# Schrödinger operators exhibiting a sudden change of the spectral character

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# Physical motivation: irreversible dynamics

While the equations of motion governing quantum dynamics are invariant with respect to time reversal, we often encounter quantum systems behaving in an irreversible way, for instance

- spontaneous decays of particles, nuclei, etc.
- inelastic scattering processes in nuclear, atomic or molecular physics
- current passing through an element attached to poles of a battery
- and, of course, an irreversible process *par excellence* is the *wave packet reduction* which is the core of Copenhagen description of a measuring process performed on a quantum system

A description of such a process is typically associated with enlarging the state Hilbert space, conventionally referred to as coupling the system to a *heat bath*.

# The folklore character of some common claims

It is generally accepted that to obtain an irreversible behavior through coupling to a heat bath, the following is needed

- the bath is a system with infinite number of degrees of freedom
- the bath Hamiltonian has a continuous spectrum
- the presence (or absence) of irreversible modes is determined by the energies involved rather than the coupling strength

While this all is true in many cases, one of our aims here is to show that *neither of the above need not be true in general*.

The idea belongs to Uzy Smilansky and may bring to mind Agatha Christie's revolt against mystery 'rules' she demonstrated in *Murder on the Orient Express*

# Mathematical motivation

A *small change of the coupling constant* can have dramatic influence on the spectrum. Consider the the one-dimensional Schrödinger operator

$$H_\lambda = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2 x^2$$

It is obvious that all $\lambda > 0$ the spectrum of such an operator is *purely discrete*, $\sigma(H_\lambda) = \{(2n+1)\omega :\ n = 0, 1, \dots\}$, while for $\lambda = 0$ and $\lambda < 0$ we have $\sigma(H_\lambda) = [0, \infty)$ and $\sigma(H_\lambda) = \mathbb{R}$, respectively.

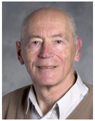
This is easy. A much more subtle question is whether similar things could happen if the potential modification concerns a *small part* of the configuration space, or even a *'set of zero measure'*.

My goal here is to give an *affirmative answer* to this question demonstrating it on Smilansky model and its modifications.

# What we will thus call here Smilansky model?

I speak here of the model originally proposed in [Smilansky'04] to describe a one-dimensional system interacting with a caricature *heat bath* represented by a harmonic oscillator.

Mathematical properties of the model were analyzed in [Solomyak'04], [Evans-Solomyak'05], [Naboko-Solomyak'06]. More recently, time evolution in such a (slightly modified) model was analyzed [Guarneri'11] and compared to its classical counterpart [Guarneri'18]



Let us also pay a memory to Michael Solomyak of the great Sankt Petersburg school, who left us unfortunately two years ago – many of us knew him

In PDE terms, the model is described through a 2D Schrödinger operator

$$H_{\mathrm{Sm}} = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right) + \lambda y \delta(x)$$

on $L^2(\mathbb{R}^2)$ with various modifications to be mentioned later.

# A summary of results about the model

- *Spectral transition:* if $|\lambda| > \sqrt{2}$ the particle can escape to infinity along the singular 'channel' in the $y$ direction. In spectral terms, it corresponds to switch from a positive to a below unbounded spectrum at $|\lambda| = \sqrt{2}$.
- At the heuristic level, the *mechanism* is easy to understand: we have an effective variable decoupling far from the $x$-axis and the oscillator potential competes there with the $\delta$ interaction eigenvalue $-\frac{1}{4}\lambda^2 y^2$.
- *Eigenvalue absence:* for any $\lambda \geq 0$ there are *no eigenvalues* $\geq \frac{1}{2}$. If $|\lambda| > \sqrt{2}$, the point spectrum of $H_{\mathrm{Sm}}$ is *empty*.
- *Existence of eigenvalues:* for $0 < |\lambda| < \sqrt{2}$ we have $H_{\mathrm{Sm}} \geq 0$. The point spectrum is nonempty and finite, and

$$N(\tfrac{1}{2}, H_{\mathrm{Sm}}) \sim \frac{1}{4\sqrt{2(\mu(\lambda)-1)}}$$

holds as $\lambda \to \sqrt{2}-$, where $\mu(\lambda) := \sqrt{2}/\lambda$.

# Further results

- *Absolute continuity:* in the supercritical case $|\lambda| > \sqrt{2}$ we have $\sigma_{\mathrm{ac}}(H_{\mathrm{Sm}}) = \mathbb{R}$
- *Extensions* of the result to a two 'channel' case with different oscillator frequencies [Evans-Solomyak'05] and other situations
- *Extension* to multiple 'channels' on a system *periodic in x* [Guarneri'11]. In this paper the time evolution generated by $H_{\mathrm{Sm}}$ is investigated and proposed as a model of *wavepacket collapse*.

I am not going to show you proofs of these claims, instead I will show you how the the discrete spectrum can be found *numerically* which can provide additional insights [E-Lotoreichik-Tater'17].

At the same the method we will use, rephrasing the task as a *spectral problem for Jacobi matrices* is the core of the proofs I will skip, so you will have at least a feeling of what is technique involved.

# Numerical search for eigenvalues

In the halfplanes $\pm x > 0$ the wave functions can be expanded using the 'transverse' base spanned by the functions

$$\psi_n(y) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \, \mathrm{e}^{-y^2/2} H_n(y)$$

corresponding to the oscillator eigenvalues $n + \frac{1}{2},\ n = 0, 1, 2, \dots$.

Furthermore, one can make use of the mirror symmetry w.r.t. $x = 0$ and divide $H_\lambda$ into the trivial odd part $H_\lambda^{(-)}$ and the even part $H_\lambda^{(+)}$ which is equivalent to the operator on $L^2(\mathbb{R} \times (0, \infty))$ with the same symbol determined by the boundary condition

$$f_x(0+, y) = \frac{1}{2} \alpha y f(0+, y) \, .$$

# Numerical solution, continued

We substitute the Ansatz

$$f(x,y) = \sum_{n=0}^{\infty} c_n \, \mathrm{e}^{-\kappa_n x} \psi_n(y)$$

with $\kappa_n := \sqrt{n + \frac{1}{2} - \epsilon}$.

This yields for solution with the energy $\epsilon$ the equation

$$B_\lambda c = 0 \,,$$

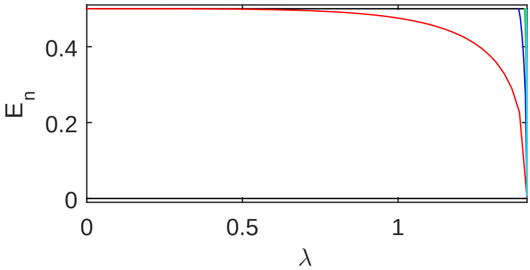
where $c$ is the coefficient vector and $B_\lambda$ is the operator in $\ell^2$ with

$$(B_\lambda)_{m,n} = \kappa_n \delta_{m,n} + \frac{1}{2} \lambda (\psi_m, y\psi_n) \,.$$

Note that the matrix is in fact tridiagonal because

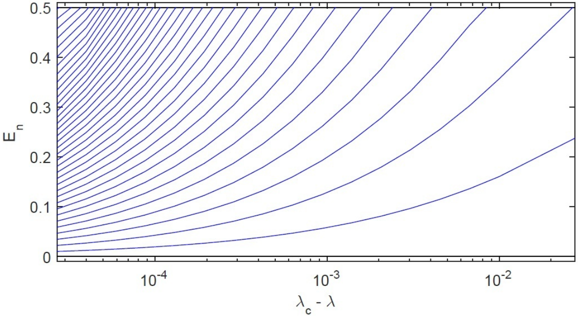
$$(\psi_m, y\psi_n) = \frac{1}{\sqrt{2}} \big( \sqrt{n+1} \, \delta_{m,n+1} + \sqrt{n} \, \delta_{m,n-1} \big) \,.$$

# The eigenvalues: got it Solomyak right?



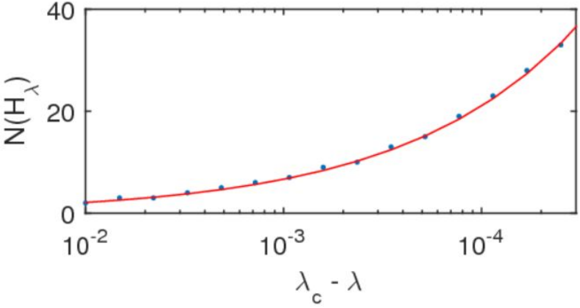

In most part of the subcritical region there is a single eigenvalue, the second one appears only at $\lambda \approx 1.387559$. The next thresholds are $1.405798, 1.410138, 1.41181626, 1.41263669, \ldots$

# A better picture near the critical value



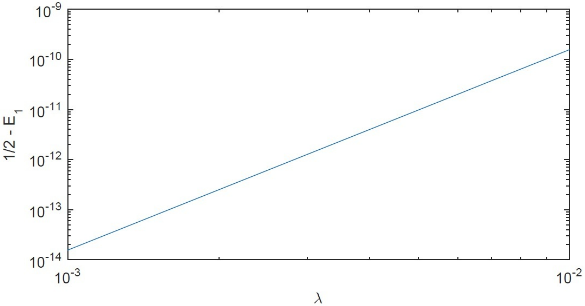

Close to the critical value, however, many eigenvalues appear which gradually fill the interval $(0, \frac{1}{2})$ as the critical value is approached

# So the answer is: he did




The dots mean the eigenvalue numbers, the red curve is the above mentioned asymptotics due to Solomyak

# Weak coupling: the ground state asymptotics




The numerical solution also indicates other properties, for instance, that the first eigenvalue behaves as $\epsilon_1 = \frac{1}{2} - c\lambda^4 + o(\lambda^4)$ as $\lambda \to 0$, with $c \approx 0.0156$.

# Theory helps: in fact, we have $c = 0.015625$

Indeed, the relation $B_\lambda c = 0$ can be written explicitly as

$$\sqrt{\mu_\lambda}\, c_0^\lambda + \frac{\lambda}{2\sqrt{2}} c_1^\lambda = 0\,,$$

$$\frac{\sqrt{k}\lambda}{2\sqrt{2}} c_{k-1}^\lambda + \sqrt{k+\mu_\lambda}\, c_k^\lambda + \frac{\sqrt{k+1}\lambda}{2\sqrt{2}} c_{k+1}^\lambda = 0\,, \quad k \geq 1\,,$$

where $\mu_\lambda := \frac{1}{2} - E_1(\lambda)$ and $c^\lambda = \{c_0^\lambda, c_1^\lambda, \dots\}$ is the corresponding normalized eigenvector of $B_\lambda$.

Using the above relations and simple estimates, we get

$$\sum_{k=1}^{\infty} |c_k^\lambda|^2 \leq \frac{3}{4}\lambda^2 \quad \text{and} \quad c_0^\lambda = 1 + \mathcal{O}(\lambda^2)$$

as $\lambda \to 0+$; hence we have in particular $c_1^\lambda = \frac{\lambda}{2\sqrt{2}} + \mathcal{O}(\lambda^2)$.

# In fact, we have $c = 0.015625$

The first of the above relation then gives

$$\mu_\lambda = \frac{\lambda^4}{64} + \mathcal{O}(\lambda^5)$$

as $\lambda \to 0+$, in other words

$$E_1(\lambda) = \frac{1}{2} - \frac{\lambda^4}{64} + \mathcal{O}(\lambda^5)\,.$$

And the mentioned coefficient 0.015625 is nothing else than $\frac{1}{64}$. $\square$.

The figure here is too big to be stored here.
You can find it in the original version of these slides:

http://gemma.ujf.cas.cz/%7Eexner/Talks/iml18p.pdf

# A regular version of Smilansky model

The question whether one can observe a similar effect for Schrödinger operators with regular potentials was asked in [Guarneri'11].

The coupling cannot be now linear in $y$ and the profile of the channel has to change with $y$. We replace the $\delta$ by a *family of shrinking potentials* whose mean matches the $\delta$ coupling constant, $\int U(x,y)\,\mathrm{d}x \sim y$. This can be achieved, e.g., by choosing $U(x,y) = \lambda y^2 V(xy)$ for a fixed function $V$.

This motivates us to investigate the following operator on $L^2(\mathbb{R}^2)$,

$$H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2 - \lambda y^2 V(xy),$$

where $\omega$, $a$ are positive constants and the potential $V$ is a nonnegative function with bounded first derivative and $\operatorname{supp} V \subset [-a, a]$.

By Faris-Lavine theorem $H$ is e.s.a. on $C_0^\infty(\mathbb{R}^2)$ and the same is true for its generalizations, with more 'decay' channels, periodicity in $x$, etc.

# Subcritical case

To state the result we employ a 1D comparison operator $L = L_V$,

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2 - \lambda V(x)$$

on $L^2(\mathbb{R})$ with the domain $H^2(\mathbb{R})$. What matters is the sign of its spectral threshold; since $V \geq 0$, the latter is a monotonous function of $\lambda$ and there is a $\lambda_{\text{crit}} > 0$ at which the sign changes.

**Theorem (Barseghyan-E'14)**

*Under the stated assumption, the spectrum of the operator $H$ is bounded from below provided the operator $L$ is positive.*

The claim is proved by *Neumann bracketing* imposing additional boundary conditions at the lines $y = \pm \ln n$, $n = 2, 3, \dots,$ and showing that the components of $H$ in these strips have a uniform lower bound by an operator unitarily equivalent to $L$.

# Supercritical case

Once the transverse channel principal eigenvalue dominates over the harmonic oscillator contribution, the spectral behavior changes:

**Theorem** (Barseghyan-E'14)

*Under our hypotheses,* $\sigma(H) = \mathbb{R}$ *holds if* $\inf \sigma(L) < 0$.

Proof relies on construction of an appropriate Weyl sequence: we have to find $\{\psi_k\}_{k=1}^\infty \subset D(H)$ such that $\|\psi_k\| = 1$ which contains no convergent subsequence, and at same time

$$\|H\psi_k - \mu\psi_k\| \to 0 \quad \text{as} \quad k \to \infty.$$

Specifically, we choose

$$\psi_k(x,y) = h(xy)\,\mathrm{e}^{i\epsilon_\mu(y)}\chi_k\left(\frac{y}{n_k}\right) + \frac{f(xy)}{y^2}\,\mathrm{e}^{i\epsilon_\mu(y)}\chi_k\left(\frac{y}{n_k}\right),$$

where $\epsilon_\mu(y) := \int_{\sqrt{|\mu|}}^{y} \sqrt{t^2+\mu}\,\mathrm{d}t$, $h$ is the normalized ground-state eigenfunction by $L$, $f(t) := -\frac{i}{2}\,t^2 h(t)$, and finally, $\chi_k$ are suitable, compactly supported mollifier functions.

# More results on Smilansky model

The regular version shares also other properties with the original [Barseghyan-E'17]:

- in the subcritical case, $\inf \sigma(L) > 0$, we have $\sigma_{\rm ess}(H) = [\omega, \infty)$ and $\emptyset \neq \sigma_{\rm disc}(H) \subset [0, \omega)$
- in the critical case, $\inf \sigma(L) = 0$, we have $\sigma(H) = \sigma_{\rm ess}(H) = [0, \infty)$

One can also consider a modification of Smilansky(-Solomyak) model in which the system is placed into a *homogeneous magnetic field*, described by the Hamiltonian

$$H = (i\nabla + A)^2 + \omega^2 y^2 + \lambda y \delta(x)\,,$$

where $A$ is a suitable vector potential. The behavior is similar, in the subcritical case we now have $\sigma_{\rm ess}(H(A)) = [\sqrt{B^2 + \omega^2}, \infty)$ but again a small nonzero $\lambda$ gives rise to a discrete spectrum which fills the interval $[0, \sqrt{B^2 + \omega^2})$ as $|\lambda|$ approaches $2\omega$, and above this value the spectrum fills the whole real line [Barseghyan-E'17b].

The effect of the magnetic field on the regular version is similar.

# Moreover, the effect is robust

To illustrate it, consider the $\delta'$ *version* of Smilansky model:

The Hamiltonian corresponds now to the differential expression

$$H_\beta\psi(x,y) = -\frac{\partial^2\psi}{\partial x^2}(x,y) + \frac{1}{2}\left(-\frac{\partial^2\psi}{\partial y^2}(x,y) + y^2\psi(x,y)\right)$$

its domain consisting of $\psi \in H^2((0,\infty)\times\mathbb{R}) \oplus H^2((-\infty,0)\times\mathbb{R})$ such that

$$\psi(0+,y) - \psi(0-,y) = \frac{\beta}{y}\frac{\partial\psi}{\partial x}(0+,y)\,, \quad \frac{\partial\psi}{\partial x}(0+,y) = \frac{\partial\psi}{\partial x}(0-,y)\,.$$

Let $\mathfrak{m}_{\rm ac}$ be the multiplicity function of the absolutely continuous spectrum.

**Theorem (E-Lipovský'18)**

*The spectrum of operator $H_0$ is purely a.c., $\sigma(H_0) = [\frac{1}{2},\infty)$ with $\mathfrak{m}_{\rm ac}(E,H_0) = 2n$ for $E \in (n-\frac{1}{2}, n+\frac{1}{2})$, $n \in \mathbb{N}$. For $\beta > 2\sqrt{2}$ the a.c. spectrum of $H_\beta$ coincides with $\sigma(H_0)$. For $\beta \le 2\sqrt{2}$ there is a new branch of continuous spectrum added to the spectrum; for $\beta = 2\sqrt{2}$ we have $\sigma(H_\beta) = [0,\infty)$ and for $\beta < 2\sqrt{2}$ the spectrum covers the whole real line.*

# $\delta'$ Smilansky model, continued

**Theorem (E-Lipovský'18)**

*Assume $\beta \in (2\sqrt{2}, \infty)$, then the discrete spectrum of $H_\beta$ is nonempty and lies in the interval $(0, \frac{1}{2})$. The number of eigenvalues is approximately given by*

$$\frac{1}{4\sqrt{2\left(\frac{\beta}{2\sqrt{2}} - 1\right)}} \quad \text{as} \quad \beta \to 2\sqrt{2}+$$

**Theorem (E-Lipovský'18)**

*For large enough $\beta$ there is a single eigenvalue which asymptotically behaves as*

$$\lambda_1 = \frac{1}{2} - \frac{4}{\beta^4} + \mathcal{O}\left(\beta^{-5}\right)$$

# Another model of this class

Now we pass to a family of systems in which the transition is even *more dramatic* passing from *purely discrete spectrum* in the subcritical case to the whole real line in the supercritical one.

To begin, recall that there are situations where *Weyl's law fails* and the spectrum is discrete even if the classically allowed phase-space volume is infinite. A classical example due to [Simon'83] is a 2D Schrödinger operator with the potential

$$V(x,y) = x^2y^2$$

or more generally, $V(x,y) = |xy|^p$ with $p \geq 1$.

Similar behavior one can observe for Dirichlet Laplacians in *regions with hyperbolic cusps* – see [Geisinger-Weidl'11] for recent results and a survey. Moreover, using the *dimensional-reduction technique* of Laptev and Weidl one can prove spectral estimates for such operators.

A common feature of these models is that the particle motion is confined into *channels narrowing towards infinity*.

# Adding potentials unbounded from below

This may remain true even for Schrödinger operators with *unbounded from below* in which a classical particle can escape to infinity with an increasing velocity.

The situation changes, however, if the *attraction is strong enough*; recall that such a behavior was noted already in [Znojil'98].

As an illustration, let us analyze the following class of operators:

$$L_p(\lambda) \,:\, L_p(\lambda)\psi = -\Delta\psi + \Big(|xy|^p - \lambda(x^2+y^2)^{p/(p+2)}\Big)\,\psi\,, \quad p \geq 1$$

on $L^2(\mathbb{R}^2)$, where $(x,y)$ are the standard Cartesian coordinates in $\mathbb{R}^2$ and the parameter $\lambda$ in the second term of the potential is non-negative; unless the value of $\lambda$ is important we write it simply as $L_p$.

Note that $\frac{2p}{p+2} < 2$ so the operator is e.s.a. on $C_0^\infty(\mathbb{R}^2)$ by Faris-Lavine theorem again; the symbol $L_p$ or $L_p(\lambda)$ will always mean its closure.

# The subcritical case

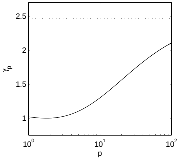
The spectral properties of $L_p(\lambda)$ depend crucially on the value of $\lambda$ and there is a *transition between different regimes* as $\lambda$ changes.

Let us start with the *subcritical case* which occurs for small values of $\lambda$. To characterize the smallness quantitatively we need an auxiliary operator which will be an (an)harmonic oscillator Hamiltonian on line,

$$\tilde{H}_p \,:\, \tilde{H}_p u = -u'' + |t|^p u$$

on $L^2(\mathbb{R})$ with the standard domain.

The principal eigenvalue $\gamma_p = \inf \sigma(H_p)$ equals one for $p = 2$; for $p \to \infty$ it becomes $\gamma_\infty = \frac{1}{4}\pi^2$; it smoothly interpolates between the two values; a numerical solution gives true minimum $\gamma_p \approx 0.998995$ attained at $p \approx 1.788$; in the semilogarithmic scale the plot is as follows:



# The subcritical case – continued

The spectrum is naturally bounded from below and discrete if $\lambda = 0$; our aim is to show that this remains to be the case provided $\lambda$ is small enough.

**Theorem (E-Barseghyan'12)**

*For any $\lambda \in [0, \lambda_{\rm crit}]$, where $\lambda_{\rm crit} := \gamma_p$, the operator $L_p(\lambda)$ is bounded from below for $p \geq 1$; if $\lambda < \gamma_p$ its spectrum is purely discrete.*

Idea of the proof: Let $\lambda < \gamma_p$. By minimax we need to estimate $L_p$ from below by a s-a operator with a purely discrete spectrum. To construct it we employ bracketing imposing additional Neumann conditions at concentric circles of radii $n = 1, 2, \dots$.

In the estimating operators the variables decouple asymptotically and the spectral behavior is determined by the angular part of the operators; to prove the discreteness one has to check that the lowest ev's in the annuli tend to infinity as $n \to \infty$.

# The supercritical case

**Theorem (E-Barseghyan'12)**

*The spectrum of $L_p(\lambda)$, $p \geq 1$, is unbounded below from if $\lambda > \lambda_{\rm crit}$.*

Idea of the proof: Similar as above with a few differences:

- now we seek an upper bound to $L_p(\lambda)$ by a below unbounded operator, hence we impose Dirichlet conditions on concentric circles
- the estimating operators have now a nonzero contribution from the radial part, however, it is bounded by $\pi^2$ independently of $n$
- the negative $\lambda$-dependent term now outweights the anharmonic oscillator part so that $\inf \sigma(L_{n,p}^{(1,D)}) \to -\infty$ holds as $n \to \infty$ □

Using suitable Weyl sequences similar to those the previous model, however, we are able to get a stronger result:

**Theorem (Barseghyan-E-Khrabustovskyi-Tater'16)**

$\sigma(L_p(\lambda)) = \mathbb{R}$ *holds for any* $\lambda > \gamma_p$ *and* $p > 1$.

# Spectral estimates: bounds to eigenvalue sums

Let us return to the subcritical case and define the following quantity:

$$\alpha := \frac{1}{2}\left(1+\sqrt{5}\right)^2 \approx 5.236 > \gamma_p^{-1}$$

We denote by $\{\lambda_{j,p}\}_{j=1}^{\infty}$ the eigenvalues of $L_p(\lambda)$ arranged in the ascending order; then we can make the following claim.

**Theorem** (E-Barseghyan'12)

*To any nonnegative $\lambda < \alpha^{-1} \approx 0.19$ there exists a positive constant $C_p$ depending on $p$ only such that the following estimate is valid,*

$$\sum_{j=1}^{N} \lambda_{j,p} \geq C_p(1-\alpha\lambda)\frac{N^{(2p+1)/(p+1)}}{(\ln^p N+1)^{1/(p+1)}} - c\lambda N, \quad N=1,2,\dots,$$

*where $c = 2\left(\frac{\alpha^2}{4}+1\right) \approx 15.7$.*

# Cusp-shaped regions

The above bounds are valid for any $p \geq 1$, hence it is natural to ask about the limit $p \to \infty$ describing the particle confined in a region with four hyperbolic 'horns', $D = \{(x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$, described by the Schrödinger operator

$$H_D(\lambda) : H_D(\lambda)\psi = -\Delta\psi - \lambda(x^2 + y^2)\psi$$

with a parameter $\lambda \geq 0$ and Dirichlet condition on the boundary $\partial D$.

**Theorem** (E-Barseghyan'12)

*The spectrum of $H_D(\lambda)$ is discrete for any $\lambda \in [0, 1)$ and the spectral estimate*

$$\sum_{j=1}^{N} \lambda_j \geq C(1-\lambda)\frac{N^2}{1 + \ln N}, \qquad N = 1, 2, \dots$$

*holds true with a positive constant $C$.*

# Proof outline

To get the estimate for cusp-shaped regions, one can check that for any $u \in H^1$ satisfying the condition $u|_{\partial D} = 0$ the inequality

$$\int_D (x^2 + y^2) u^2(x,y)\, \mathrm{d}x\, \mathrm{d}y \le \int_D |(\nabla u)(x,y)|^2\, \mathrm{d}x\, \mathrm{d}y$$

is valid which in turn implies

$$H_D(\lambda) \ge -(1-\lambda)\Delta_D\,,$$

where $\Delta_D$ is the Dirichlet Laplacian on the region $D$.

The result then follows from the eigenvalue estimates on $\Delta_D$ known from [Simon'83], [Jakšić-Molchanov-Simon'92].

The proof for $p \in (1,\infty)$ is more complicated, using splitting of $\mathbb{R}^2$ into rectangular domains and estimating contributions from the channel regions, the middle part, and the rest. We will not discuss it here, because we are able to demonstrate a stronger result *à la* Lieb and Thirring.

# Better spectral estimates

**Theorem** (Barseghyan-E-Khrabustovskyi-Tater'16)

*Given* $\lambda < \gamma_p$, *let* $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ *be eigenvalues of* $L_p(\lambda)$. *Then for* $\Lambda \ge 0$ *and* $\sigma \ge 3/2$ *the following inequality is valid,*

$$\mathrm{tr}\,(\Lambda - L_p(\lambda))_+^\sigma \le C_{p,\sigma}\left(\frac{\Lambda^{\sigma+(p+1)/p}}{(\gamma_p-\lambda)^{\sigma+(p+1)/p}}\ln\left(\frac{\Lambda}{\gamma_p-\lambda}\right) + C_\lambda^2\left(\Lambda + C_\lambda^{2p/(p+2)}\right)^{\sigma+1}\right),$$

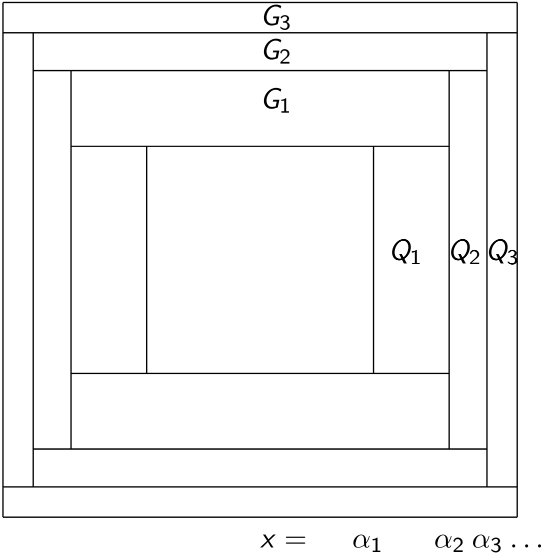
*where the constant* $C_{p,\sigma}$ *depends on* $p$ *and* $\sigma$ *only and*

$$C_\lambda =: \max\left\{\frac{1}{(\gamma_p-\lambda)^{(p+2)/(p(p+1))}},\, \frac{1}{(\gamma_p-\lambda)^{(p+2)^2/(4p(p+1))}}\right\}.$$

*Sketch of the proof:* By minimax principle we can estimate $L_p(\lambda)$ from below by a self-adjoint operator with a purely discrete negative spectrum and derive a bound to the momenta of the latter.

We split $\mathbb{R}^2$ again, now in a 'lego' fashion using a monotone sequence $\{\alpha_n\}_{n=1}^\infty$ such that $\alpha_n \to \infty$ and $\alpha_{n+1} - \alpha_n \to 0$ holds as $n \to \infty$.

# Proof sketch



# Proof sketch, continued

Estimating the 'transverse' variables by by their extremal values, we reduce the problem essentially to assessment of the spectral threshold of the anharmonic oscillator with Neumann cuts.

**Lemma**

*Let* $l_{k,p} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^p$ *be the Neumann operator on* $[-k, k]$, $k > 0$. *Then*

$$\inf \sigma\left(l_{k,p}\right) \geq \gamma_p + o\left(k^{-p/2}\right) \quad \textit{as} \;\; k \to \infty.$$

Combining it with the 'transverse' eigenvalues $\left\{\frac{\pi^2 k^2}{(\alpha_{n+1}-\alpha_n)^2}\right\}_{k=0}^{\infty}$, using Lieb-Thirring inequality for this situation [Mickelin'16], and choosing properly the sequence $\{\alpha_n\}_{n=1}^{\infty}$, we are able to prove the claim. □

# The critical case

Let us return to $L := -\Delta + |xy|^p - \gamma_p (x^2+y^2)^{p/(p+2)}$ and the conjectures we made about its spectrum. Concerning the essential spectrum:

**Theorem** (Barseghyan-E-Khrabustovskyi-Tater'16)

*We have* $\sigma_{\rm ess}(L) \supset [0,\infty)$.

This can be proved in the same as above using suitable Weyl sequences.

**Theorem** (Barseghyan-E-Khrabustovskyi-Tater'16)

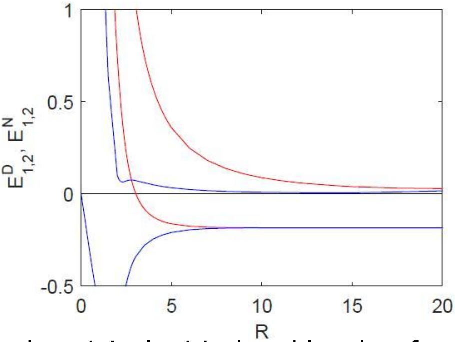
*The negative spectrum of L is discrete.*

The proof uses a 'lego' estimate similar to the one presented above.

For the moment, however, we cannot prove that $\sigma_{\rm disc}(L)$ is nonempty. We conjecture that it is the case having a *strong numerical evidence* for that.

# Bracketing: numerical analysis

We solve our spectral problem with $p = 2$ in a disc of radius $R$ with Dirichlet and Neumann condition at the boundary, and plot the first two eigenvalues as a function of $R$.

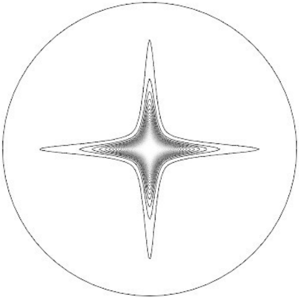



This indicates that the original critical problem has for $p = 2$ a single eigenvalue $E_1 \approx -0.18365$.

Numerics also suggests ground state existence for *other values of p* but it becomes unreliable for $p \gtrsim 20$.

# Ground state eigenfunction

We also find the eigenfunction, note that with the $R = 20$ cut-off the Dirichlet and Neumann ones are practically identical; the outer level marks the $10^{-3}$ value.



# Back to Smilansky model: resonances

There are other interesting effects here. Let us show, for instance, that Smilansky model can exhibit *resonances*, cf. [E-Lotoreichik-Tater'16].

The first question in this respect is *which resonances* we speak about. There are *resolvent resonances* associated with poles in the analytic continuation of the resolvent over the cut(s) corresponding to the continuous spectrum, and *scattering resonances* identified with singularities of the scattering matrix.

The former are found using the same Jacobi matrix problem as before, of course, this time with a 'complex energy'.

Let is look at the latter. Suppose the incident wave comes in the $m$-th channel from the left. We use the Ansatz

$$f(x,y) = \begin{cases} \sum_{n=0}^{\infty} \left( \delta_{mn} \mathrm{e}^{-ipx} \psi_n(y) + r_{mn} \, \mathrm{e}^{ix\sqrt{p^2+\epsilon_m-\epsilon_n}} \psi_n(y) \right) \\ \sum_{n=0}^{\infty} t_{mn} \, \mathrm{e}^{-ix\sqrt{p^2+\epsilon_m-\epsilon_n}} \psi_n(y) \end{cases}$$

for $\mp x > 0$, respectively, where $\epsilon_n = n + \frac{1}{2}$ and the incident wave energy is assumed to be $p^2 + \epsilon_m =: k^2$.

# Smilansky model resonances, continued

It is straightforward to compute from here the boundary values $f(0\pm,y)$ and $f'(0\pm,y)$. The continuity requirement at $x=0$ together with the orthonormality of the basis $\{\psi_n\}$ yields

$$t_{mn} = \delta_{mn} + r_{mn}\,.$$

Furthermore, we substitute the boundary values coming from the Ansatz into

$$f'(0+,y) - f'(0-,y) - \lambda y f(0,y) = 0$$

and integrate the obtained expression with $\int \mathrm{d}y\, \psi_l(y)$. This yields

$$\sum_{n=0}^{\infty} \Big(2p_n\delta_{ln} - i\lambda(\psi_l, y\psi_n)\Big) r_{mn} = i\lambda(\psi_l, y\psi_m)\,,$$

where we have denoted $p_n = p_n(k) := \sqrt{k^2 - \epsilon_n}$.

# Smilansky model resonances, continued

In particular, poles of the scattering matrix are associated with the kernel of the $\ell^2$ operator on the left-hand side. This is the same condition, however, we had before, thus we have

**Proposition**

*The resolvent and scattering resonances coincide in the Smilansky model.*

*Remarks:* (a) The on-shell scattering matrix is a $\nu \times \nu$ matrix where $\nu := \left[k^2 - \frac{1}{2}\right]$ whose elements are the transmission and reflection amplitudes; they have common singularities.

(b) The resonance condition may have (and it has) numerous solutions, but only those 'not far from the physical sheet' are of interest.

(c) The Riemann surface of energy has infinite number of sheets determined by the choices *branches of the square roots*. The interesting resonances on the $n$-th sheet are obtained by *flipping sign of the first* $n-1$ *of them*.

# Smilansky model resonances: weak coupling

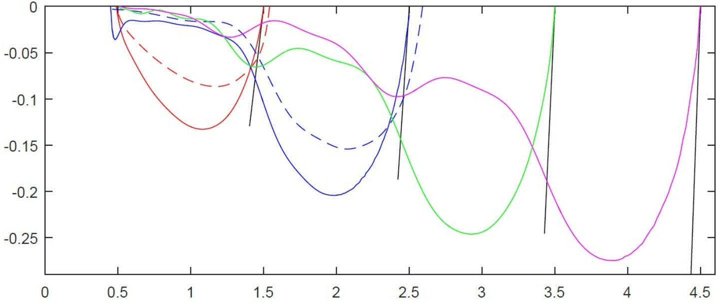
The weak-coupling analysis follows the route as for the discrete spectrum – in fact it includes the eigenvalue case if we stay on the 'first' sheet – and shows that for small $\lambda$ a resonance poles splits of each threshold according to the asymptotic expansion

$$\mu_n(\lambda) = -\frac{\lambda^4}{64}\big(2n+1+2in(n+1)\big) + o(\lambda^4)\,.$$

Hence the distance for the corresponding threshold is proportional to $\lambda^4$ and the trajectory asymptote is the 'steeper' the larger $n$ is.

Numerically, however, one can go beyond the weak coupling regime – and *the picture becomes more intriguing*

# Examples of resonance trajectories




Resonance trajectories as $\lambda$ changes for zero to $\sqrt{2}$. The weak-coupling asymptotes are shown. The 'non-threshold' resonances at the second and third sheet appear at $\lambda = 1.287$ and $\lambda = 1.19$, respectively.

# Open questions

There are many, for instance,

- is the supercritical spectrum always *absolutely continuous*?
- what is its *spectral multiplicity*?
- what can one say of *classical motion* in the regular versions of the model?
- can one *classify the resonances*?
- etc., etc.

# The talk was based on

[EB12] P.E., D. Barseghyan: Spectral estimates for a class of Schrödinger operators with infinite phase space and potential unbounded from below, *J. Phys. A: Math. Theor.* **45** (2012), 075204.

[BE14] D. Barseghyan, P.E.: A regular version of Smilansky model, *J. Math. Phys.* **55** (2014), 042194.

[BE17a] D. Barseghyan, P.E.: A regular analogue of the Smilansky model: spectral properties, *Rep. Math. Phys.* **80** (2017), 177–192.

[ELT16] P.E., V. Lotoreichik, M. Tater: On resonances and bound states of Smilansky Hamitonian, *Nanosystems: Phys. Chem. Math.* **7** (2016), 789–802.

[ELT17] P.E., V. Lotoreichik, M. Tater: Spectral and resonance properties of Smilansky Hamiltonian, *Phys. Lett.* **A381** (2017), 756–761.

[BEKT16] D. Barseghyan, P.E., A. Khrabustovskyi, M. Tater: Spectral analysis of a class of Schrödinger operators exhibiting a parameter-dependent spectral transition, *J. Phys. A: Math. Theor.* **49** (2016), 165302.

[BE17b] D. Barseghyan, P.E.: A magnetic version of the Smilansky-Solomyak model, *J. Phys. A: Math. Theor.* **50** (2017), 485203

[EL18] P.E., J. Lipovský: Smilansky-Solomyak model with a $\delta'$-interaction, *Phys. Lett.* **A382** (2018), 1207–1213.

# It remains to say

Thank you for your attention!