A Faber-Krahn inequality for the Robin Laplacian on exterior domains

Vladimir Lotoreichik

in collaboration with D. Krejčiřík

Czech Academy of Sciences, Řež near Prague

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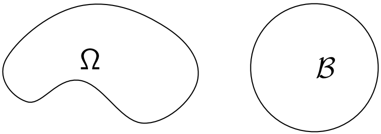
Outline

1. Motivation & background
   - Optimization in bounded domains
   - The Robin Laplacian on an exterior domain
2. Spectral optimization in exterior domains
   - Two dimensions and the role of connectedness
   - Higher dimensions and the Willmore energy
   - Planes with cuts
3. Summary and open questions

Classical isoperimetric inequalities

Geometric setting

Bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with $\mathcal{C}^\infty$-boundary $\partial\Omega$; ball $\mathcal{B} = \mathcal{B}_R \subset \mathbb{R}^d$.



The results for non-smooth domains can be deduced via approximation.

Dirichlet eigenvalues of the Laplacian on Ω

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad \Longrightarrow \quad 0 < \lambda_1^{\rm D}(\Omega) \leq \lambda_2^{\rm D}(\Omega) \leq \lambda_3^{\rm D}(\Omega) \leq \dots$$

Classical geometric and spectral isoperimetric inequalities

$$\begin{cases} |\Omega| = |\mathcal{B}| \\ \Omega \ncong \mathcal{B} \end{cases} \quad \Longrightarrow \quad \begin{cases} |\partial\Omega| > |\partial\mathcal{B}|, & \texttt{J. Steiner, H. Schwarz,...} \\ \lambda_1^{\rm D}(\Omega) > \lambda_1^{\rm D}(\mathcal{B}), & \texttt{Faber-1923, Krahn-1926} \end{cases}$$

Faber-Krahn inequality for other boundary conditions?

Dirichlet BC: $u = 0$ on $\partial\Omega$ (*quantum mechanics,...*)

One of many that give well-posed spectral problem for $-\Delta$ in Ω.

Could one generalise the Faber-Krahn inequality for other BC?

Neumann BC: $\partial_n u = 0$ on $\partial\Omega$ (*heat insulators,..*)

Trivial setting: the lowest eigenvalue $= 0$.

Robin BC: $\partial_n u + \alpha u = 0$ on $\partial\Omega$, $\alpha \in \mathbb{R}$ (*elasticity, superconductivity*)

Non-trivial! In physics, searching for the shape minimizing the critical temperature of the superconductivity (Giorgi-Smits-07).

$\alpha > 0$: quite complete

Bossel-86, Daners-06

Daners-Kennedy-07

Bucur-Giacomini-10

$\alpha < 0$: partial results

Ferone-Nitsch-Trombetti-15

Freitas-Krejčiřík-15

Antunes-Freitas-Krejčiřík-17

The Robin Laplacian on a bounded domain

Robin eigenvalues of the Laplacian on Ω

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \partial_n u + \alpha u = 0, & \text{on } \partial\Omega, \end{cases} \quad \Longrightarrow \quad \lambda_1^\alpha(\Omega) \le \lambda_2^\alpha(\Omega) \le \lambda_3^\alpha(\Omega) \le \dots$$

Underlying self-adjoint operator $-\Delta_\alpha^\Omega$ induced by quadratic form in $L^2(\Omega)$:

$$H^1(\Omega) \ni u \mapsto \int_\Omega |\nabla u|^2 \mathrm{d}x + \alpha \int_{\partial\Omega} |u|^2 \mathrm{d}\sigma(x).$$

$\alpha \mapsto \lambda_1^\alpha(\Omega)$ is increasing with the properties

$\alpha > 0$: $\lambda_1^\alpha(\Omega) \in \big(0, \lambda_1^{\mathrm{D}}(\Omega)\big)$,

$\alpha < 0$: $\lambda_1^\alpha(\Omega) < 0$,

$\alpha \to +\infty$: $\lambda_1^\alpha(\Omega) \to \lambda_1^{\mathrm{D}}(\Omega)$,

$\alpha \to -\infty$: $\lambda_1^\alpha(\Omega) \to -\infty$.

FK-inequality for Robin Laplacian on a bounded domain

The original Faber-Krahn technique fails!

The Bossel-Daners inequality ($\boxed{\alpha > 0}$, Bossel-86, Daners-06)

$$\begin{cases} |\Omega| = |\mathcal{B}| \\ \Omega \not\cong \mathcal{B} \end{cases} \qquad \Longrightarrow \qquad \boxed{\lambda_1^\alpha(\Omega) > \lambda_1^\alpha(\mathcal{B})}$$

Flipped inequality ($d = 2$, $\boxed{\alpha < 0}$, Antunes-Freitas-Krejčiřík-17)

$$|\partial\Omega| = |\partial\mathcal{B}| \qquad \Longrightarrow \qquad \boxed{\lambda_1^\alpha(\Omega) \leq \lambda_1^\alpha(\mathcal{B})}$$

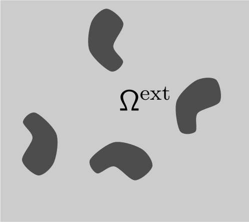
"Hot" open questions for $\alpha < 0$:

- $|\Omega| = |\mathcal{B}|$: the inequality is wrong for $d \geq 2$, might be true for simply connected domains in $\mathbb{R}^2$.
- $|\partial\Omega| = |\partial\mathcal{B}|$: the inequality is wrong for $d \geq 3$, might be true for convex domains in $\mathbb{R}^d$, $d \geq 3$.

The Robin Laplacian on an exterior domain

Exterior domain

$\Omega^{\rm ext} := \mathbb{R}^d \setminus \overline{\Omega}$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain, having $N_\Omega < \infty$ simply connected components.



$\Omega^{\rm ext}$ is connected, unbounded and with compact boundary.

Self-adjoint operator $-\Delta_\alpha^{\Omega^{\rm ext}}$ induced by the quadratic form in $L^2(\Omega^{\rm ext})$:

$$H^1(\Omega^{\rm ext}) \ni u \mapsto \int_{\Omega^{\rm ext}} |\nabla u|^2 {\rm d}x + \alpha \int_{\partial\Omega} |u|^2 {\rm d}\sigma(x).$$

For $\alpha < 0$, spectral optimization is also meaningful for $\Omega^{\rm ext}$.

Spectral portrait of $-\Delta_\alpha^{\Omega^{\rm ext}}$

$$\lambda_1^\alpha(\Omega^{\rm ext}) := \inf \sigma(-\Delta_\alpha^{\Omega^{\rm ext}}).$$

- $\sigma_{\rm ess}(-\Delta_\alpha^{\Omega^{\rm ext}}) = [0,\infty)$.
- $\lambda_1^\alpha(\Omega^{\rm ext}) \to -\infty$ as $\alpha \to -\infty$.

Proposition

(i) *$d = 2$: $\lambda_1^\alpha(\Omega^{\rm ext}) < 0$ if, and only if, $\alpha < 0$.*

(ii) *$d \geq 3$: $\lambda_1^\alpha(\Omega^{\rm ext}) < 0$ if, and only if, $\alpha < \alpha_\star(\Omega^{\rm ext}) < 0$.*

(i) Min-max with an explicit test function.

(ii) Gagliardo-Nirenberg-Sobolev inequality for $\Omega^{\rm ext}$ (Lu-Ou-05).

Why spectral shape optimization for $-\Delta_\alpha^{\Omega^{\rm ext}}$?

New geometric setting: optimization in unbounded domains

Most of the known results are for bounded domains.
(A. Henrot, Birkhäuser, 2006 & De Gruyter Open, 2017).

Robin boundary condition with $\alpha < 0$ is crucial

For Dirichlet/Neumann/Robin with $\alpha > 0$ the problem is meaningless because the lowest spectral point is always zero.

Interplay with the continuous spectrum

Optimization of novel spectral quantities like $\alpha_\star(\Omega^{\rm ext})$.

Spectral isoperimetric inequality for exterior planar domains

Theorem (Krejčiřík-VL-17, $d = 2$, $\boxed{\alpha < 0}$)

$$\frac{|\partial\Omega|}{N_\Omega} = |\partial\mathcal{B}| \quad \Longrightarrow \quad \boxed{\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\mathcal{B}^{\rm ext})}$$

$\lambda_1^\alpha(\Omega^{\rm ext}) < \lambda_1^\alpha(\mathcal{B}^{\rm ext})$ for convex $\Omega \not\cong \mathcal{B}$ (convexity might be redundant).

For all $u_\star \in L^2(\Omega^{\rm ext})$, $u_\star \ne 0$, with $\nabla u_\star \in L^2(\Omega^{\rm ext})$

$$\lambda_1^\alpha(\Omega^{\rm ext}) \le \frac{\displaystyle\int_{\Omega^{\rm ext}} |\nabla u_\star|^2 + \alpha \int_{\partial\Omega^{\rm ext}} |u_\star|^2}{\displaystyle\int_{\Omega^{\rm ext}} |u_\star|^2}$$

(The min-max principle)

How to find $u_\star$ such that the Rayleigh quotient $\le \lambda_1^\alpha(\mathcal{B}^{\rm ext})$?

Method inspired by the upper bound on the 1$^{\rm st}$ Dirichlet eigenvalue given in Payne-Weinberger-61.

Sketch of the proof (in two dimensions)

$\Omega, \mathcal{B} \subset \mathbb{R}^2$, $N := N_\Omega$, $\frac{|\partial\Omega|}{N} = |\partial\mathcal{B}|$, and $\alpha < 0$.

- Ground-state $v\colon \mathcal{B}^{\rm ext} \to (0,\infty)$ of $-\Delta_\alpha^{\mathcal{B}^{\rm ext}}$ is radial $v(x) = F(|x| - R)$.
- Define $u_\star\colon \Omega^{\rm ext} \to (0,\infty)$ by $u_\star(x) := F(\mathrm{dist}(x, \partial\Omega))$.
- The co-area formula and $\int_{\partial\Omega} \kappa = 2\pi N$ yield (after some work)

$$\int_{\Omega^{\rm ext}} |u_\star|^2 \le N \int_{\mathcal{B}^{\rm ext}} |v|^2, \qquad \int_{\Omega^{\rm ext}} |\nabla u_\star|^2 \le N \int_{\mathcal{B}^{\rm ext}} |\nabla v|^2,$$

$$\int_{\partial\Omega^{\rm ext}} |u_\star|^2 = N \int_{\partial\mathcal{B}^{\rm ext}} |v|^2.$$

- The min-max principle gives

$$\lambda_1^\alpha(\Omega^{\rm ext}) \le \frac{\int_{\Omega^{\rm ext}} |\nabla u_\star|^2 + \alpha \int_{\partial\Omega^{\rm ext}} |u_\star|^2}{\int_{\Omega^{\rm ext}} |u_\star|^2} \le \frac{\int_{\mathcal{B}^{\rm ext}} |\nabla v|^2 + \alpha \int_{\partial\mathcal{B}^{\rm ext}} |v|^2}{\int_{\mathcal{B}^{\rm ext}} |v|^2} = \lambda_1^\alpha(\mathcal{B}^{\rm ext}).$$

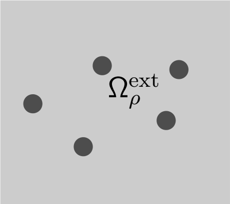
Necessity of N_Ω in the constraint

It is impossible to replace $\frac{|\partial\Omega|}{N_\Omega} = |\partial\mathcal{B}_R|$ by $|\partial\Omega| = |\partial\mathcal{B}_R|$.

Union of $N \geq 2$ disjoint disks

$\Omega_\rho = \cup_{n=1}^{N} \mathcal{B}_\rho(x_n)$; $|x_n - x_m| > 2\rho$, $n \neq m$

$|\partial\Omega_\rho| = |\partial\mathcal{B}_R| \implies \rho = \frac{R}{N}$



Strong coupling $\boxed{\alpha \to -\infty}$ (Pankrashkin-Popoff-16)

$$\lambda_1^\alpha(\Omega_\rho^{\mathrm{ext}}) - \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}}) = |\alpha| \left(\frac{1}{\rho} - \frac{1}{R}\right) + o(\alpha) = |\alpha| \frac{N-1}{R} + o(\alpha).$$

For sufficiently large $|\alpha|$

The inequality flips $\lambda_1^\alpha(\Omega_\rho^{\mathrm{ext}}) > \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}})$.

Spectral isochoric inequality for exterior planar domains

Proposition (Krejčiřík-VL-17, $d = 2$, $\boxed{\alpha < 0}$)

$$\begin{cases} |\Omega| = |\mathcal{B}| \\ N_\Omega = 1, \Omega \ncong \mathcal{B} \end{cases} \qquad \Longrightarrow \qquad \boxed{\lambda_1^\alpha(\Omega^{\rm ext}) < \lambda_1^\alpha(\mathcal{B}^{\rm ext})}$$

Proof.

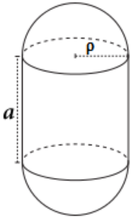
- Let $\widehat{\mathcal{B}}$ be a disk such that $|\partial\Omega| = |\partial\widehat{\mathcal{B}}|$.
- Then $|\widehat{\mathcal{B}}| > |\mathcal{B}|$ and $\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\widehat{\mathcal{B}}^{\rm ext})$.
- Explicit computations give $\lambda_1^\alpha(\widehat{\mathcal{B}}^{\rm ext}) < \lambda_1^\alpha(\mathcal{B}^{\rm ext})$. □

Trick fails for bounded domains: reverse monotonicity $\lambda_1^\alpha(\widehat{\mathcal{B}}) > \lambda_1^\alpha(\mathcal{B})$.

The constraint $|\partial\Omega| = |\partial\mathcal{B}|$ is "wrong" for $d \geq 3$

Long cylinder with 2 hemispherical caps

$\Omega_{\rho,a} = \mathrm{Conv}(\mathcal{B}_\rho(x_0) \cup \mathcal{B}_\rho(x_1))$, where $|x_0 - x_1| = a$.



For any $\rho < R$ exists $a > 0$ such that $|\partial\Omega_{\rho,a}| = |\partial\mathcal{B}_R|$.

Strong coupling $\boxed{\alpha \to -\infty}$

$$\lambda_1^\alpha(\Omega_{\rho,a}^{\mathrm{ext}}) - \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}}) = |\alpha| \left(\frac{d-2}{\rho} - \frac{d-1}{R}\right) + o(\alpha)\,.$$

$\rho < \frac{d-2}{d-1} R$ and $|\alpha|$ sufficiently large: $\lambda_1^\alpha(\Omega_{\rho,a}^{\mathrm{ext}}) > \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}})$.

Convex $\Omega_\star \subset \mathbb{R}^d$, $|\partial\Omega_\star| = |\partial\mathcal{B}_R|$, mean curvature of $\partial\Omega_\star$ attains zero: $\lambda_1^\alpha(\Omega_\star^{\mathrm{ext}}) < \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}})$ for large $|\alpha|$.

In contrast to that, for bounded case, $|\partial\Omega| = |\partial\mathcal{B}|$ is conjectured to be a suitable constraint under convexity.

Curvatures

$\Omega \subset \mathbb{R}^d$, $d \geq 3$ – bounded domain.

Principal curvatures of $\partial\Omega$

$\kappa_1, \kappa_2, \ldots, \kappa_{d-1} \colon \partial\Omega \to \mathbb{R}$ – non-negative for convex Ω.

The mean curvature of $\partial\Omega$

$$M := \frac{\kappa_1 + \kappa_2 + \cdots + \kappa_{d-1}}{d-1} \colon \partial\Omega \to \mathbb{R}\,.$$

Averaged $(d-1)^{\mathrm{st}}$-power of the mean curvature

$$\mathcal{M}(\partial\Omega) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} M^{d-1}(s)\mathrm{d}\sigma(s)\,.$$

For $d = 3$, $\mathcal{M}(\partial\Omega)$ is the ratio of the Willmore energy of $\partial\Omega$ and its area.

Spectral shape optimization for $d \geq 3$

Theorem (Krejčiřík-VL-17, $d \geq 3$, $\alpha < 0$)

$$\begin{cases} \mathcal{M}(\partial\Omega) = \mathcal{M}(\partial\mathcal{B}) \\ \Omega \text{ convex} \end{cases} \quad \Longrightarrow \quad \begin{cases} \lambda_1^\alpha(\Omega^{\rm ext}) \leq \lambda_1^\alpha(\mathcal{B}^{\rm ext}) \\ \alpha_\star(\Omega^{\rm ext}) \geq \alpha_\star(\mathcal{B}^{\rm ext}) \end{cases}$$

Key points in the proof

- Common ideas with the two-dimensional case.
- Higher dimension complicates, but convexity simplifies.
- Gauss-Bonnet formula, Steiner polynomials,...
- Properties of convex bodies: Alexandrov-Fenchel inequality,...

The Robin Laplacian on a plane with a cut

$\Sigma \subset \mathbb{R}^2$ – smooth open arc. $\mathcal{S} \subset \mathbb{R}^2$ – a line segment.

The self-adjoint operator $-\Delta_\alpha^{\mathbb{R}^2\setminus\Sigma}$ in $L^2(\mathbb{R}^2)$ induced by:

$$H^1(\mathbb{R}^2 \setminus \Sigma) \ni u \mapsto \int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \alpha \int_\Sigma \left(|u|_{\Sigma_+}|^2 + |u|_{\Sigma_-}|^2 \right) \mathrm{d}\sigma(x)$$

Basic spectral properties

$\sigma_{\rm ess}(-\Delta_\alpha^{\mathbb{R}^2\setminus\Sigma}) = [0,\infty)$ and $\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma) := \inf \sigma(-\Delta_\alpha^{\mathbb{R}^2\setminus\Sigma}) < 0$, $\forall \alpha < 0$.

Spectral isoperimetric inequality for the plane with a cut

Theorem (VL-18, $d = 2$, $\boxed{\alpha < 0}$)

$$\begin{cases} |\Sigma| = |\mathcal{S}| \\ \Sigma \not\cong \mathcal{S} \end{cases} \quad \Longrightarrow \quad \boxed{\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma) < \lambda_1^\alpha(\mathbb{R}^2 \setminus \mathcal{S})}$$

Key tools for the proof

- Min-max principle.
- Reduction to integral operators in $L^2(\Sigma)$ and $L^2(\mathcal{S})$.
- Line segment is the shortest path connecting two endpoints.

The method inspired by the proof of isoperimetric inequality for the 1^{st}-eigenvalue of Schrödinger operator with δ-interaction on a loop (Exner-Harrell-Loss-06).

In the two-dimensional setting ($d = 2$, $\alpha < 0$)

For connected Ω, the inequality $\lambda_1^\alpha(\Omega^{\mathrm{ext}}) \le \lambda_1^\alpha(\mathcal{B}^{\mathrm{ext}})$ holds if

$$\texttt{length of } \partial\Omega = \texttt{length of } \partial\mathcal{B} \quad \text{or} \quad \texttt{area of } \Omega = \texttt{area of } \mathcal{B}$$

For possibly disconnected Ω, $\lambda_1^\alpha(\Omega^{\mathrm{ext}}) \le \lambda_1^\alpha(\mathcal{B}^{\mathrm{ext}})$ holds if

$$\frac{\texttt{length of } \partial\Omega}{\texttt{number of components in } \Omega} = \texttt{length of } \partial\mathcal{B}$$

For an arc Σ & a line segment $\mathcal{S}$, $\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma) \le \lambda_1^\alpha(\mathbb{R}^2 \setminus \mathcal{S})$ holds if

$$\texttt{length of } \Sigma = \texttt{length of } \mathcal{S}$$

Open direction

Results for higher eigenvalues are missing.

Higher dimensions ($d \geq 3$, $\alpha < 0$)

The constraint $|\partial\Omega| = |\partial\mathcal{B}|$ is "wrong" as a counterexample shows.

For convex Ω, $\lambda_1^\alpha(\Omega^{\rm ext}) \leq \lambda_1^\alpha(\mathcal{B}^{\rm ext})$ & $\alpha_\star(\Omega^{\rm ext}) \geq \alpha_\star(\mathcal{B}^{\rm ext})$ hold if

$$\frac{\texttt{Willmore-type energy of } \partial\Omega}{\texttt{the area of } \partial\Omega} = \frac{\texttt{Willmore-type energy of } \partial\mathcal{B}}{\texttt{the area of } \partial\mathcal{B}}$$

Open problem

Is the result still true for (a class of) non-convex Ω?

Thank you

D. Krejčiřík and V. L., Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, J. Convex Anal. **25** (2018), 319–337.

D. Krejčiřík and V. L., Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, II: non-convex domains and higher dimensions, arXiv:1707.02269.

V. L., Spectral isoperimetric inequalities for singular interactions on open arcs, to appear in Appl. Anal. arXiv:1609.07598.

Thank you for your attention!