Eigenvalues of a Robin Laplacian with a large parameter in the boundary condition: recent results on non-smooth domains

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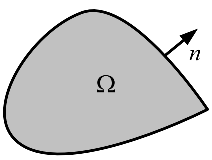
Problem setting

Robin Laplacian

$\Omega \subset \mathbb{R}^d$ suitably regular open set, $d \geq 2$, $\alpha \in \mathbb{R}$

The Robin Laplacian Q^Ω_α in $L^2(\Omega)$ is



$$Q^\Omega_\alpha u = -\Delta u \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = \alpha u \text{ on } \partial\Omega.$$

The associated sesquilinear form q^Ω_α is given by

$$q^\Omega_\alpha(u,u) = \int_\Omega |\nabla u|^2\, dx - \alpha \int_{\partial\Omega} |u|^2\, ds, \quad u \in H^1(\Omega).$$

Spectral properties, in particular, the eigenvalues $E_j(Q^\Omega_\alpha)$, as $\alpha \to +\infty$?

By the min-max principle, under suitable assumptions (e.g. bounded Lipschitz) the eigenvalues $E_j(Q^\Omega_\alpha)$ are given by

$$E_j(Q^\Omega_\alpha) = \inf_{S \subset H^1(\Omega),\, \dim S = j} \ \sup_{u \in S,\, u \not\equiv 0} \frac{q^\Omega_\alpha(u,u)}{\|u\|^2_{L^2(\Omega)}},$$

in particular, $E_j \to -\infty$ at fixed j as $\alpha \to +\infty$.

First eigenvalue

Lower bound (Sobolev estimate)

Classical result on Sobolev spaces: If Ω is bounded Lipschitz, then for some $K > 0$ and all $\varepsilon \in (0,1)$ and $u \in H^1(\Omega)$ there holds

$$\int_{\partial\Omega} |u|^2\, ds \le K\Big(\varepsilon \int_\Omega |\nabla u|^2\, dx + \frac{1}{\varepsilon}\int_\Omega |u|^2\, dx\Big)$$

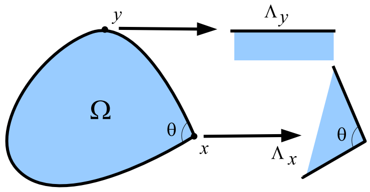
$\Longrightarrow \quad E_1(Q^\Omega_\alpha) \ge -C\alpha^2$, $C > 0$, $\alpha \to +\infty$.

Asymptotics

Under slightly stronger assumptions one has (Lacey–Ockedon–Sabina '1998, Levitin–Parnovski '2008, Bruneau–Popoff '2016):

$$E_1(Q^\Omega_\alpha) \sim -C_\Omega \alpha^2, \quad C_\Omega \ge 1, \quad -C_\Omega = \inf_{x\in\partial\Omega} E_1(Q^{\Lambda_x}_1),$$

with Λ_x being the tangent cone at x, and $C_\Omega = 1$ for smooth Ω



Domains with peaks (Kovařík–P '2018)

If Ω has a peak of the type $\sqrt{x_1^2 + \cdots + x_{d-1}^2} < x_d^p$ with $1 < p < 2$, then $E_j \sim -\varepsilon_j \alpha^{\frac{2}{2-p}}$, where with $(-\varepsilon_j)$ are the eigenvalues of an explicit 1D operator depending on d and p.

Smooth domains

An "almost separation of variables" near the boundary, essentially the main term is governed by the 1D operator T_α in $L^2(0,+\infty)$ in the normal direction:

$$f \mapsto -f'', \quad -f'(0) = \alpha f(0), \quad \operatorname{spec} T_\alpha = \{-\alpha^2\} \cup [0,+\infty).$$

- C^1 domains (Lou–Zhu '2004, Daners–Kennedy '2010): $E_j \sim -\alpha^2$,
- C^3 domains (P '2013, Exner–Minakov–Parnovski '2014, P–Popoff '2015):

  $$E_j(Q^\Omega_\alpha) = -\alpha^2 - (d-1)H_{\max}(\Omega)\,\alpha + o(\alpha), \quad H_{\max}(\Omega) = \sup \text{ of the mean curvature } H \text{ of } \partial\Omega.$$

- there is a version for Robin p-Laplacians (Kovařík–P '2017),
- effective boundary operator (P–Popoff '2016): if Ω is C^3, then

  $$E_j(Q^\Omega_\alpha) = -\alpha^2 + E_j(L_\alpha) + O(1), \quad L_\alpha = -\Delta_{\partial\Omega} - \alpha(d-1)H \text{ in } L^2(\partial\Omega).$$

- The effective operator L_α is of a semiclassical nature,

  $$L_\alpha = \alpha(-h^2\Delta_{\partial\Omega} + V), \quad h = \alpha^{-\frac{1}{2}}, V = -(d-1)H,$$

  and further terms in the asymptotics can be obtained under additional assumptions, e.g. Helffer–Kachmar'2014: 2D domains with a curvature admitting a single non-degenerate maximum,
- Various extensions, e.g. eigenvalue counting function, tunneling effect: Helffer–Kachmar–Raymond '2017, Kachmar–Keraval–Raymond '2016, ...

Spin-off: a link to isoperimetric inequalities

Bareket'77: if $|\Omega| = |B|$ and B is a ball, do we have $E_1(Q^B_\alpha) \ge E_1(Q^\Omega_\alpha)$ for all $\alpha > 0$?

Freitas-Krejčiřík'2015: NO! (and a counterexample with $\alpha \to +\infty$!)

Observation: if $E_1(Q^B_\alpha) \ge E_1(Q^\Omega_\alpha)$ holds for a smooth Ω and a large $\alpha > 0$, then, due to $E_j = -\alpha^2 - (d-1) H_{\max} \alpha + o(\alpha)$ one has

$$H_{\max}(B) \le H_{\max}(\Omega) \qquad (H).$$

Hence, if B maximize E_1 in a class of domains Ω, it also satisfies (H) in the same class.

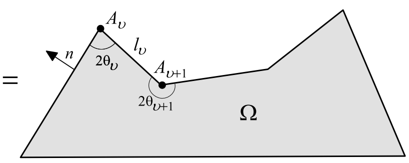
- (H) does not hold in the class of all smooth domains (spherical shells are "better" than balls),
- (H) holds in the class of the star-shaped smooth domains (folkloric, proof e.g. in P-Popoff '2015),
- (H) holds in the class of the *simply connected* smooth 2D domains (follows from a more general result by Pestov–Ionin'1955, many alternative proofs, e.g. Kawohl '2007, P '2015),
- (H) does not hold in the class of 3D domains diffeomorphic to a ball (Ferone–Nitsch–Trombetti '2016).

The estimate (H) plays a role in many other problems, e.g. magnetic Neumann laplacians (Fournais–Helffer '2017), Steklov-type eigenvalues, norms of extensions operators ...

Now, revenons à nos moutons:

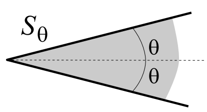
Non-smooth domains: beyond the first eigenvalue

We continue with:
$\Omega \subset \mathbb{R}^2$ polygon with V vertices A_v, side lengths $l_v := |A_{v+1} - A_v|$, and *half*-angles θ_v at A_v



Question: Given $j \in \mathbb{N}$, what is the behavior of $E_j(Q^\Omega_\alpha)$ as $\alpha \to +\infty$?

Model operator: Robin Laplacian $Q^{S_\theta}_\alpha \simeq \alpha^2 Q^{S_\theta}_1$ in an infinite sector S_θ of aperture 2θ, and $T_\theta := Q^{S_\theta}_1$



Theorem (Khalile–P '2018): $\sigma_{\mathrm{ess}}(T_\theta) = [-1, +\infty)$, the spectrum in $(-\infty, -1)$ consists of $K_\theta < \infty$ eigenvalues, which are increasing in θ, and

$$K_\theta = 0 \text{ for } \theta \ge \tfrac{\pi}{2}, \qquad K_\theta \ge 1 \text{ for } \theta < \tfrac{\pi}{2} \qquad \text{and} \qquad E_1 = -\frac{1}{\sin^2\theta},$$

$$K_\theta = 1 \text{ for } \tfrac{\pi}{6} \le \theta < \tfrac{\pi}{2}, \qquad K_\theta \to \infty \text{ for } \theta \to 0 \qquad \text{and} \qquad E_j \sim -\frac{1}{(2j-1)^2\theta^2},$$

and the first K_θ eigenfunctions are exponentially localized near the vertex.

Corner-induced eigenvalues

Open problem (Bucur–Freitas–Kennedy '2017): For a convex polygon Ω with V vertices, do we have $E_j \simeq -\dfrac{1}{\sin^2\theta_j}\alpha^2$ for the first V eigenvalues? (Spoiler: NO!)

Theorem (Khalile '2018):
For $j \le K = K_{\theta_1} + \cdots + K_{\theta_V}$ one has

$$E_j(Q_\alpha^\Omega) = E_j(T_{\theta_1} \oplus \cdots \oplus T_{\theta_V})\alpha^2 + O(e^{-c\alpha}), \quad c > 0, \quad \text{as } \alpha \to +\infty.$$

The respective eigenfunctions are exponentially localized near the corners and are close (in a rigorously defined sense) to linear combinations of the (suitably truncated and rotated) eigenfunctions of T_{θ_v} corresponding to the convex corners (i.e. with $\theta_v \le \frac{\pi}{2}$).

For any fixed $j > K$ one has $E_j(Q_\alpha^\Omega) \simeq -\alpha^2$.

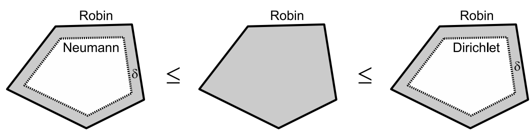
- The result predicted in the above open problem is false, e.g. for $E_2(T_{\theta_u}) < E_1(T_{\theta_v})$ (which happens for θ_u is small enough). Nevertheless, it holds true if $\theta_v \ge \frac{\pi}{6}$ for all v.
- If there are two or more equal angles, one may be interested in the tunneling effect between them (which leads to calculating the exponentially small remainder in a more precise way). Up to know, only a very particular configuration with two corners was studied (Helffer–P '2015).

Edge-induced eigenvalues (1)

What about a more precise asymptotics for $E_{K+j}(Q_\alpha^\Omega) = -\alpha^2 + o(\alpha^2)$?

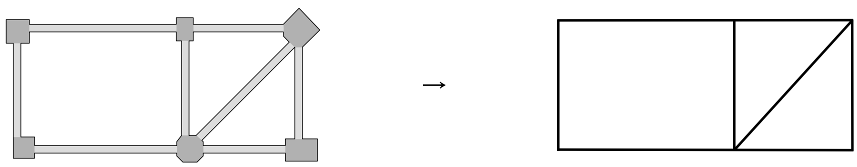
To estimate the negative eigenvalues one may work with

$$\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$$



and $\delta = \delta(\alpha) \to 0$ as $\alpha \to +\infty$

Geometrically, the problem looks close the analysis of Dirichlet laplacians in domains converging to graphs (e.g. Post '2005, Molchanov-Vainberg '2007, Grieser '2008):

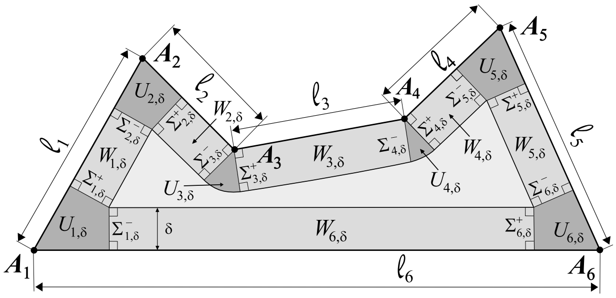


Modulo a common diverging term and vertex-induced eigenvalues, the eigenvalues (of the Dirichlet Laplacians) are determined by quantum graph laplacians with transmission conditions determined by local scattering matrices at the vertices. In "almost all cases", the effective transmission condition at each vertex is the Dirichlet one. (Recently: necessary and sufficient conditions in terms of eigenvalues of the central part: Nazarov '2014+, P '2017, Bakharev–Nazarov '2017)

Adaptation to the Robin case?

Edge-induced eigenvalues (2)

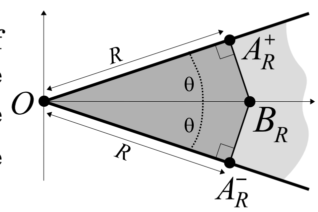
One can further decompose Ω_δ into corner neighborhoods $U_{v,\delta}$ and (slightly shortened) edge neighborhoods $W_{v,\delta}$, separated by segments $\Sigma_v^{\pm,\delta}$:



$$U := \bigcup U_{v,\delta},\quad W := \bigcup W_{v,\delta},\quad \Sigma := \bigcup \Sigma^{\pm}_{v,\delta}$$

First idea: by imposing Dirichlet/Neumann at Σ one arrives at a direct sum of operators in U and W, but: no reasonable bound for the operators in the convex parts of U!

Definition: A corner of opening angle $\theta \in \left(0, \frac{\pi}{2}\right)$ is *non-resonant* if the Laplacian $L_{\theta,R}$ in the truncated sector (see Figure), with the Robin boundary condition $\partial_n u = u$ at $OA_R^{\pm}$ and the Neumann one on the other sides, satisfies $E_{K_\theta+1}(L_{\theta,R}) \geq -1 + CR^{-2}$ with some $C > 0$ as $R \to +\infty$.



Observation (separation of variables + a kind of monotonicity): all $\theta \geq \frac{\pi}{4}$ are non-resonant.

Edge-induced eigenvalues (3): main result

Theorem (Khalile, Ourmières-Bonafos, P ’2018+):

If all corners of Ω are concave or non-resonant ones (in particular, if $\theta_v \geq \frac{\pi}{4}$ for all v), then, for any fixed j,

$$E_{K+j}(Q_\alpha^\Omega) = -\alpha^2 + E_j(D_1 \oplus \dots \oplus D_V) + O\left(\frac{\log\alpha}{\sqrt{\alpha}}\right),$$

where D_v is $f \mapsto -f''$ on $(0, \ell_v)$ with the *Dirichlet* boundary conditions.

Remark:
We do not expect our result to be optimal. Nevertheless, without any additional assumption on the corners the result does not hold: for the equilateral triangle ($\theta_v \equiv \frac{\pi}{6}$), the separation of variables (McCartin ’2011) gives $K = 3$ (i.e. 3 corner-induced eigenvalues) and $E_4 = -\alpha^2 + o(1)$, while $E_1(\oplus D_v) > 0$.

Still in progress: An analogous result is expected for curvilinear polygons Ω with half-angles $\geq \frac{\pi}{4}$:

$$E_{K+j}(Q_\alpha^\Omega) = -\alpha^2 + E_j(L_\alpha^D) + r(\alpha),$$

$$L_\alpha^D = -\partial^2 - \alpha H \text{ on } \partial\Omega \text{ with the Dirichlet boundary condition at the corners,}$$

but with a rather involved remainder $r(\alpha)$.

Edge-induced eigenvalues: scheme of the proof (1)

We work in Ω_δ with $\delta = c_0 \dfrac{\log\alpha}{\alpha}$ with $c_0 > 0$ chosen sufficiently large.

For the upper bound it is sufficient to impose Dirichlet boundary condition at all artificial boundaries.

For the lower bound, the following lemma appears to be very useful:

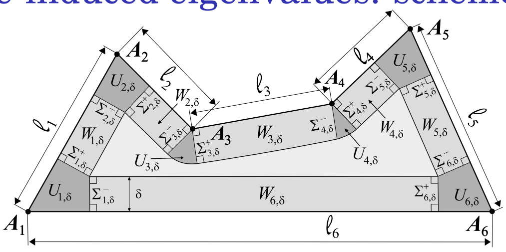
Lemma (Exner–Post '2003): Let Q and Q' be self-adjoint, non-negative, with compact resolvents in $\mathscr{H}$ and $\mathscr{H}'$ and generated by sesquilinear forms q and q'. Let $j \in \mathbb{N}$ and assume that there exists $J : D(q) \to D(q')$ such that, for some $\varepsilon_1 \le \big(1 + E_j(Q)\big)^{-1}$ and $\varepsilon_2 > 0$,

$$\|u\|^2 - \|Ju\|^2 \le \varepsilon_1\big(q(u,u) + \|u\|^2\big),$$
$$q'(Ju,Ju) - q(u,u) \le \varepsilon_2\big(q(u,u) + \|u\|^2\big),$$

then

$$E_j(Q') \le E_j(Q) + \frac{\big(E_j(Q)\varepsilon_1 + \varepsilon_2\big)\big(1 + E_j(Q)\big)}{1 - \big(1 + E_j(Q)\big)\varepsilon_1}$$

Edge-induced eigenvalues: scheme of the proof (2)



$$U := \bigcup U_{v,\delta},$$
$$W := \bigcup W_{v,\delta},$$
$$\Sigma := \bigcup \Sigma^{\pm}_{v,\delta}$$

One obtains the lower bound for the operator B in $L^2(\Omega_\delta)$ given by the form

$$b(u,u) = \int_{\Omega_\delta} |\nabla u|^2 dx - \alpha \int_{\partial\Omega} |u|^2 ds, \quad u \in H^1(\Omega_\delta), \quad b(u,u) = b^U(u,u) + b^W(u,u).$$

One denotes $\Lambda \subset L^2(\Omega_\delta)$ the subspace spanned by the first K eigenfunctions of B.
Furthermore, let L the 1D operator $f \mapsto -f''$ on $(0,\delta)$ with $f'(0) + \alpha f(0) = f'(\delta) = 0$, then one denotes (E,Ψ) its first eigenpair with a normalized Ψ. (I.e. L is action of B with respect to the normal coordinate.)
One uses the preceding lemma with

$$\mathscr{H} = L^2(\Omega_\delta) \ominus \Lambda, \quad Q := B - E, \quad \mathscr{H}' = \oplus L^2(\lambda_v\delta, \ell_v - \lambda_{v+1}\delta), \quad Q' = \oplus D_v$$

and the identification operator J by

$$(Ju)(s) = (Pu)(s) - P(\ell_v\delta)\rho(s) - (Pu)(\ell_v - \lambda_v\delta, \ell_v - \lambda_{v+1}\delta)\rho(\lambda_v, \ell_v - \lambda_{v+1}\delta - s),$$

$$(Pu)(s) = \int_0^\delta u(s,t)\Psi(t)\,dt, \qquad \rho \text{ is a cut-off function supported near } 0$$

(an adaptation of a construction by Post '2005 for a special class of waveguide junctions)

Edge-induced eigenvalues: scheme of the proof (3)

$$(Ju)(s) = (Pu)(s) - P(\ell_v\delta)\rho(s) - (Pu)(\ell_v - \lambda_v\delta, \ell_v - \lambda_{v+1}\delta)\rho(\lambda_v, \ell_v - \lambda_{v+1}\delta - s),$$

$$(Pu)(s) = \int_0^\delta u(s,t)\Psi(t)\,dt, \qquad \rho \text{ is a cut-off function supported near } 0.$$

The two main technical ingredients are:

- Helffer-Sjöstrand-type estimates for distances between various subspaces (based on Agmon-type decay estimates), as there are several "almost orthogonal" subspaces in play,
- The non-resonance condition, which allows to show that the terms with ρ in the expression for Ju are small in a suitable sense. It appears through the control

$$\int_\Sigma |u|^2 ds \le C\delta^2\alpha\big(b^U(u,u) - E\|u\|^2_{L^2(U)} + \|u\|^2_{L^2(\Omega_\delta)}\big),$$

which is trivial for concave corners, but requires some work for convex ones.

Concluding remarks

- Our definition of "non-resonance corners" is a naive adaptation of a sufficient condition for the absence of threshold resonances in waveguide junctions. The latter condition can be reformulated in several equivalent forms (e.g. non existence of non-trivial bounded solutions to some problem, a condition for the scattering matrix at the threshold). Our condition is still applicable for an explicit range of corners.
- A "correct" definition of (non-)resonance corners should probably make use of (suitably defined) scattering matrices associated with sectors. We are not aware of any development in this direction (any comment would be welcome!). The expected result: if $\theta \to K_\theta$ is constant near $\theta = \theta_0$, then θ_0 is non-resonant.