\Omega$: $\mathcal{S}(\Sigma) := \bigcap_{x \in \Sigma} \mathcal{T}_x$.

Theorem [Lotoreichik, R. '17]

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, bounded, Lipschitz. If $\dim \mathcal{S}(\Gamma_{\mathrm{N}}) \geq 1$ then

$$\mu_{k+1} \leq \lambda_k^\Gamma \quad \text{for all } k \in \mathbb{N}.$$

Mixed and Neumann eigenvalues

Tangential space at $x \in \partial\Omega$:

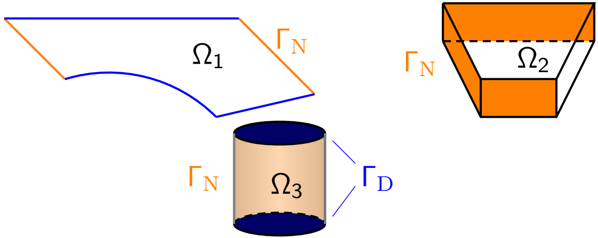
$$\mathcal{T}_x = \left\{ \tau \in \mathbb{R}^d : \tau \cdot \nu(x) = 0 \right\} \subset \mathbb{R}^d.$$

For $\Sigma \subset \partial\Omega$: $\mathcal{S}(\Sigma) := \bigcap_{x \in \Sigma} \mathcal{T}_x$.

Theorem [Lotoreichik, R. '17]

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, bounded, Lipschitz. If $\dim \mathcal{S}(\Gamma_{\mathrm{N}}) \geq 1$ then

$$\mu_{k+1} \leq \lambda_k^{\Gamma} \quad \text{for all } k \in \mathbb{N}.$$



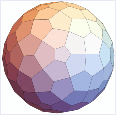
Mixed and Dirichlet eigenvalues

Mixed and Dirichlet eigenvalues

Ω polyhedral

$\Leftrightarrow$

$\partial\Omega$ piecewise $(d-1)$-dim. hyperplane

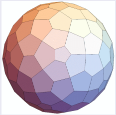


Mixed and Dirichlet eigenvalues

Ω polyhedral

$\Leftrightarrow$

$\partial\Omega$ piecewise $(d-1)$-dim. hyperplane



Theorem [Lotoreichik, R. '17]

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, convex and polyhedral. Then

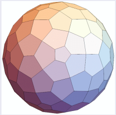
$$\lambda^{\Gamma}_{k+\dim \mathcal{S}(\Gamma_{\mathrm{D}})} \leq \lambda_k \quad \text{for all } k \in \mathbb{N}.$$

Mixed and Dirichlet eigenvalues

Ω polyhedral

$\Leftrightarrow$

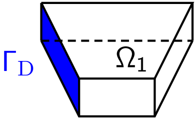
$\partial\Omega$ piecewise $(d-1)$-dim. hyperplane



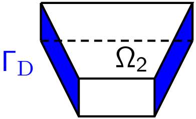
Theorem [Lotoreichik, R. '17]

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, convex and polyhedral. Then

$$\lambda^{\Gamma}_{k+\dim \mathcal{S}(\Gamma_{\mathrm{D}})} \leq \lambda_k \quad \text{for all } k \in \mathbb{N}.$$



$$\lambda^{\Gamma}_{k+2} \leq \lambda_k$$



$$\lambda^{\Gamma}_{k+1} \leq \lambda_k$$

Proofs

Variational: plug suitable test functions into the Rayleigh quotient.

Proofs

Variational: plug suitable test functions into the Rayleigh quotient.

Comparison with Neumann

- Use $x \mapsto e^{i\omega x}$ with $\omega \in \mathbb{R}^d$, $|\omega|^2 = \lambda_k^\Gamma$ (Filonov '04)
- Extra requirement: ω tangential to all of Γ_{N}

Proofs

Variational: plug suitable test functions into the Rayleigh quotient.

Comparison with Neumann

- Use $x \mapsto e^{i\omega x}$ with $\omega \in \mathbb{R}^d$, $|\omega|^2 = \lambda_k^\Gamma$ (Filonov '04)
- Extra requirement: ω tangential to all of Γ_N

Comparison with Dirichlet

- Use partial derivatives of Dirichlet eigenfunctions (Levine, Weinberger '86)
- $\tau_j \cdot \nabla u$ with $\{\tau_1, \dots, \tau_\ell\}$ basis of $\mathcal{S}(\Gamma_\mathrm{D})$ belongs to form domain
- Grisvard's integration by parts for convex polyhedra

Some applications of mixed eigenvalue problems

- Analysis of nodal domains for Neumann eigenfunctions

Some applications of mixed eigenvalue problems

- Analysis of nodal domains for Neumann eigenfunctions
- Construction of isospectral domains answering negatively to 'Can one hear the shape of the drum?' (Sunada '85, Gordon, Webb, Wolpert '92)

Some applications of mixed eigenvalue problems

- Analysis of nodal domains for Neumann eigenfunctions
- Construction of isospectral domains answering negatively to 'Can one hear the shape of the drum?' (Sunada '85, Gordon, Webb, Wolpert '92)
- Inequalities for mixed eigenvalues appear in approaches to the

hot spots conjecture (J. Rauch)

Hottest point on insulated plate Ω moves towards $\partial\Omega$ as $t \nearrow \infty$.

Reformulation: Global extrema for the 2^{nd} Neumann eigenfunction are attained on the boundary.

Some applications of mixed eigenvalue problems

- Analysis of nodal domains for Neumann eigenfunctions
- Construction of isospectral domains answering negatively to 'Can one hear the shape of the drum?' (Sunada '85, Gordon, Webb, Wolpert '92)
- Inequalities for mixed eigenvalues appear in approaches to the

hot spots conjecture (J. Rauch)

Hottest point on insulated plate Ω moves towards $\partial\Omega$ as $t \nearrow \infty$.

Reformulation: Global extrema for the 2^{nd} Neumann eigenfunction are attained on the boundary.

Judge, Mondal '18 (preprint): True for all triangles!

Some applications of mixed eigenvalue problems

- Analysis of nodal domains for Neumann eigenfunctions
- Construction of isospectral domains answering negatively to 'Can one hear the shape of the drum?' (Sunada '85, Gordon, Webb, Wolpert '92)
- Inequalities for mixed eigenvalues appear in approaches to the

hot spots conjecture (J. Rauch)

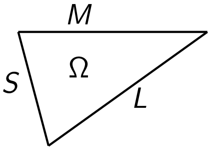
Hottest point on insulated plate Ω moves towards $\partial\Omega$ as $t \nearrow \infty$.

Reformulation: Global extrema for the 2^{nd} Neumann eigenfunction are attained on the boundary.

Judge, Mondal '18 (preprint): True for all triangles!
Uses our Neumann comparison theorem.

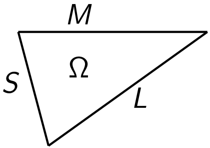
Inequalities for the first eigenvalue

Ω triangle with side lengths $S \leq M \leq L$.



Inequalities for the first eigenvalue

Ω triangle with side lengths $S \leq M \leq L$.



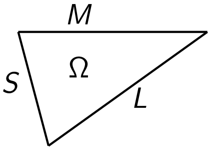
Theorem [Siudeja '15]

For any **right triangle** with smallest angle in $(\frac{\pi}{6}, \frac{\pi}{4})$

$$\lambda_1^S < \lambda_1^M < \lambda_1^L < \lambda_1^{M \cup S}.$$

Inequalities for the first eigenvalue

Ω triangle with side lengths $S \leq M \leq L$.



Theorem [Siudeja '15]

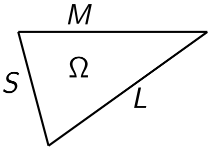
For any **right triangle** with smallest angle in $(\frac{\pi}{6}, \frac{\pi}{4})$

$$\lambda_1^S < \lambda_1^M < \lambda_1^L < \lambda_1^{M \cup S}.$$

Conjecture: This is true for **any triangle** if $S < M < L$.

Inequalities for the first eigenvalue

Ω triangle with side lengths $S \leq M \leq L$.



Theorem [SIUDEJA '15]

For any **right triangle** with smallest angle in $(\frac{\pi}{6}, \frac{\pi}{4})$

$$\lambda_1^S < \lambda_1^M < \lambda_1^L < \lambda_1^{M \cup S}.$$

Conjecture: This is true for **any triangle** if $S < M < L$.

Theorem [R. '18]

- For any triangle, $\lambda_1^L \leq \lambda_1^{M \cup S}$.
- For any right triangle, $\max\{\lambda_1^S, \lambda_1^M\} \leq \lambda_1^L$.