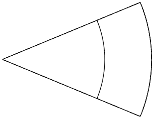
2-partition

$p = 1$

- $\Omega = \Box, \ \bigcirc$?

-  is a ∞-minimal 2-partition but not a 1-minimal 2-partition

- Angular sector with opening $\pi/4$



φ_2: symmetric eigenfunction associated with $\lambda_2(\Omega)$

$$0.37 \simeq \int_{D_1} |\varphi_2|^2 < \int_{D_2} |\varphi_2|^2 \simeq 0.63$$

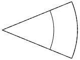
$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$$

2-partition

$p = 1$

- $\Omega = \square, \ \bigcirc$?

-  is a ∞-minimal 2-partition but not a 1-minimal 2-partition

-  is a ∞-minimal 2-partition but not a 1-minimal 2-partition

- The inequality $\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$ is "generically" satisfied

[*Helffer–Hoffmann-Ostenhof*]

Lower bounds

Square, equilateral triangle, disk

$$\left(\frac{1}{k}\sum_{i=1}^{k}\lambda_i(\Omega)^p\right)^{1/p} \leq \mathfrak{L}_{k,p}(\Omega) \leq L_k(\Omega)$$

Explicit eigenvalues for $\square$, $\triangle$, $\bigcirc$

Ω	$\lambda_{m,n}(\Omega)$	m, n
$\square$	$\pi^2(m^2+n^2)$	$m, n \geq 1$
$\triangle$	$\frac{16}{9}\pi^2(m^2+mn+n^2)$	$m, n \geq 1$
$\bigcirc$	$j_{m,n}^2$	$m \geq 0,\ n \geq 1$ (multiplicity)

Upper bounds

Square, disk

$$\mathfrak{L}_{k,p}(\Omega) \leq \Lambda_{k,\infty}(\mathcal{D}_\star)$$

Explicit upper bound for $\bigcirc$

$$\mathfrak{L}_{k,p}(\bigcirc) \leq \lambda_1(\Sigma_{2\pi/k})$$

with $\Sigma_{2\pi/k}$: angular sector of opening $2\pi/k$

Explicit upper bound for $\square$

$$\mathfrak{L}_{k,p}(\square) \leq \inf_{m,n\geq 1}\{\lambda_{m,n}(\square) | mn = k\} \leq \lambda_{k,1}(\square)$$

Properties

Dichotomy for the case $p = \infty$

Let $k > 2$

To determine a ∞-minimal k-partition,

we consider the eigenspace E_k associated with λ_k

Two cases:

- If there exists $u \in E_k$ with k nodal domains, then u produces a minimal k-partition and any minimal k-partition is nodal

$$\mathfrak{L}_{k,\infty}(\Omega) = \lambda_k(\Omega) = L_k(\Omega)$$

[Bipartite case]

- If $\mu(u) < k$ for any $u \in E_k$...

...we have to find another strategy

[Non bipartite case]

Known results in the non bipartite case, $p = \infty$

Sphere and fine flat torus

Theorem

The minimal 3-partition for the sphere is

[*Helffer–Hoffmann-Ostenhof–Terracini*]

Theorem

Let $0 < b \leq a$ *and* $T(a, b) = (\mathbf{R}/a\mathbf{Z}) \times (\mathbf{R}/b\mathbf{Z})$ *the flat torus*

$$\mathcal{D}_k(a, b) = \{]\tfrac{i-1}{k}a, \tfrac{i}{k}a[\times]0, b[, 1 \leq i \leq k\}$$

- k *even and* $\frac{b}{a} \leq \frac{2}{k} \Rightarrow \mathcal{D}_k(a, b)$ *is minimal*
- k *odd and* $\frac{b}{a} < \frac{1}{k} \Rightarrow \mathcal{D}_k(a, b)$ *is minimal* [*Helffer–Hoffmann-Ostenhof*]
- k *odd and* $\frac{1}{k} \leq \frac{b}{a} \leq \ell_* \Rightarrow \mathcal{D}_k(a, b)$ *is minimal* [*BN-Léna 16*]

The question is open for any other domain (in the non bipartite case)

Topological configurations

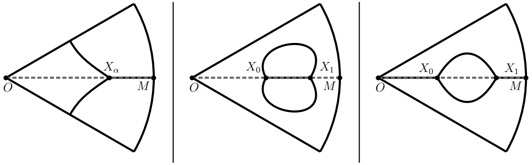
Euler formula

$$k = 1 + b_1 - b_0 + \sum_{\mathbf{x}_i \in X(\partial\mathcal{D})} \left(\frac{\nu(\mathbf{x}_i)}{2} - 1\right) + \frac{1}{2} \sum_{\mathbf{y}_i \in Y(\partial\mathcal{D})} \rho(\mathbf{y}_i)$$

with

b_0	number of components of $\partial\Omega$
b_1	number of components of $\partial\mathcal{D} \cup \partial\Omega$
$\nu(\mathbf{x}_i)$	number of curves ending at $\mathbf{x}_i \in X(\partial\mathcal{D})$
$\rho(\mathbf{y}_i)$	number of curves ending at $\mathbf{y}_i \in Y(\partial\mathcal{D})$

$\Rightarrow$ 3 types of configurations

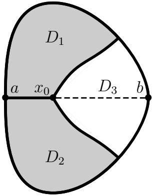


Question

If Ω is symmetric, does it exist a symmetric minimal 3-partition ?

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis



$\mathcal{D} = (D_1, D_2, D_3)$ minimal 3-partition

$\Rightarrow (D_1, D_3)$ minimal 2-partition for $\mathrm{Int}(\overline{D_1} \cup \overline{D_3})$

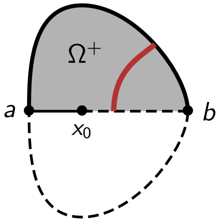
$\Rightarrow$ nodal partition on $\mathrm{Int}(\overline{D_1} \cup \overline{D_3})$

[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis

Introduce a mixed Dirichlet-Neumann problem



$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

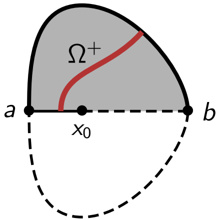
- $(\lambda_2(x_0), \varphi_{x_0})$ second eigenmode
- $x_0 \mapsto \lambda_2(x_0)$ is increasing
- the nodal line starts from (a, b) and reaches the boundary

[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis

Introduce a mixed Dirichlet-Neumann problem



$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

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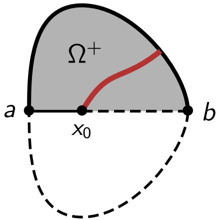
[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis

Introduce a mixed Dirichlet-Neumann problem



$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

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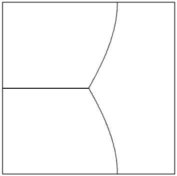
- $x_0 \mapsto \lambda_2(x_0)$ is increasing
- the nodal line starts from (a, b) and reaches the boundary

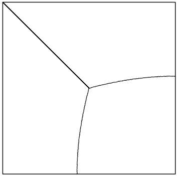
[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

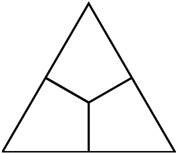
First configuration: examples



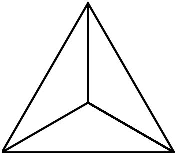
$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 66.581$



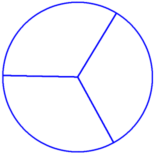
$\Lambda_{3,\infty}(\mathcal{D}_1) \simeq 66.581$

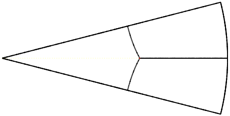


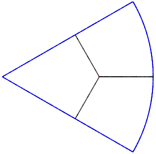
$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 61.872$

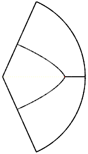


$\Lambda_{3,\infty}(\mathcal{D}_1) \simeq 93.156$



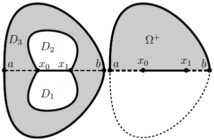






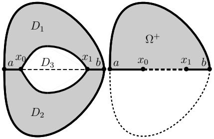
Non bipartite symmetric ∞-minimal 3-partition

Second and third configurations: Two critical points on the symmetry axis



Mixed Neumann-Dirichlet-Neumann problem

$$\left\{\begin{array}{rcll} -\Delta\varphi & = & \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi & = & 0 & \text{on } [a,x_0]\cup[x_1,b] \\ \varphi & = & 0 & \text{elsewhere} \end{array}\right.$$



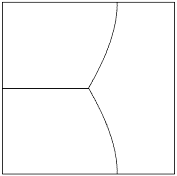
Mixed Dirichlet-Neumann-Dirichlet problem

$$\left\{\begin{array}{rcll} -\Delta\varphi & = & \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi & = & 0 & \text{on } [x_0,x_1] \\ \varphi & = & 0 & \text{elsewhere} \end{array}\right.$$

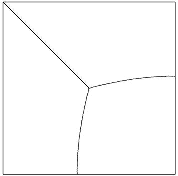
No candidate for the square, disk, angular sectors with two critical points!

∞-minimal 3-partition

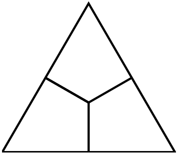
Candidates



$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 66.58$



$\Lambda_{3,\infty}(\mathcal{D}_1) \simeq 66.58$



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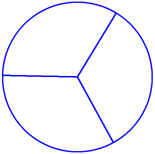
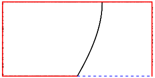
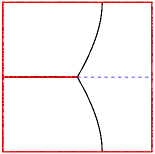


$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 20.20$

Applications



$$0.75 \simeq \int_{D_1} |\varphi_2|^2 > 2 \int_{D_2} |\varphi_2|^2 \simeq 0.51$$



$$\mathfrak{L}_{3,1}(\square) < \Lambda_{3,\infty}(\mathcal{D})$$

Numerical simulations

p-minimal 3-partition for the square

Since $\Lambda_3^{DN} \simeq 66.581$ and $L_3 = 10\pi^2 \simeq 98.696$

$$\lambda_3 < \mathfrak{L}_{3,\infty} < \Lambda_3^{DN}, \qquad \left(\frac{1}{3}\sum_{j=1}^{3}\lambda_j(\square)^p\right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN}$$

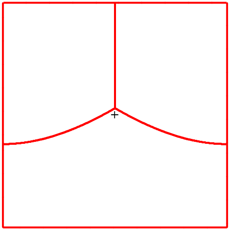
Numerical simulations

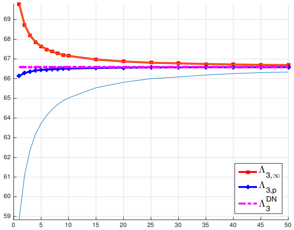
p-minimal 3-partition for the square

$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2 \left(\frac{2^p+5^p+5^p}{3}\right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \qquad \Rightarrow \qquad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 1$





[*Bogosel-BN16*]

Numerical simulations

p-minimal 3-partition for the square

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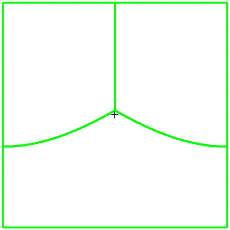
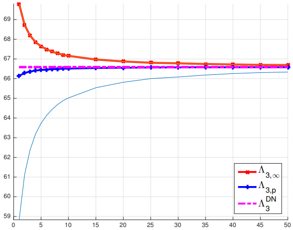
$p = 2$





[*Bogosel-BN16*]

Numerical simulations

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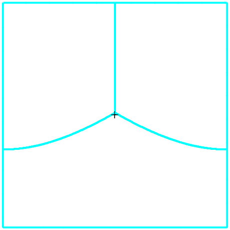
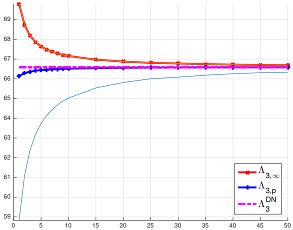
$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2 \left(\frac{2^p+5^p+5^p}{3}\right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \quad \Rightarrow \quad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 5$





[*Bogosel-BN16*]

Numerical simulations

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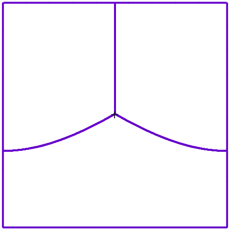
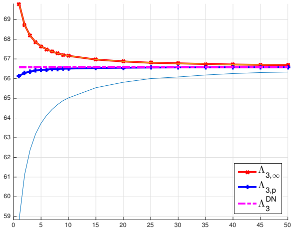
$p = 10$





[Bogosel-BN16]

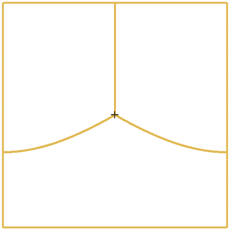
Numerical simulations

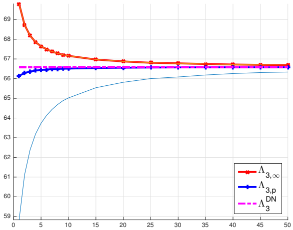
p-minimal 3-partition for the square

$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2 \left(\frac{2^p+5^p+5^p}{3}\right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \qquad \Rightarrow \qquad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 50$





[Bogosel-BN16]

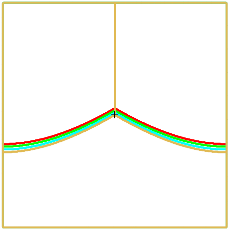
Numerical simulations

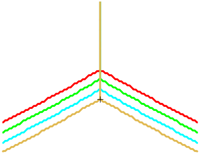
p-minimal 3-partition for the square

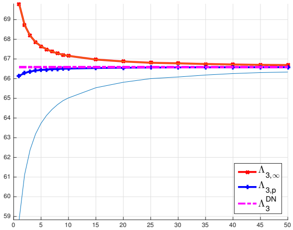
$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2 \left(\frac{2^p+5^p+5^p}{3}\right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \quad \Rightarrow \quad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 1, 2, 5, 50$







[Bogosel-BN16]

Numerical simulations

p-minimal 3-partition

Conjecture

For the square :

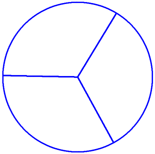
- $p \mapsto \mathfrak{L}_{3,p}(\square)$ is increasing
- $p_\infty(\square, 3) = +\infty$

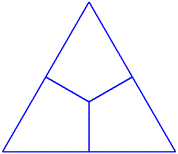
For the disk:

- $p_\infty(\bigcirc, 3) = 1$

For the equilateral triangle:

- $p_\infty(\triangle, 3) = 1$





is a p-minimal 3-partition for any $p \geq 1$

Iterative methods

Penalization

1. Instead of looking for k domains $(D_1, \ldots, D_K)$, we look for a k-upple of functions $(\varphi_1, \ldots, \varphi_k) \in M$ with

$$M = \left\{ (\varphi_1, \ldots, \varphi_k), \varphi_i : \Omega \to [0,1] \text{ measurable }, \sum_{i=1}^{k} \varphi_i = 1 \text{ a.e. } \Omega \right\}.$$

2. Penalized eigenvalue problem on Ω

$$-\Delta v_i + \frac{1}{\varepsilon}(1 - \varphi_i) v_i = \lambda(\varepsilon, \varphi_i) v_i \qquad \text{in } \Omega$$

3. Penalized optimization problem

$$\mathcal{M}(\varepsilon, k) = \inf \left\{ \left(\frac{1}{k} \sum_{i=1}^{k} \lambda_1^p(\varepsilon, \varphi_i) \right)^{1/p}, (\varphi_1, \ldots, \varphi_k) \in M \right\}$$

In some sense

$$\lim_{\varepsilon \to 0} \mathcal{M}(\varepsilon, k) = \mathfrak{L}_{k,p}(\Omega)$$

[Bourdin-Bucur-Oudet 09]

Iterative methods

Penalization

1. Instead of looking for k domains $(D_1, \ldots, D_K)$, we look for a k-upple of functions $(\varphi_1, \ldots, \varphi_k) \in M$ with

$$M = \left\{ (\varphi_1, \ldots, \varphi_k), \varphi_i : \Omega \to [0,1] \text{ measurable }, \sum_{i=1}^{k} \varphi_i = 1 \text{ a.e. } \Omega \right\}.$$

2. Penalized eigenvalue problem on Ω

$$-\Delta v_i + \frac{1}{\varepsilon}(1-\varphi_i) v_i = \lambda(\varepsilon, \varphi_i) v_i \qquad \text{in } \Omega$$

3. Penalized optimization problem

$$\mathcal{M}(\varepsilon, k) = \inf \left\{ \left(\frac{1}{k} \sum_{i=1}^{k} \lambda_1^p(\varepsilon, \varphi_i) \right)^{1/p}, (\varphi_1, \ldots, \varphi_k) \in M \right\}$$

4. Projected-gradient descent with adaptive step

[Bourdin-Bucur-Oudet 09]