On Aharonov-Bohm operators with two colliding poles

Joint works with L. Abatangelo, L. Hillairet, C. Léna

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Aharonov–Bohm potential

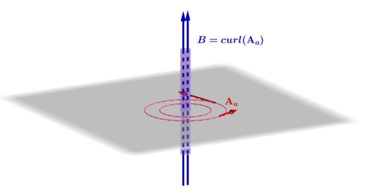
For $\boldsymbol{a} = (a_1, a_2) \in \mathbb{R}^2$, the Aharonov-Bohm magnetic potential with pole $\boldsymbol{a}$ and circulation $1/2$ is

$$\mathbf{A}_{\boldsymbol{a}}(x_1, x_2) = \frac{1}{2}\left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right).$$

$\mathbf{A}_{\boldsymbol{a}}$ generates the Aharonov-Bohm magnetic field in $\mathbb{R}^2$ with pole $\boldsymbol{a}$ and circulation $\frac{1}{2}$.

Aharonov–Bohm potential

The AB magnetic field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane (x_1, x_2) at the point $\boldsymbol{a}$, as the



radius of the solenoid goes to zero and the magnetic flux remains constantly equal to $1/2$.

Negletting the irrelevant coordinate along the solenoid, the problem becomes 2-dimensional.

Aharonov-Bohm effect [Aharonov-Bohm, Phys. Rev. (1959)]

The AB magnetic field is a δ-like magnetic field: a quantum particle moving in $\mathbb{R}^2 \setminus \{\boldsymbol{a}\}$ is affected by the magnetic potential, despite being confined to a region in which the magnetic field is zero.

Aharonov–Bohm potential

The Schrödinger operators with AB vector potential:

$$(i\nabla + \mathbf{A}_{\boldsymbol{a}})^2 u = -\Delta u + 2i\mathbf{A}_{\boldsymbol{a}} \cdot \nabla u + |\mathbf{A}_{\boldsymbol{a}}|^2 u.$$

In $\Omega \subset \mathbb{R}^2$ bounded, open and simply connected, $\forall\, \boldsymbol{a} \in \Omega$ the eigenvalue problem

$$\begin{cases} (i\nabla + A_{\boldsymbol{a}})^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

admits a sequence of real diverging eigenvalues $\{\lambda_k(\boldsymbol{a})\}_{k \geq 1}$

$$\lambda_1(\boldsymbol{a}) \leq \lambda_2(\boldsymbol{a}) \leq \cdots \leq \lambda_k(\boldsymbol{a}) \leq \dots$$

The eigenvalue function

What is known on the function

$$\boldsymbol{a} \mapsto \lambda_k(\boldsymbol{a})?$$

One-pole case.

Sharp asymptotics for simple eigenvalues when the pole a is moving in $\overline{\Omega}$:

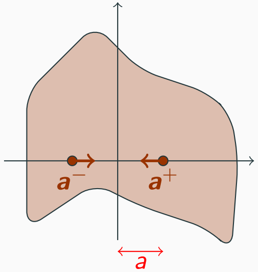
Bonnaillie-Noël-Noris-Nys-Terracini, Analysis & PDE (2014)
Noris-Nys-Terracini, Comm. Math. Physics (2015)
Abatangelo-F., Calc. Var. PDEs (2015)
Abatangelo-F., SIAM J. Math. Anal. (2016)
Abatangelo-F.-Noris-Nys, JFA (2017)

Multi-pole case.

For vector potentials which are sum of AB potentials with poles located at different points in the domain: **[Léna, J. Math. Physics (2015)]** proves continuity of eigenvalues (as functions of configuration of poles) even for coalescing poles.

Aharonov-Bohm operators with two colliding poles

For $a > 0$, let
$\boldsymbol{a}^- = (-a, 0)$ and $\boldsymbol{a}^+ = (a, 0)$
be the poles of the AB potential



$$\mathbf{A}_{\boldsymbol{a}^-,\boldsymbol{a}^+}(x) := -\mathbf{A}_{\boldsymbol{a}^-}(x) + \mathbf{A}_{\boldsymbol{a}^+}(x)$$
$$= -\frac{1}{2}\frac{(-x_2, x_1 + a)}{(x_1 + a)^2 + x_2^2} + \frac{1}{2}\frac{(-x_2, x_1 - a)}{(x_1 - a)^2 + x_2^2}.$$

Aharonov-Bohm operators with two colliding poles

$$\mathbf{A}_{\boldsymbol{a}^-,\boldsymbol{a}^+}(x) := -\mathbf{A}_{\boldsymbol{a}^-}(x) + \mathbf{A}_{\boldsymbol{a}^+}(x)$$

Let $\Omega \subseteq \mathbb{R}^2$ be open, bounded and connected with $0 \in \Omega$.

Let $\{\lambda_k^a\}_{k\geq 1}$ be the eigenvalues of $(i\nabla + \mathbf{A}_{a^-,a^+})^2$ in Ω with homogenous Dirichlet boundary conditions.

Let $\{\lambda_k\}_{k\geq 1}$ be the eigenvalues of the Dirichlet Laplacian $-\Delta$ in Ω.

Theorem [Léna, J. Math. Physics (2015)]

For every $k \geq 1$,

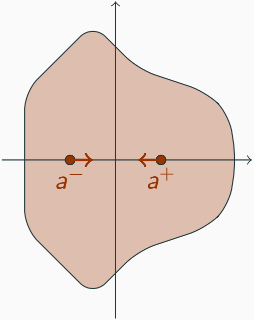
$$\lim_{a\to 0} \lambda_k^a = \lambda_k.$$

Problem: sharp asymptotics for the eigenvalue variation $\lambda_k^a - \lambda_k$ as the two poles $\boldsymbol{a}^-, \boldsymbol{a}^+$ coalesce towards a point?

Symmetric case

Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma(x_1, x_2) = (x_1, -x_2)$. Let Ω be such that

$$\sigma(\Omega) = \Omega \quad \text{and} \quad 0 \in \Omega.$$



Let λ_N be a simple eigenvalue of the Dirichlet Laplacian on Ω and u_N be an associated $L^2(\Omega)$-normalized eigenfunction

$\rightsquigarrow \exists\, k \in \mathbb{N}$, $\beta \neq 0$, $\alpha \in [0, \pi)$ s.t.

$$u_N(r(\cos t, \sin t)) \underset{r \to 0^+}{\sim} \beta\, r^k \sin(\alpha - k t)$$

- If $k = 0$, u_N does not vanish near 0 and $\beta \sin\alpha = u_N(0)$.
- $k = 1$: 0 is a regular point in the nodal set of u_N and $\beta^2 = |\nabla u_N(0)|^2$.
- If $k \geq 1$, the nodal set of u_N near 0 consists of $2k$ regular half-curves meeting at 0 with equal angles; the minimal slope of half-curves is $\frac{\alpha}{k}$.

Symmetric case

Symmetry (and simplicity of λ_N) $\rightsquigarrow$ u_N is either $\underset{\alpha=\frac{\pi}{2}}{\overset{\uparrow}{\text{even}}}$ or $\underset{\alpha=0}{\overset{\uparrow}{\text{odd}}}$ in x_2

Theorem [Abatangelo-F.-Hillairet-Léna, J. Spectr. T., to appear]

Let u_N be even in x_2. Then

$$\text{if } k=0, \quad \lambda_N^a = \lambda_N + \frac{2\pi |u_N(0)|^2}{|\log a|} + o\left(\frac{1}{|\log a|}\right), \qquad \text{as } a \to 0^+,$$

$$\text{if } k \geq 1, \quad \lambda_N^a = \lambda_N + \frac{k\pi\beta^2}{4^{k-1}} \binom{k-1}{\left\lfloor \frac{k-1}{2} \right\rfloor}^2 a^{2k} + o\left(a^{2k}\right), \quad \text{as } a \to 0^+.$$

Theorem [Abatangelo-F.-Léna, preprint 2018]

Let u_N be odd in x_2. Then

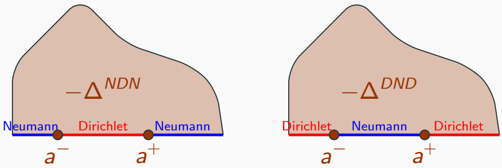
$$\lambda_N^a = \lambda_N - \frac{k\pi\beta^2}{4^{k-1}} \binom{k-1}{\left\lfloor \frac{k-1}{2} \right\rfloor}^2 a^{2k} + o\left(a^{2k}\right), \quad \text{as } a \to 0^+.$$

Idea of the proof.

1. Isospectrality. The sequence $\{\lambda_k^a\}_{k\geq 1}$ is the union, counted with multiplicities, of sequences $\{\lambda_k^{NDN}(a^+, a^-)\}_{k\geq 1}$, $\{\lambda_k^{DND}(a^+, a^-)\}_{k\geq 1}$.

$$\{\lambda_k^{NDN}(a^+, a^-)\}_{k\geq 1} = \left\{ \begin{array}{l} \text{eigenvalues of Neumann-Dirichlet-Neumann} \\ \text{Laplacian } -\Delta^{NDN} \text{ on the half domain} \end{array} \right\}$$

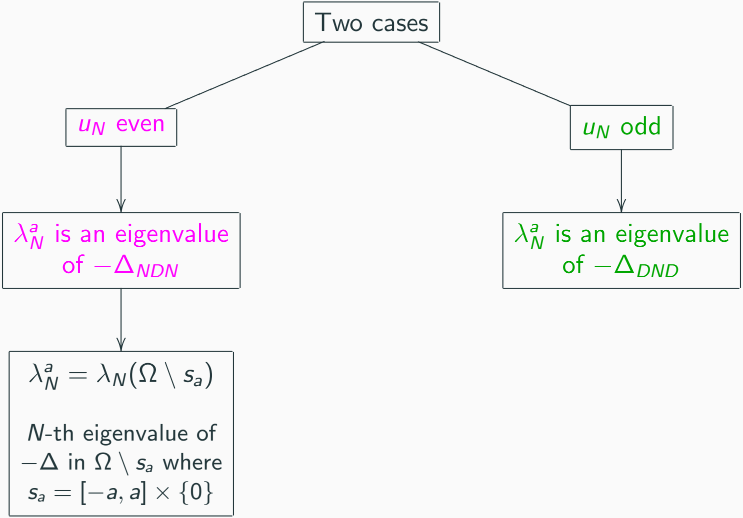
$$\{\lambda_k^{DND}(a^+, a^-)\}_{k\geq 1} = \left\{ \begin{array}{l} \text{eigenvalues of Dirichlet-Neumann-Dirichlet} \\ \text{Laplacian } -\Delta^{DND} \text{ on the half domain} \end{array} \right\}$$



See **[Bonnaillie-Noël-Helffer-Hoffmann-Ostenhof, J. Phys. A (2009)]** for isospectrality results for a single pole.

Idea of the proof.

2.



Even case

In the even case, the problem reduces to the study of the asymptotics of

$$\lambda_N(\Omega \setminus s_a) \quad \text{as } a \to 0^+.$$

Problem: spectral theory for boundary value problems for the Laplacian (with homogeneous Dirichler boundary conditions) on domains which are perturbed by removing a "small" set (a tubular neighborhood of a submanifold, a small ball, a crack...) from a fixed Ω.

What is the good notion of "smallness"?

Eigenvalues under removal of small capacity sets

Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set and $K \subset \Omega$ compact.

(Condenser) capacity of K in Ω

$$\mathrm{Cap}_\Omega(K) = \inf \left\{ \int_\Omega |\nabla f|^2 : \ f \in H^1_0(\Omega) \text{ and } f - \eta_K \in H^1_0(\Omega \setminus K) \right\},$$

η_K being a fixed smooth function such that $\operatorname{supp} \eta_K \subset \Omega$, $\eta_K \equiv 1$ in a neighborhood of K.

The infimum is achieved by the capacitary potential V_K, unique solution to

$$\begin{cases} -\Delta V_K = 0, & \text{in } \Omega \setminus K, \\ V_K = 0, & \text{on } \partial\Omega, \\ V_K = 1, & \text{on } K. \end{cases}$$

Eigenvalues under removal of small capacity sets

Rauch-Taylor, JFA (1975):

- the spectrum of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$ does not change when a zero capacity compact set is removed from Ω
- *Fireman's pole problem:* if $\Omega_\varepsilon = \Omega \setminus K_\varepsilon$ with K_ε *"concentrating"* at some closed subset K, then the resolvents of $-\Delta$ on Ω_ε converge to the resolvents of $-\Delta$ on Ω if K has zero capacity

  (**Example**: firehouse $= \Omega \subset \mathbb{R}^3$, $K_\varepsilon =$one fireman's pole $=$ cylinder of radius ε, $K =$a line)

Perturbation theory

Courtois, JFA (1995):

perturbation theory for the Dirichlet spectrum of $\Omega \setminus K$ (being K a "small" compact set) with the capacity of K as the perturbation parameter

- if $K \subset \Omega$ is a compact set, the N-th Dirichlet eigenvalue $\lambda_N(\Omega \setminus K)$ in $\Omega \setminus K$ is close to $\lambda_N(\Omega)$ if (and only if) the capacity of K in Ω is close to zero
- if the capacity of K in Ω is close to zero then the function $\lambda_N(\Omega \setminus K) - \lambda_N(\Omega)$ is even differentiable with respect to the capacity of K in Ω.

Perturbation theory

Theorem [Courtois (1995)]

$$\lambda_N(\Omega \setminus K) = \lambda_N(\Omega) + \mathrm{Cap}_\Omega(K)\mu_K(u_N^2) + o(\mathrm{Cap}_\Omega(K))$$

as $\mathrm{Cap}_\Omega(K) \to 0$, where μ_K is a finite positive probability measure supported in K defined as the renormalized singular part of ΔV_K.

- Proved in [Flucher, JMAA (1995)] in that case of a family of compact sets $\{K_\varepsilon\}_{\varepsilon>0}$ concentrating to a point.
- Sharp asymptotics of $\lambda_N(\Omega \setminus K)$ as $\mathrm{Cap}_\Omega(K) \to 0$ if $\mu_K(u_N^2) \not\to 0$
  Example: if $\{K_\varepsilon\}_{\varepsilon>0}$ concentrates to a point x_0, $\mu_{K_\varepsilon}(u_N^2) \to u_N^2(x_0)$, so the above theorem gives the sharp asymptotics only if $u_N(x_0) \neq 0$.
- **Problem**: sharp asymptotic expansion of the eigenvalue variation when $\mu_K(u_N^2) \to 0$? E.g., when $\{K_\varepsilon\}_{\varepsilon>0}$ concentrates to a point x_0 at which the eigenfunction vanishes?

The u-capacity (Dirichlet capacity)

Let $u \in H^1_0(\Omega)$.

u-capacity of K in Ω

$$\mathrm{Cap}_\Omega(K,u) = \inf_{\substack{f \in H^1_0(\Omega) \\ f-u \in H^1_0(\Omega \setminus K)}} \int_\Omega |\nabla f|^2 = \int_\Omega |\nabla V_{K,u}|^2$$

The infimum is achieved by $V_{K,u}$, unique solution to

$$\begin{cases} -\Delta V_{K,u} = 0, & \text{in } \Omega \setminus K, \\ V_{K,u} = 0, & \text{on } \partial\Omega, \\ V_{K,u} = u, & \text{on } K, \end{cases} \quad \text{i.e.} \quad \int_{\Omega\setminus K} \nabla V_{K,u} \cdot \nabla \varphi \, dx = 0 \ \forall \varphi \in H^1_0(\Omega \setminus K)$$

The u-capacity (Dirichlet capacity)

Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in Ω.

We say that K_ε *is concentrating to a compact set* $K \subset \Omega$
if $\forall$ U open set s.t. $K \subseteq U \subseteq \Omega$ $\exists$ $\varepsilon_U > 0$ s.t. $U \supset K_\varepsilon$ $\forall$ $\varepsilon < \varepsilon_U$.

Let $\{K_\varepsilon\}_{\varepsilon>0}$ concentrate to a compact set $K \subset \Omega$ such that one of the two following conditions hold:

(i) $\mathrm{Cap}_\Omega(K) = 0$;

(ii) $K = \bigcap_{\varepsilon>0} K_\varepsilon$ where K_ε is decreasing as $\varepsilon \to 0$

Then, for all $f \in H^1_0(\Omega)$, $\lim_{\varepsilon\to 0^+} \mathrm{Cap}_\Omega(K_\varepsilon, f) = \mathrm{Cap}_\Omega(K, f)$ (and $H^1_0(\Omega)$-convergence of the potentials).
In particular $\lim_{\varepsilon\to 0^+} \mathrm{Cap}_\Omega(K_\varepsilon) = \mathrm{Cap}_\Omega(K)$.

Sharp asymptotic expansion in terms of the u_N-capacity

Theorem [Courtois (1995)]

Let $\lambda_N(\Omega)$ be a simple eigenvalue of the Dirichlet Laplacian in a bounded, connected, and open set $\Omega \subset \mathbb{R}^n$ and let u_N be a $L^2(\Omega)$-normalized eigenfunction associated to $\lambda_N(\Omega)$. Let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact sets in Ω concentrating to a compact set K with $\mathrm{Cap}_\Omega(K) = 0$. Then

$$\lambda_N(\Omega \setminus K_\varepsilon) = \lambda_N(\Omega) + \mathrm{Cap}_\Omega(K_\varepsilon, u_N) + o\left(\mathrm{Cap}_\Omega(K_\varepsilon, u_N)\right), \quad \text{as } \varepsilon \to 0.$$

[Bertrand-Colbois, JFA (2006)]: estimates from above and below of the eigenvalue variation in terms of the u_1-capacity.

Asymptotics of $\mathrm{Cap}_\Omega(K_\varepsilon, u_N)$ as $\varepsilon \to 0$?

Compact sets concentrating to a point $\{x_0\}$ $\Big(\mathrm{Cap}_\Omega(\{x_0\}) = 0$ if $n \geq 2\Big)$

If the eigenfunction u_N does not vanish at the limit point, then the u_N-capacity is asymptotic to the condenser capacity (up to a constant):

Proposition

If $(K_\varepsilon)_{\varepsilon>0}$ concentrates to a point $x_0 \in \Omega$ such that $u_N(x_0) \neq 0$, then

$$\mathrm{Cap}_\Omega(K_\varepsilon, u_N) = u_N^2(x_0)\mathrm{Cap}_\Omega(K_\varepsilon) + o(\mathrm{Cap}_\Omega(K_\varepsilon)), \quad \text{as } \varepsilon \to 0.$$

Then, if $u_N(x_0) \neq 0$, the problem of asymptotics of $\lambda_N(\Omega \setminus K_\varepsilon) - \lambda_N(\Omega)$ is reduced to the study of the behaviour of $\mathrm{Cap}_\Omega(K_\varepsilon)$.

Asymptotics of $\mathrm{Cap}_\Omega(K_\varepsilon, u_N)$ as $\varepsilon \to 0$?

In dimension 2: sharp asymptotics of $\mathrm{Cap}_\Omega(K_\varepsilon)$ for compact connected sets concentrating to a point in terms of their diameter

Proposition [Abatangelo-F.-Hillairet-Léna, JST, to appear]

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set. Let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact connected sets concentrating to a point. Then

$$\mathrm{Cap}_\Omega(K_\varepsilon) = \frac{2\pi}{|\log(\mathrm{diam}\, K_\varepsilon)|} + O\left(\frac{1}{\log^2(\mathrm{diam}\, K_\varepsilon)}\right), \quad \text{as } \varepsilon \to 0^+.$$

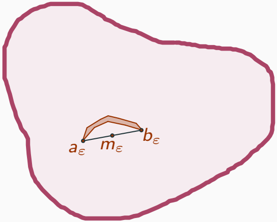
$$\Downarrow$$

Theorem [Abatangelo-F.-Hillairet-Léna, JST, to appear]

If $\lambda_N(\Omega)$ is simple, u_N a $L^2(\Omega)$-normalized associated eigenfunction u_N, and $(K_\varepsilon)_{\varepsilon>0}$ concentrates to a point $x_0 \in \Omega$ such that $u_N(x_0) \neq 0$, then

$$\lambda_N(\Omega \setminus K_\varepsilon) - \lambda_N(\Omega) = \frac{2\pi u_N^2(x_0)}{|\log(\mathrm{diam}\, K_\varepsilon)|} + o\left(\frac{1}{|\log(\mathrm{diam}\, K_\varepsilon)|}\right), \text{ as } \varepsilon \to 0^+.$$

Idea of the proof. Let $a_\varepsilon, b_\varepsilon \in K_\varepsilon$ be such that $|b_\varepsilon - a_\varepsilon| = \operatorname{diam} K_\varepsilon =: \delta_\varepsilon$. Let $m_\varepsilon = \frac{1}{2}(a_\varepsilon + b_\varepsilon) \to x_0$ as $\varepsilon \to 0$.



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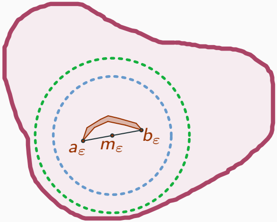

Let $R > 0$ s.t. $B(m_\varepsilon, R) \subset \Omega$.
$K_\varepsilon \subseteq \overline{B}(m_\varepsilon, \delta_\varepsilon) \Rightarrow$

$$\mathrm{Cap}_\Omega(K_\varepsilon) \leq \mathrm{Cap}_{B(m_\varepsilon,R)} \overline{B}(m_\varepsilon, \delta_\varepsilon) = \frac{2\pi}{\log(R/\delta_\varepsilon)}$$

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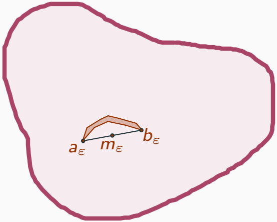

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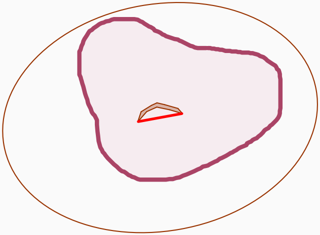


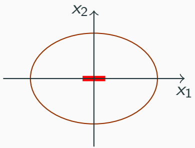
$\exists L > 0$ such that $\Omega \subset \mathcal{E}_\varepsilon$, where $\mathcal{E}_\varepsilon$ is the interior of the ellipse

- centered at m_ε
- major semi-axis $= \sqrt{L^2 + \frac{1}{4}\delta_\varepsilon^2}$
- major axis on the straight line $a_\varepsilon b_\varepsilon$
- minor semi-axis of length L

$\mathrm{Cap}_{\mathcal{E}_\varepsilon}(K_\varepsilon) \leq \mathrm{Cap}_\Omega(K_\varepsilon)$, Steiner symmetrization on the potential V_{K_ε} $\Rightarrow \mathrm{Cap}_{\mathcal{E}_\varepsilon}(s_\varepsilon) \leq \mathrm{Cap}_{\mathcal{E}_\varepsilon}(K_\varepsilon)$

Then $\mathrm{Cap}_\Omega(K_\varepsilon) \geq \mathrm{Cap}_{\mathcal{E}_\varepsilon}(s_\varepsilon)$.

Evaluate $\mathrm{Cap}_{\mathcal{E}_\varepsilon}(s_\varepsilon)$.



After a roto-translation, we can assume that $s_\varepsilon = \left[-\frac{\delta_\varepsilon}{2}, \frac{\delta_\varepsilon}{2}\right] \times \{0\}$ and that the major-axis of the ellipse is the x_1-axis in the Cartesian coordinates (x_1, x_2).

Elliptic coordinates (ξ, η): $\begin{cases} x_1 = \frac{\delta_\varepsilon}{2}\cosh(\xi)\cos(\eta), \\ x_2 = \frac{\delta_\varepsilon}{2}\sinh(\xi)\sin(\eta). \end{cases}$

$s_\varepsilon \rightsquigarrow \{\xi = 0\}$, $\mathcal{E}_\varepsilon \rightsquigarrow \{\xi < \xi_\varepsilon\}$, with $\xi_\varepsilon = \log\left(\frac{2L}{\delta_\varepsilon} + \sqrt{1 + \frac{4L^2}{\delta_\varepsilon^2}}\right)$.

The mapping $\Phi : (\xi, \eta) \mapsto (x_1, x_2)$ is conformal.

Let $W(\xi, \eta) = V_{s_\varepsilon}(\Phi(\xi, \eta))$

$$\begin{cases} -\Delta V_{s_\varepsilon} = 0, & \text{in } \Omega \setminus s_\varepsilon, \\ V_{s_\varepsilon} = 0, & \text{on } \partial\mathcal{E}_\varepsilon, \\ V_{s_\varepsilon} = 1, & \text{on } s_\varepsilon, \end{cases} \quad \rightsquigarrow \quad \begin{cases} -\Delta W = 0, & \text{in } (0, \xi_\varepsilon) \times (0, 2\pi), \\ W = 0, & \text{on } \xi = \xi_\varepsilon, \\ W = 1, & \text{on } \xi = 0, \end{cases}$$

$$W(\xi, \eta) = 1 - \frac{\xi}{\xi_\varepsilon} \;\Rightarrow\; \mathrm{Cap}_{\mathcal{E}_\varepsilon}(s_\varepsilon) = \int_{\mathcal{E}_\varepsilon} |\nabla V_{s_\varepsilon}|^2 \, dx_1 \, dx_2 = \int_{(0,\xi_\varepsilon)\times(0,2\pi)} |\nabla W|^2 \, d\xi \, d\eta$$

$$= \frac{2\pi}{\xi_\varepsilon} = \frac{2\pi}{|\log \delta_\varepsilon|}\left(1 + O\left(\frac{1}{|\log \delta_\varepsilon|}\right)\right) \quad \text{as } \varepsilon \to 0^+.$$

If $u_N(x_0) = 0$? $\mathrm{Cap}_\Omega(K_\varepsilon, u_N)$ for segments in dimension 2

If the eigenfunction u_N vanishes at the limit point, then the asymptotics of $\mathrm{Cap}_\Omega(K_\varepsilon, u_N)$ depends on the order of vanishing of u_N at that point.

Let u_N be an eigenfunction of the Dirichlet Laplacian in Ω, $x_0 = 0 \in \Omega$,
$\Rightarrow \exists\ k \in \mathbb{N}$, $\beta \in \mathbb{R} \setminus \{0\}$ and $\alpha \in [0, \pi)$ such that

$$u_N(r(\cos t, \sin t)) \sim \beta r^k \sin(\alpha - kt), \quad \text{as } r \to 0^+.$$

Theorem [Abatangelo-F.-Hillairet-Léna, JST, to appear]

Let $s_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\}$.

(i) If $\alpha \neq 0$ (s_ε is not tangent to a nodal line of u_N), then

$$\mathrm{Cap}_\Omega(s_\varepsilon, u_N) = \begin{cases} \frac{2\pi}{|\log \varepsilon|}\, u_N^2(0)\,(1 + o(1)), & \text{if } k = 0, \\ \varepsilon^{2k}\, \pi\, \beta^2 \sin^2 \alpha\, C_k(1 + o(1)), & \text{if } k \geq 1, \end{cases}$$

as $\varepsilon \to 0^+$, being $C_k > 0$.

(ii) If $\alpha = 0$, then $\mathrm{Cap}_\Omega(s_\varepsilon, u_N) = O\left(\varepsilon^{2k+2}\right)$ as $\varepsilon \to 0^+$.

Eigenvalue asymptotics in $\Omega \setminus s_\varepsilon$

Theorem [Abatangelo-F.-Hillairet-Léna, JST, to appear]

Let $\lambda_N(\Omega)$ be simple and let u_N be a $L^2(\Omega)$-normalized associated eigenfunction satisfying

$$u_N(r(\cos t, \sin t)) \sim \beta r^k \sin(\alpha - kt)$$

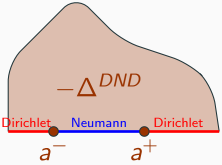
as $r \to 0^+$. Let $s_\varepsilon = [-\varepsilon, \varepsilon] \times \{0\}$. Then, as $\varepsilon \to 0^+$,

$$\lambda_N(\Omega \setminus s_\varepsilon) - \lambda_N(\Omega) = \begin{cases} \frac{2\pi}{|\log \varepsilon|} \, u_N^2(0)\,(1 + o(1)), & \text{if } k = 0,\ \alpha \neq 0, \\ \varepsilon^{2k} \, \pi \, \beta^2 \sin^2 \alpha \, C_k (1 + o(1)), & \text{if } k \geq 1,\ \alpha \neq 0, \\ O\left(\varepsilon^{2k+2}\right), & \text{if } \alpha = 0. \end{cases}$$

Odd case

In the odd case, the problem reduces to the study of the asymptotics of

$$\{\lambda_k^{DND}(a^+, a^-)\}_{k\geq 1} = \left\{ \begin{array}{l} \text{eigenvalues of Dirichlet-Neumann-Dirichlet} \\ \text{Laplacian } -\Delta^{DND} \text{ on the half domain} \end{array} \right\}$$



DND problem

Let Ω be a bounded Lipschitz open set in $\mathbb{R}^2$ such that

$$\exists\, \varepsilon_0 > 0 \text{ such that } \Gamma_{\varepsilon_0} := [-\varepsilon_0, \varepsilon_0] \times \{0\} \subset \partial\Omega.$$

For each $a \in (0, \varepsilon_0]$, we consider the eigenvalue problem

$$\begin{cases} -\Delta\, u = \lambda\, u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \setminus \Gamma_a, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_a := [-a, a] \times \{0\}. \end{cases}$$

Let $(\lambda_j(a))_{j\geq 1}$ be the eigenvalues.

Proposition

For each integer $N \geq 1$, $\lambda_N(a) \to \lambda_N$ as $a \to 0$.

Related spectral stability results: **[Colorado and Peral, JFA (2003)]**, first eigenvalue under mixed Dirichlet-Neumann boundary conditions on a smooth bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 3$), both for vanishing Dirichlet boundary portion and for vanishing Neumann boundary portion.

DND problem

Let us assume that λ_N is simple and let u_N be an associate normalized eigenfunction, i.e.

$$\begin{cases} -\Delta u_N = \lambda_N u_N, & \text{in } \Omega, \\ u_N = 0, & \text{on } \partial\Omega, \\ \int_\Omega u_N^2(x)\,dx = 1. \end{cases}$$

Then there exist $k \in \mathbb{N} \setminus \{0\}$ and $\beta \in \mathbb{R} \setminus \{0\}$ such that

$$u_N(r\cos t, r\sin t) \underset{r\to 0}{\sim} \beta\, r^k \sin(kt).$$

Theorem [Abatangelo-F.-Léna, preprint 2018]

$$\lim_{a\to 0^+} \frac{\lambda_N - \lambda_N(a)}{a^{2k}} = \beta^2 \frac{k\pi}{2^{2k-1}} \binom{k-1}{\lfloor \frac{k-1}{2} \rfloor}^2$$

Idea of the proof

- We estimate from above and below the Rayleigh quotient, with proper test functions constructed by suitable manipulation of eigenfunctions.
- We obtain upper and lower bounds whose limit as $a \to 0$ can be explicitly computed taking advantage of a blow-up analysis for scaled eigenfunctions.

Idea of the proof

Blow-up Theorem [Abatangelo-F.-Léna, preprint 2018]

For a small, let u_N^a be an eigenfunction associated to $\lambda_N(a)$ satisfying $\int_\Omega |u_N^a|^2\,dx = 1$ and $\int_\Omega u_N^a\, u_N\, dx \geq 0$. Then

$$a^{-k} u_N^a(ax) \underset{a\to 0}{\to} \beta(\psi_k + W_k \circ F^{-1}) \quad \begin{array}{l} \text{in } H^1(D_R^+)\ \forall R > 1 \\ \text{in } C^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+} \setminus \{(\pm 1, 0)\}) \end{array}$$

where $\psi_k(r\cos t, r\sin t) = r^k \sin(kt)$, for $t \in [0,\pi]$ and $r > 0$,

$F(\xi,\eta) = (\cosh(\xi)\cos(\eta), \sinh(\xi)\sin(\eta))$, for $\xi \geq 0,\ \eta \in [0, 2\pi)$,

$$W_k(\xi,\eta) = \frac{1}{2^{k-1}} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{j} \exp(-(k-2j)\xi)\sin((k-2j)\eta).$$

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F is the transformation from elliptic to cartesian coordinates (C^∞ conformal diffeomorphism from $(0,+\infty)\times(0,\pi)$ to $\mathbb{R}^2_+$).

Idea of the proof

Blow-up Theorem [Abatangelo-F.-Léna, preprint 2018]

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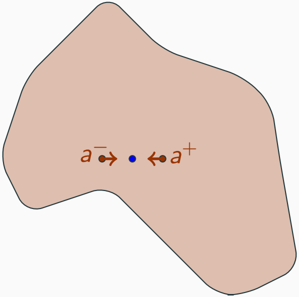
$w_k = W_k \circ F^{-1}$ is the unique finite energy weak solution to

$$\begin{cases} -\Delta w_k = 0, & \text{in } \mathbb{R}^2_+, \\ w_k = 0, & \text{on } \partial\mathbb{R}^2_+ \setminus \Gamma_1, \\ \frac{\partial w_k}{\partial \nu} = -\frac{\partial \psi_k}{\partial \nu}, & \text{on } \Gamma_1. \end{cases}$$

Non symmetric case



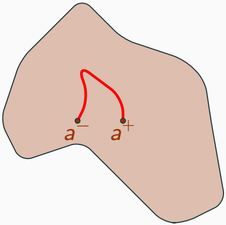
Let Ω be an open, bounded, and connected set in $\mathbb{R}^2$ such that $0 \in \Omega$ (no symmetry assumption).
Let us assume that $\exists\ N \geq 1$ such that the N-th eigenvalue λ_N of the Dirichlet Laplacian in Ω is simple.
Let u_N be a $L^2(\Omega)$-normalized eigenfunction associated to λ_N.

Theorem [Abatangelo-F.-Léna, Advanced Nonlin. Studies (2017)]

If $u_N(0) \neq 0$ (i.e. $k = 0$) then

$$\lambda_N^a - \lambda_N = \frac{2\pi\, u_N^2(0)}{|\log a|}(1 + o(1)) \quad \text{as } a \to 0^+.$$

Idea of the proof.



1. If a is small and u_N^a is an eigenfunction associated with λ_N^a, then, in a neighborhood of 0, the nodal set of u_N^a consists in a single regular curve K_a connecting a^- and a^+ and concentrating around 0.
[Noris-Terracini (2010), Helffer-Hoffmann-Ostenhof-Hoffmann-Ostenhof-Owen (1999), Alziary-Fleckinger-Pellé-Takáč (2003)]

2. For all $a > 0$ sufficiently small, $\lambda_N^a = \lambda_N(\Omega \setminus K_a)$ (Gauge invariance).

3. We denote as $d_a := \operatorname{diam} K_a$ the diameter of K_a. We already know that

$$\lambda_N(\Omega \setminus K_a) - \lambda_N = u_N^2(0)\frac{2\pi}{|\log d_a|} + o\left(\frac{1}{|\log d_a|}\right), \quad \text{as } a \to 0^+.$$

It remains to estimate d_a, i.e. the diameter of nodal lines of magnetic eigenfunctions near the collision point.

Estimate of d_a.

Estimates from above the Rayleigh quotient for λ_N^a (computed at some proper test functions constructed by suitable manipulation of limit eigenfunctions) $\rightsquigarrow$

$$\forall \tau \in (0,1) \quad \lambda_N^a \le \lambda_N + \frac{2\pi}{(1-\tau)|\log a|}\Big(u_N^2(0) + o(1)\Big) \quad \text{as } a \to 0^+.$$

Combining with the estimate

$$\lambda_N^a - \lambda_N = u_N^2(0)\frac{2\pi}{|\log d_a|} + o\left(\frac{1}{|\log d_a|}\right), \quad \text{as } a \to 0^+, \qquad (*)$$

we obtain $\frac{1}{|\log d_a|}\big(1+o(1)\big) \le \frac{1}{(1-\tau)|\log a|}\big(1+o(1)\big)$ and then

$$\frac{|\log a|}{|\log d_a|} \le \frac{1}{(1-\tau)}\big(1+o(1)\big), \quad \text{as } a \to 0^+.$$

$a^-, a^+ \in K_a \Rightarrow d_a \ge 2a \Rightarrow |\log a| \ge |\log d_a| + \log 2 \Rightarrow \frac{|\log a|}{|\log d_a|} \ge 1 + o(1)$.

Hence $1 \le \liminf_{a\to 0^+} \frac{|\log a|}{|\log d_a|} \le \limsup_{a\to 0^+} \frac{|\log a|}{|\log d_a|} \le \frac{1}{(1-\tau)}$ $\forall \tau \in (0,1)$ and then, letting $\tau \to 0^+$,

$$\lim_{a\to 0^+} \frac{|\log a|}{|\log d_a|} = 1.$$

Replacing into $(*)$ we conclude.