CERTAIN FAMILIES OF POLYNOMIALS ARISING IN THE STUDY OF HYPERELLIPTIC LIE ALGEBRAS

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ABSTRACT. The associative ring $R(P(t)) = \mathbb{C}[t^{\pm 1}, u \,|\, u^2 = P(t)]$, where $P(t) = \sum_{i=0}^{n} a_i t^i = \prod_{k=1}^{n}(t - \alpha_i)$ with $\alpha_i \in \mathbb{C}$ pairwise distinct, is the coordinate ring of a hyperelliptic curve. The Lie algebra $\mathcal{R}(P(t)) = \mathrm{Der}(R(P(t)))$ of derivations is called the hyperelliptic Lie algebra associated to $P(t)$. In this paper we describe the universal central extension of $\mathrm{Der}(R(P(t)))$ in terms of certain families of polynomials which in a particular case are associated Legendre polynomials. Moreover we describe certain families of polynomials that arise in the study of the group of units for the ring $R(P(t))$ where $P(t) = t^4 - 2bt^2 + 1$. In this study pairs of Chebychev polynomials (U_n, T_n) arise as particular cases of a pairs (r_n, s_n) with $r_n + s_n\sqrt{P(t)}$ a unit in $R(P(t))$. We explicitly describe these polynomial pairs as coefficients of certain generating functions and show certain of these polynomials satisfy particular second order linear differential equations.

1. INTRODUCTION

Throughout this paper we will take the set of natural numbers to be $\mathbb{N} = \{1, 2, \dots\}$, the set of nonnegative integers will be denoted by $\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$, and we will assume all vector spaces and algebras are defined over complex numbers $\mathbb{C}$.

The associative ring $R(P(t)) = \mathbb{C}[t^{\pm 1}, u \,|\, u^2 = P(t)]$, where $P(t) = \sum_{i=0}^{n} a_i t^i = \prod_{k=1}^{n}(t - \alpha_i)$ with $\alpha_i \in \mathbb{C}$ pairwise distinct, is the coordinate ring of a hyperelliptic curve. The Lie algebra $\mathcal{R}(P(t)) = \mathrm{Der}(R(P(t)))$ of derivations is called the hyperelliptic Lie algebra associated to $P(t)$. In this paper we describe the universal central extension of $\mathrm{Der}(R(P(t)))$ in terms of certain families of polynomials which in a particular case are associated Legendre polynomials. Moreover we describe certain families of polynomials that arise in the study of the group of units for the ring $R(P(t))$ where $P(t) = t^4 - 2bt^2 + 1$. In this study pairs of Chebychev polynomials (U_n, T_n) arise as particular cases of a pairs (r_n, s_n) with $r_n + s_n\sqrt{P(t)}$ a unit in $R(P(t))$. We explicitly describe these polynomial pairs as coefficients of certain generating functions and show certain of these polynomials satisfy particular second order linear differential equations.

Let us give some background to this paper and more precise information of what is in the paper. The Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ can be considered as the ring of rational functions on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with poles allowed only in $\{\infty, 0\}$. From geometric point of view one can

have a natural generalization of the loop algebra construction. Instead of
the sphere with two punctures, one can use any complex algebraic curve
X of genus g with a fixed subset P of n distinct points. This arrives at
M. Schlichenmaier's definition of multipoint algebras of Krichever-Novikov
affine type if we replace $\mathbb{C}[t, t^{-1}]$ with the ring R of meromorphic functions
on X with poles allowed only in P in the construction of affine Kac-Moody
algebras (see [Sch03a], [KN87b], [KN87a], and [KN89]). The n-point affine
Lie algebras which are a type of Krichever-Novikov algebra of genus zero
also appeared in the work of Kazhdan and Lusztig ([KL91, Sections 4 &
7],[FBZ01, Chapter 12]). Krichever-Novikov algebras are used to constuct
analogues of important mathematical objects used in string theory but in
the setting of a Riemann surface of arbitrary genus. Moreover Wess-Zumino-
Witten-Novikov theory and analogues of the Knizhnik-Zamolodchikov equa-
tions are developed for analogues of the affine and Virasoro algebras (see the
survey article [She05], and for example [SS99], [She03], [Sch03a], [Sch03b],
and [SS98]).

In a recent paper [CGLZ1] the n-point Virasoro algebras $\tilde{\mathcal{V}}_a$ were stud-
ied, which are natural generalizations of the classical Virasoro algebra and
have as quotients multipoint genus zero Krichever-Novikov type algebras.
Necessary and sufficient conditions for the latter two such Lie algebras to
be isomorphic, and their automorphisms, their derivation algebras were ob-
tained, their universal central extensions, and some other properties were
also determined. Also a large class of modules which were called modules
of densities were constructed, and necessary and sufficient conditions for
them to be irreducible were obtained. The n-point Virasoro algebras all are
coordinate rings of the rings the Riemann sphere with n-points removed.

In another recent paper [CGLZ2], m-th superelliptic Lie algebras $\mathcal{R}_m(P)$
associated to $P(t) \in \mathbb{C}[t]$ were defined, the necessary and sufficient condi-
tions for such Lie algebras to be simple were obtained, and their universal
central extensions and their derivation algebras were determined. They are
examples of genus greater than one Krichever-Novikov type algebras.

The present paper is a sequel to [CGLZ2]. The particular class we look at
are hyperelliptic curves, i.e., $m = 2$. Let $P(t) \in \mathbb{C}[t]$. Then we have the Rie-
mann surfaces (commutative associative algebras) $R(P) = \mathbb{C}[t^{\pm 1}, u \,|\, u^2 = P(t)]$. The Lie algebras $\mathcal{R}(P) = \text{Der}(R(P))$ are called *hyperelliptic Lie alge-
bras* due to the fact that $u^2 = P(t)$ is a hyperelliptic curve.

In the second section, we give a basis for the universal central extension
of the Lie algebras $\mathcal{R}_m(P)$ and in the particular case that 0 is not a root of
$P(t)$ explicitly describe a set of basis of two cocycles. We also explicitly give
examples the value of any 2-cocyle on a basis of $\mathcal{R}_m(P)$. Our description
uses Braá di Bruno's formula and Bell polynomials to describe generating
functions of families of polynomials appearing as coefficients of the basis
elements. In the particular case of $P(t) = t^{2r} - 2bt^r + 1$, $r \in \mathbb{N}$, associated
Legendre polynomials naturally arise. Having such an explicit description
of the two cocycles will allow, one using conformal field theoretic tools, to
study free field type representations of these algebras.

To study isomorphisms and automorphisms between the Lie algebras $\mathcal{R}_m(P)$, from [Skr88] we know that it is equivalent to considering isomorphisms and automorphisms of the Riemann surfaces $R_m(P)$. This is in general a very hard problem. In particular it is known that many sporadic simple groups can appear. For some results on higher genus Riemann surfaces, see [BJ05, Bre00] and the reference therein.

In the third section, we describe the group of units of $R_2(P)$ which will be used in the last section, in particular we explicitly describe this group in cases of $P(t) = t^4 - 2bt + 1$, $b \neq \pm 1$ which is the most interesting case studied by Date, Jimbo, Kashiwara and Miwa [DJKM]. In this later paper they investigated integrable systems arising from Landau-Lifshitz differential equation. When determining the unit group we find that it requires one find solutions of the polynomial Pell equation $f^2 - g^2 P = 1$ for a given $P \in \mathbb{C}[t]$ which is a very famous and very hard problem. See [DS04]. Then Theorem 4 describes the possible units in $\mathcal{R}_m(P)$ as $\mathcal{R}_m(P)^* \cong \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. One may want to compare this to the description of the automorphism group of a hyperellptic curve given in [Sha03]. The group of units of $R_2(P)$ are described in terms of pairs of polynomials u_n, v_n satisfying $u_n + v_n\sqrt{P} = (u_1 + v_0\sqrt{P})^n$ and are similar in some sense to the Chebyshev polynomials of the first and second kinds. We explicitly describe these polynomial pairs as coefficients of generating function and show certain of these polynomials satisfy particular second order linear differential equations.

2. THE HYPERELLIPTIC LIE ALGEBRAS $\mathcal{R}(P(t))$

Let $P(t) = \sum_{i=l}^{n+l} b_i t^i = t^l(t - a_1)\ldots(t - a_n)$ for some $l = 0$ or 1, $n \in \mathbb{N}$ and pairwise distinct nonzero $a_1, \ldots, a_n \in \mathbb{C}$. Note that $b_{n+l} = 1$. It is easy to see from [CGLZ2] that $R(P) = \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}[t^{\pm 1}]u$ and if we set $\Delta := P'\frac{\partial}{\partial u} + 2u\frac{\partial}{\partial t}$ where $P' = \frac{\partial P}{\partial t}$ we have $\mathcal{R}(P) = R(P)\Delta$. One of the reasons we are interested in the Lie algebra $\mathcal{R}(P) := \mathrm{Der}R(P(t))$ is the following

Theorem 1. [CGLZ2, Theorem 4] *The Lie algebra $\mathcal{R}(P(t))$ is simple.*

Let $R = R(P)$ and $\partial = \sqrt{P}\frac{d}{dt}$. We know

Theorem 2. [CGLZ2, Theorem 9] *Suppose that $R = R(P)$ and $P = t^l(t - a_1)\ldots(t - a_n)$ for $l = 0$ or 1, $n \in \mathbb{N}$ and pairwise distinct nonzero $a_1, \ldots, a_n \in \mathbb{C}$. Then $\dim R/\partial(R) = n + 1$ and the universal central extension of $\mathcal{R}(P)$ is $\mathcal{R}(P) \oplus R/\partial(R)$ with brackets*

$$[f\partial, g\partial] = f\partial(g)\partial - g\partial(f)\partial + \overline{\partial(f)\partial(\partial(g))}, \forall\ f, g \in R.$$

Note that $\mathcal{R}(P)$ is the classical centerless Virasoro algebra if $n = 0$, and $\mathcal{R}(P)$ with $l > 1$ is isomorphic to one of the the Lie algebras in the above theorem. So our Lie algebra $\mathcal{R}(P)$ with the given restrictions on P cover all cases of P up to isomorphisms.

To study representations of the Lie algebras $\mathcal{R}(P) \oplus R/\partial(R)$ with nontrivial central action, it is important to have a basis for its center $R/\partial(R)$ with concrete formulae for their brackets. This is the main purpose of this section, which turns out to be complicated.

From the proof of [CGLZ2, Theorem 9] we suppose a basis is given by $\omega_0 := \overline{t^{-1}\sqrt{P}}$, and

$$\{\overline{t^{-1}}, \ldots, \overline{t^{n-2}}\}.$$

In the rest of this section we will derive some formulae for the 2-cocycles of $\mathcal{R}(P)$ in terms of this precise basis.

2.1. **The general case.** We can calculate any 2-cocyle as being a linear combination of the coefficients of ψ where $\psi(f\partial, g\partial) = \overline{\partial(f)\partial^2(g)}$. In particular

$$(2.1) \quad \psi(t^r\partial, t^s\partial) = rs\sum_{i=0}^{n}\left(s - 1 + \frac{i}{2}\right) b_i\delta_{r+s+i-2,0}\omega_0$$

$$= \begin{cases} (rs(r-s)b_{-r-s+2}\omega_0)/2, & \text{if } 0 \leqslant -r - s + 2 \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

and

$$(2.2) \quad \psi(t^r\sqrt{P}\partial, t^s\sqrt{P}\partial)$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{n}\left(r + \frac{i}{2}\right)\left(r + i - 2 + \frac{j}{2}\right)(r + i - 1)b_ib_j\delta_{r+s+i+j-2,0}\omega_0$$

For the cross term $\psi(t^r\partial, t^s\sqrt{P}\partial)$, the formula is more complicated and we don't have a simple description of what it is but we have the following in the case $l = 0$ (for $l = 1$ a similar description can be made). For all $r, s \in \mathbb{Z}$, we have

$$(2.3) \quad \psi(t^r\partial, t^s\sqrt{P}\partial)$$

$$= \overline{rt^{r-1}\sqrt{P}\partial\left(\sum_{j=0}^{n}\left(s + \frac{j}{2}\right)b_jt^{s+j-1}\right)}$$

$$= \overline{rt^{r-1}\sqrt{P}\left(\sum_{j=0}^{n}\left(s + \frac{j}{2}\right)(s + j - 1)b_jt^{s+j-2}\sqrt{P}\right)}$$

$$= r\sum_{i=0}^{n}\left(\sum_{j=0}^{n}\left(s + \frac{j}{2}\right)(r + s + j - 2)b_j\right)b_i\overline{t^{r+s+i+j-3}}.$$

We now consider the $\overline{t^{s+i+j-2}}$: As for any $r \in \mathbb{Z}$ we have

$$\partial(t^r\sqrt{P}) = \sum_{i=0}^{n}\left(r + \frac{i}{2}\right)b_it^{r+i-1}.$$

so in the quotient $R/\partial R$ we have the recursion relation

$$\sum_{i=0}^{n}\left(r + \frac{i}{2}\right)b_i\overline{t^{r+i-1}} = 0.$$

As $b_n = 1$, this proves that for $r \geqslant 0$

$$\overline{t^{r+n-1}} = -\left(r + \frac{n}{2}\right)^{-1} \sum_{i=0}^{n-1} \left(r + \frac{i}{2}\right) b_i \overline{t^{r+i-1}}.$$

This implies that $\{\overline{t^k} \,|\, k \geqslant n - 1\}$ can be written as a linear combination of $\{\overline{t^{-1}}, \ldots, \overline{t^{n-2}}\}$. More precisely we can find polynomials $p_{r,i} = p_{r,i}(b_0, \ldots, b_{n-1})$, $-1, \leqslant i \leqslant n - 2$, such that

$$(2.4) \qquad \overline{t^k} = \sum_{i=-1}^{n-2} p_{k,i} \overline{t^i}$$

for all $k \geqslant n - 1$. We will set $p_{r,i} = \delta_{r,i}$ for $-1 \leqslant i, r \leqslant n - 2$ so (2.4) holds for all $k \geqslant -1$. As $b_0 \neq 0$ we have

$$(2.5) \qquad \overline{t^{r-1}} = -\frac{1}{rb_0} \sum_{i=1}^{n} \left(r + \frac{i}{2}\right) b_i \overline{t^{r+i-1}}.$$

This means that $\{\overline{t^{-k}} \,|\, k \geqslant 2\}$ can be written as a linear combination of $\{\overline{t^{-1}}, \ldots, \overline{t^{n-2}}\}$. More precisely we can find rational functions $q_{r,i} = q_{r,i}(b_0, \ldots, b_{n-1})$, $-1, \leqslant i \leqslant n - 2$, such that

$$(2.6) \qquad \overline{t^{-k}} = \sum_{i=-1}^{n-2} q_{k,i} \overline{t^i}$$

for all $k \geqslant 1$.

As a consequence of (2.3), (2.4) and (2.6), we have

(2.7)

$$\psi(t^r \partial, t^s \sqrt{P} \partial)$$

$$= r \sum_{k=-1}^{n-2} \left(\sum_{i,j=0}^{n} \Theta(r + s + i + j - 2) \left(\left(s + \frac{j}{2}\right)(r + s + j - 2)b_j\right) b_i p_{r+s+i+j-3,k}(b) \right) \overline{t^k}$$

$$+ r \sum_{k=-1}^{n-2} \left(\sum_{i,j=0}^{n} \Theta(-(r + s + i + j - 1)) \left(\left(s + \frac{j}{2}\right)(r + s + j - 2)b_j\right) b_i q_{-(r+s+i+j-3),k}(b) \right) \overline{t^k}$$

where

$$\Theta(k) = \begin{cases} 1 & \text{if } k \geqslant 0 \\ 0 & \text{for } k < 0, \end{cases}$$

is the Heaviside function. Because of this we then turn to giving very general formulae for the polynomials $p_{k,i}$ and $q_{k,i}$.

If we let $p_{i,r} = p_{r,i}(b_0, \ldots, b_{n-1})$ be polynomials in $b_0, \ldots, b_{n-1}$, $i = -1, 0, 1, 2 \ldots, n - 2$, $r \geqslant -1$ that satisfy the recursion relation

$$(2.8) \qquad \sum_{j=0}^{n} \left(r + \frac{j}{2}\right) p_{r+j-1,i}(b_0, \ldots, b_{n-1})b_j = 0 \quad \text{for all } r \geqslant n - 1,$$

with initial conditions $p_{r,i} = \delta_{i,r}$ for $-1 \leqslant i, r \leqslant n - 2$. Set

$$(2.9) \qquad P_i(b, z) := \sum_{k \geqslant 0} p_{k-1,i} z^k = \sum_{k \geqslant -1} p_{k,i} z^{k+1},$$

and

$$\bar{P}(z) := \sum_{j=0}^{n} b_j z^{n-j}, \quad Q(z) := z\bar{P}'(z) - n\bar{P}(z),$$

$$R_i(z) := z^n \sum_{j=0}^{n} \sum_{-j \leqslant l < n-1} (2l + j)b_j p_{l+j-1,i} z^l,$$

where $-1 \leqslant i \leqslant n - 2$. Note that $\bar{P}(z)$, $Q(z)$ and $R_i(z)$ are all polynomials in z with coefficients polynomials in the b_i. Then we have

$$(2.10) \qquad 2z\bar{P}(z)\frac{d}{dz}P_i(b, z) + Q(z)P_i(b, z) = R_i(z),$$

for $i = 1, \ldots, n$. This has integrating factor

$$(2.11) \qquad \mu = \exp\left(\int \frac{\bar{P}'(z)}{2\bar{P}(z)} - \frac{n}{2z}\, dz\right) = \sqrt{\bar{P}(z)} z^{-n/2}.$$

Here $z^{-n/2}$ is only to be used formally in the following when integrating and is not to be thought of as a multivalued function. For $i = -1, \ldots, n - 2$ we obtain generating functions as solutions to the family of n equations (2.10):

$$(2.12) \qquad P_i(b, z) = \frac{z^{n/2}}{\sqrt{\bar{P}(z)}} \int \frac{R_i(z)}{2z^{\frac{n+2}{2}} \sqrt{\bar{P}(z)}}\, dz$$

and they are given by hyperelliptic integrals. In the above formula we expand $\dfrac{1}{\bar{P}(z)^{1/2}}$ as a Laurent series in z, multiply by $R_i(z)/2z^{(n+2)/2}$ and then integrate formally term by term. Let us explain this more precisely: If we expand $\dfrac{1}{\sqrt{\bar{P}(t)}} = \sum_{k=0}^{\infty} h_k(b_0, \ldots, b_{n-1})t^k$, then (2.12) becomes

(2.13)
$$P_i(b, z) = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{n}\sum_{-j \leqslant l < n-1} h_k(b_0, \ldots, b_{n-1})(2l + j)b_j p_{l+j-1,i}$$

$$\times \left(\int h_m(b_0, \ldots, b_{n-1})z^{\frac{n-2}{2}+m+l}\, dz\right)z^{k+(n/2)}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{n}\sum_{-j \leqslant l < n-1} (2l + j)b_j p_{l+j-1,i} h_k(b_0, \ldots, b_{n-1})h_m(b_0, \ldots, b_{n-1})C z^{k+(n/2)}$$

$$+ \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{n}\sum_{-j \leqslant l < n-1} \frac{(2l + j)b_j p_{l+j-1,i}}{n + 2(m + l)} h_k(b_0, \ldots, b_{n-1})h_m(b_0, \ldots, b_{n-1})z^{m+n+l+k}.$$

where C is a constant of integration (which is zero if n is odd as the left hand power series has no fractional powers of z). Then (2.13) gives us a (rather complicated) description of $p_{a,i}$ as coefficient of z^a in the above

power series. So we are left with providing a description of $h_k(b_0, \ldots, b_{n-1})$ (see (2.15) below).

One can expand $1/\sqrt{P(z)}$ using Bell polynomials and Faà di Bruno's formula as follows. The Bell polynomials in the variables $z_1, z_2, z_3, \ldots$ are defined to be

$$B_{m,k}(z_1, \ldots, z_{m-k+1}) := \sum \frac{m!}{l_1! l_2! \cdots l_{m-k+1}!} \left(\frac{z_1}{1!}\right)^{l_1} \cdots \left(\frac{z_{m-k+1}}{(m-k+1)!!}\right)^{l_{m-k+1}}$$

where the sum is over $l_1 + l_2 + \cdots = k$ and $l_1 + 2l_2 + 3l_3 + \cdots = m$.

Now Faà di Bruno's formula for the m-derivative of $f(g(x))$ is

$$\frac{d^m}{dx^m} f(g(x)) = \sum_{l=0}^{m} f^{(l)}(g(x)) B_{m,l}(g'(x), g''(x), \ldots, g^{(m-l+1)}(x)).$$

Setting $f(x) = 1/\sqrt{x}$, $g(x) = \bar{P}(x)$ we get

(2.14) $$f^{(m)}(x) = \frac{(-1)^m (2m-1)!!}{2^m x^{(2m+1)/2}}$$

and $\bar{P}^{(k)}(0) = k! b_{n-k}$ so that

$$\frac{d^m}{dx^m} f(g(x))|_{x=0} = \sum_{l=0}^{m} \frac{(-1)^l (2l-1)!!}{2^l b_n^{(2l+1)/2}} B_{m,l}(b_{n-1}, 2b_{n-2}, \ldots, (m-l+1)! b_{n-m+l-1}),$$

where we set $b_k = 0$ for $k \leqslant -1$. As a consequence

$$\frac{1}{\sqrt{\bar{P}(t)}} = \sum_{k=0}^{\infty} \frac{d^k}{dt^k} f(g(t))|_{t=0} t^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{l=0}^{k} \frac{(-1)^l (2l-1)!!}{2^l b_n^{(2l+1)/2}} B_{k,l}(b_{n-1}, 2b_{n-2}, \ldots, (k-l+1)! b_{n-k+l-1}) \right) t^k,$$

and hence
(2.15)

$$h_k(b_0, \ldots, b_{n-1}) = \frac{1}{k!} \left(\sum_{l=0}^{k} \frac{(-1)^l (2l-1)!!}{2^l b_n^{(2l+1)/2}} B_{k,l}(b_{n-1}, 2b_{n-2}, \ldots, (k-l+1)! b_{n-k+l-1}) \right).$$

Lastly we derive the formulae for the $q_{k,i}$ in (2.6) in terms of generating functions. For $-1 \leqslant i \leqslant n-2$ set

(2.16) $$Q_i(b, z) := \sum_{k \geqslant -n+2} q_{k,i} z^{k+n-2} = \sum_{k \geqslant 0} q_{k-n+2,i} z^k$$

where $q_{k,i}$, $k \geqslant -n+2$, are rational functions in b_i and satisfy the recursion relation

(2.17) $$\overline{(t^{-1})^{1-r}} = -\frac{1}{rb_0} \sum_{i=1}^{n} \left(r + \frac{i}{2}\right) b_i \overline{t^{r+i-1}}.$$

or

(2.18) $$\overline{(t^{-1})^k} = -\frac{1}{(1-k)b_0} \sum_{i=1}^{n} \left(1 - k + \frac{i}{2}\right) b_i \overline{(t^{-1})^{k-i}}.$$

This gives us a recursion relation

$$(2.19) \qquad \sum_{j=0}^{n} (2k - (j+2)) \, b_j q_{k-j,i} = 0.$$

for all $k \geqslant 2$. These polynomials satisfy the initial condition $q_{k,i} = \delta_{k,-i}$ for $-n+2 \leqslant i, k \leqslant 1$. Set

$$(2.20) \qquad P(z) = \sum_{i=0}^{n} b_i z^i, \qquad Q(z) := zP'(z) - 2(n-1)P(z),$$

$$S_i(z) := \sum_{j=0}^{n} \sum_{j-n+2 \leqslant l < 2} (2l - (j+2)) b_j q_{l-j,i} z^{l+n-2}.$$

Then

$$2zP(z)\frac{d}{dz}Q_i(b,z) + Q(z)Q_i(b,z) = S_i(z)$$

which has integrating factor

$$\mu = \exp \int \frac{Q(z)}{2zP(z)} \, dz = z^{-(n-1)}\sqrt{P(z)}$$

Thus

$$(2.21) \qquad Q_i(b,z) = \frac{z^{n-1}}{\sqrt{P(z)}} \int \frac{S_i(z)}{2z^n \sqrt{P(z)}} \, dz.$$

As in the case of $P_i(b,z)$, one can use Fáa de Bruno's formula and Bell polynomials to find the Taylor series expansion of $Q_i(b,z)$ at $z=0$.

Next we illustrate the above generating functions in the examples below. First we can directly obtain

Lemma 3. *If $P(t) = t^2 - 2bt$ with $b \neq 0$, then the center of the universal central extension of $\mathcal{R}(P)$ is two dimensional and we can write any 2-cocycle ψ as*

$$\psi(t^r\partial, t^s\partial) = \left(r^3\delta_{r+s,0} - br(r-1)(2r-1)\delta_{r+s-1,0}\right)\omega_0,$$

$$\psi(t^r\partial, t^s\sqrt{P}\partial) = 3rb^{r+s+1}\left(r\left(4s^2 + 5s + 2\right) + 4s^3 + 10s^2 + 9s + 3\right)$$
$$\times \frac{(2r+2s-3)!!}{(r+s+1)!}\overline{1},$$

$$\psi(t^r\sqrt{P}\partial, t^s\sqrt{P}\partial) = (r+1)^3\delta_{r+s+2,0}\omega_0$$
$$- b(2r+1)(2r^2 + 2r + 1)\delta_{r+s+1,0}\omega_0$$
$$+ b^2(4r^2 - 1)r\delta_{r+s,0}\omega_0.$$

where, by definition, $(2r + 2s - 1)!! = (2r + 2s - 1) \cdot (2r + 2s - 5) \cdots 5 \cdot 3 \cdot 1$ and

$$\frac{(2r+2s-3)!!}{(r+s+1)!}\overline{1} = \begin{cases} 0 & \text{if } r+s \leqslant -2 \\ \frac{1}{3} & \text{if } r+s = -1 \\ -1 & \text{if } r+s = 0. \end{cases}$$

2.2. The case $R = \mathbb{C}[t^{\pm 1}, u \mid u^2 = t^2 - 2bt + c]$ **with** $c \neq 0$**.** Up to isomorphism we may assume that $c = 1$. In the case $P(t) = t^2 - 2bt + 1$, $l = 0$, the center of the universal central extension is three dimensional and in terms of the basis elements $\omega_0 := \overline{t^{-1}\sqrt{P}}$, $\overline{t^{-1}}$, $\overline{1}$,

$$\psi(t^r \partial, t^s \partial) = \left(r^3 \delta_{r+s,0} - br(r-1)(2r-1)\delta_{r+s-1,0}\right.$$
$$\left. + r(r-1)(r-2)\delta_{r+s-2,0}\right)\omega_0$$

$$\psi(t^r \sqrt{P}\partial, t^s \sqrt{P}\partial) = (r+1)^3 \delta_{r+s+2,0}\omega_0$$
$$- b(2r+1)\left(2r^2 + 2r + 1\right)\delta_{r+s+1,0}\omega_0$$
$$+ \left(b^2 r \left(4r^2 - 1\right) + 2r\left(r^2 + 1\right)\right)\delta_{r+s,0}\omega_0$$
$$- 2b(r-1)r(2r-1)\delta_{r+s-1,0}\omega_0$$
$$+ r(r-1)(r-2)\delta_{r+s-2,0}\omega_0.$$

In this case

$$\partial(t^k \sqrt{P}) = (k+1)t^{k+1} - b\left(2k+1\right)t^k + kt^{k-1}$$

so in the quotient $R/\partial R$ we have

(2.22) $$(k+1)\overline{t^{k+1}} - b\left(2k+1\right)\overline{t^k} + k\overline{t^{k-1}} = 0.$$

This is the recursion relation for the Legendre polynomials $p_k(b)$. Thus for $k \geqslant 0$

$$\overline{t^k} = p_k(b)\overline{1},$$

For $k \leqslant 1$ we have

$$\overline{t^{-k}} = p_{k-1}(b)\overline{t^{-1}}.$$

Lastly we calculate

$$\psi(t^r \partial, t^s \sqrt{P}\partial) = r\left(s^2 - s\right)\overline{t^{r+s-3}}$$
$$+ rb\left(s - 4s^2\right)\overline{t^{r+s-2}}$$
$$+ r\left((4s^2 + 2s)b^2 + 2s^2 + s + 1\right)\overline{t^{r+s-1}}$$
$$- rb\left(4s^2 + 5s + 2\right)\overline{t^{r+s}}$$
$$+ r\left(s+1\right)^2 \overline{t^{r+s+1}}.$$

When $r + s \geqslant 3$ we get

$$\psi(t^r \partial, t^s \sqrt{P}\partial)$$
$$= r\left(s^2 - s\right)p_{r+s-3}(b)\overline{1} + rb\left(s - 4s^2\right)p_{r+s-2}(b)\overline{1}$$
$$+ r\left((4s^2 + 2s)b^2 + 2s^2 + s + 1\right)p_{r+s-1}(b)\overline{1}$$
$$- rb\left(4s^2 + 5s + 2\right)p_{r+s}(b)\overline{1} + r\left(s+1\right)^2 p_{r+s+1}(b)\overline{1}.$$

If $r + s = 2$ we get

$$\psi(t^r \partial, t^s \sqrt{P}\partial)$$
$$= r(r-1)(r-2)\overline{t^{-1}} + \frac{1}{2}br\left(b^2\left(r^2 - 3r + 1\right) - 3r^2 + 9r - 5\right)\overline{1}.$$

If $r + s = 1$ one has

$$\psi(t^r \partial, t^s \sqrt{P}\partial) = -3b(r-1)^2 r \overline{t^{-1}} + \frac{1}{2}r\left(b^2\left(3r^2 - 6r + 2\right) + 3r^2 - 6r + 4\right)\overline{1}.$$

When $r + s = 0$ we get

$$\psi(t^r \partial, t^s \sqrt{P}\partial)$$
$$= \frac{1}{2}r\left(3r(r-1)b^2 + 3r^2 - 3r + 2\right)\overline{t^{-1}} - br\left(3r^2 - 3r + 1\right)\overline{1}.$$

If $r + s = -1$ one obtains

$$\psi(t^r \partial, t^s \sqrt{P}\partial) = \frac{1}{2}br\left(b^2\left(r^2 - 1\right) - 3r^2 + 1\right)\overline{t^{-1}} + r^3\overline{1}.$$

When $r + s \leqslant -2$ we get

$$\psi(t^r \partial, t^s \sqrt{P}\partial)$$
$$= r\left(s^2 - s\right)p_{-(r+s)+2}(b)\overline{t^{-1}} + rb\left(s - 4s^2\right)p_{-(r+s)+1}(b)\overline{t^{-1}}$$
$$+ r\left((4s^2 + 2s)b^2 + 2s^2 + s + 1\right)p_{-(r+s)}(b)\overline{t^{-1}}$$
$$- rb\left(4s^2 + 5s + 2\right)p_{-(r+s)-1}(b)\overline{t^{-1}} + r\left(s+1\right)^2 p_{-(r+s)-2}(b)\overline{t^{-1}}.$$

2.3. The case $R = \mathbb{C}[t^{\pm 1}, u \,|\, u^2 = t^3 + at + b]$ with $b \neq 0$.

Up to isomorphism we may assume that $b = 1$. The center of the universal central extension is four dimensional and in terms of the basis elements $\omega_0 := \overline{t^{-1}\sqrt{P}}$, $\overline{t^{-1}}$, $\overline{1}$, $\overline{t}$. We will express $\overline{t^r}$ in terms of this basis.

Here we have $P(t) = t^3 + at + 1$ and

$$\partial(t^k\sqrt{P}) = \sqrt{P}\partial(t^k\sqrt{P}) = kt^{k-1}P + \frac{1}{2}t^k P'$$

$$(2.23) \qquad\qquad = \left(k + \frac{3}{2}\right)t^{k+2} + \left(k + \frac{1}{2}\right)at^k + kbt^{k-1}$$

so that

$$\overline{t^2} = -\frac{a}{3}\overline{1}$$

$$\overline{t^3} = -\frac{3}{5}a\overline{t} - \frac{2}{5}\overline{1}$$

$$\overline{t^4} = -\frac{4}{7}\overline{t} + \frac{5}{21}a^2\overline{1}$$

$$\overline{t^5} = \frac{7}{15}a^2\overline{t} + \frac{8a}{15}\overline{1}$$

$$\vdots \qquad \vdots$$

If we write

$$(2.24) \qquad\qquad \overline{t^r} = p_{r,1}\overline{t} + p_{r,0}\overline{1} + p_{r,-1}\overline{t^{-1}}$$

for some polynomials $p_{r,i}$, $i = -1, 0, 1$ in a so that $p_{k,i}$, $i = -1, 0, 1$ satisfy the same recurrence relation as t^k but with initial conditions

$$p_{-1,-1} = 0, \quad p_{0,-1} = 0, \quad p_{1,-1} = 0,$$
$$p_{-1,0} = 0, \quad p_{0,0} = 1, \quad p_{1,0} = 0,$$
$$p_{-1,1} = 0, \quad p_{0,1} = 0, \quad p_{1,1} = 1.$$

So we get $Q(z) = -3 - az^2$,

(2.25) $$R_{-1}(z) = 0$$

(2.26) $$R_0(z) = z^3 \sum_{j=0}^{n} \sum_{-j \leqslant l \leqslant 0} (2l+j)b_j p_{l+j-1,0} z^l = -z,$$

(2.27) $$R_1(z) = z^3 \sum_{j=0}^{3} \sum_{-j \leqslant l \leqslant 0} (2l+j)b_j p_{l+j-1,1} z^l = z^2,$$

and the generating series are given by (2.12) or rather (2.13)

$$P_0(z) = \frac{z^{3/2}}{\sqrt{1+az^2+z^3}} \int \frac{-z}{2z^{5/2}\sqrt{1+az^2+z^3}} \, dz$$

$$= z - \frac{az^3}{3} - \frac{2z^4}{5} + \frac{5a^2z^5}{21} + \frac{8}{15}az^6 + \left(\frac{16}{55} - \frac{15a^3}{77}\right)z^7 + \cdots$$

$$P_1(z) = \frac{z^{3/2}}{\sqrt{1+az^2+z^3}} \int \frac{z^2}{2z^{5/2}\sqrt{1+az^2+z^3}} \, dz$$

$$= z^2 - \frac{3az^4}{5} - \frac{4z^5}{7} + \frac{7a^2z^6}{15} + \frac{348}{385}az^7 + \cdots$$

Similarly one has from (2.5)

$$\overline{t^{-1}} = \overline{t^{-1}},$$

$$\overline{t^{-2}} = -\frac{a}{2}\overline{t^{-1}} + \frac{1}{2}\bar{t},$$

$$\overline{t^{-3}} = \frac{3a^2}{8}\overline{t^{-1}} - \frac{1}{4}\bar{1} - \frac{3a}{8}\bar{t},$$

$$\overline{t^{-4}} = -\frac{15a^3 + 24b^2}{48}\overline{t^{-1}} + \frac{5a}{24}\bar{1} + \frac{5a^2}{16}\bar{t},$$

$$\vdots \qquad \vdots$$

$$\overline{t^{-k}} = q_{k,-1}\overline{t^{-1}} + q_{k,0}\bar{1} + q_{k,1}\bar{t}.$$

Then one needs to solve the differential equations

$$2zP(z)\frac{d}{dz}Q_i(b,z) + Q(z)Q_i(b,z) = S_i(z), \quad i = -1, 0, 1.$$

From (2.20)

$$S_{-1}(z) = 0$$
$$S_0(z) = -2z - az^2$$
$$S_1(z) = -4 - 3az$$

so that by (2.21) (and Fáa de Bruno's formula and Bell polynomials)

$$Q_{-1}(z) = \frac{z^2}{\sqrt{z^3 + az + 1}}$$

$$= z^2 - \frac{a}{2}z^3 + \frac{3a^2}{8}z^4 - \frac{15a^3 + 24}{48}z^5 + \cdots$$

$$Q_0(z) = \frac{z^2}{\sqrt{z^3 + az + 1}} \int \frac{-2z - az^2}{2z^3\sqrt{1 + az^2 + z^3}} \, dz$$

$$= z - \frac{1}{4}z^4 + \frac{5a}{24}z^5 + \cdots$$

$$Q_1(z) = \frac{z^2}{\sqrt{z^3 + az + 1}} \int \frac{-4 - 3az}{2z^3\sqrt{1 + az^2 + z^3}} \, dz$$

$$= 1 + \frac{1}{2}z^3 - \frac{3a}{8}z^4 + \frac{5a^2}{16}z^5 + \cdots$$

where one needs to use a constant of integration 1, $a/2$, and $-a^2/8$ respectively. Recall that (2.26) looks like the same recursion relation as (2.22) but allows one to solve for the negative powers of $\overline{t^{-k}}$ as a linear combination of $\left\{\overline{t^{-1}}, \overline{1}, \overline{t}\right\}$. Thus we needed to use a different set of initial conditions $q_{k,i} = \delta_{k,-i}$, $-1 \leqslant k \leqslant 1$, $-1 \leqslant i \leqslant 1$ than for the $p_{k,i}$. Using (2.20) and (2.21) one is lead to the calculation given above for Q_i, $i = -1, 0, 1$.

2.4. **The case** $R = \mathbb{C}[t^{\pm 1}, u \,|\, u^2 = t^{2r} - 2bt^r + 1]$, $r \in \mathbb{N}$. We assume $r \in \mathbb{N}$ is fixed throughout this subsection. The center of the universal central extension is $2r + 1$ dimensional and in terms of the basis elements $\omega_0 := \overline{t^{-1}\sqrt{P}}$, $\overline{t^i}$ for $i = -1, 0, 1, \ldots 2r - 2$. We will explain how to express $p_k = \overline{t^k}$ in terms of this basis. In this case we have

$$\partial(t^k\sqrt{P}) = (k + r)t^{2r+k-1} - b(2k + r)t^{r+k-1} + kt^{k-1}$$

which leads one to the recursion relation

$$(k + r)p_{2r+k-1} - b(2k + r)p_{r+k-1} + kp_{k-1} = 0$$

Let now $\alpha, \beta \in \mathbb{C}$, $c \in \mathbb{R}$, $c > 0$ and $\gamma := \alpha + \beta + 1$. The associated Jacobi polynomials $\rho_k = P_k^{(\alpha,\beta)}(x; c)$, $k \in \mathbb{Z}_+$, $\rho_{-1} = 0$, $\rho_0 = 1$ satisfy the recursion relation

$$2(k + c + 1)(k + c + \gamma)(2k + 2c + \gamma - 1)\rho_{k+1}$$
$$= (2k + 2c + \gamma)\Big((2k + 2c + \gamma - 1)(2k + 2c + \gamma + 1)x$$
$$+ (\gamma - 1)(\gamma - 2\beta - 1)\Big)\rho_k - 2(k + c + \gamma - \beta - 1)$$
$$\times (k + c + \beta)(2k + 2c + \gamma + 1)\rho_{k-1}$$

for all $k \in \mathbb{Z}_+$. Setting $c = 1$ gives the usual recursion relation for the Jacobi polynomials. Setting $\alpha = \beta = 0$ into the associated Jacobi recursion relation above gives us

$$(k + c + 1)\rho_{k+1} = (2k + 2c + 1)x\rho_k - (k + c)\rho_{k-1}$$

for $k \geqslant 0$. The recursion relation we have is

$$(k+r)p_{2r+k-1} - b(2k+r)p_{r+k-1} + kp_{k-1} = 0$$

Let $k = rs + q$ where $0 \leqslant q \leqslant r-1$. Then this becomes

$$(rs+q+r)p_{2r+rs+q-1} - b(2(rs+q)+r)p_{r+rs+q-1} + (rs+q)p_{rs+q-1} = 0$$

For $l \in \mathbb{Z}$ with r and $0 \leqslant q \leqslant r-1$ setting $P_l^{q/r}(b) := p_{(l+1)r+q-1}$ we get

$$\left(s + \frac{q}{r} + 1\right) P_{s+1}^{q/r}(b) - b\left(2s + 2\left(\frac{q}{r}\right) + 1\right) P_s^{q/r}(b) + \left(s + \frac{q}{r}\right) P_{s-1}^{q/r}(b) = 0.$$

Thus the $P_l^{q/r}(b)$ and hence the $p_{(l+1)r+q-1}$ are associated Legendre polynomials (or functions as the case may be). Similarly the $q_{(l+1)r+q-1}$ are associated Legendre polynomials but with a different set of initial conditions. Then one can rewrite (2.7) in terms of the associate Legendre polynomials and the basis $\{\overline{t^{-1}}, \ldots, \overline{t^{n-2}}\}$.

3. Chebyshev polynomials and some related polynomials

We recall below our description of the group of units $R^*(P)$ for the interesting case of $P(t)$ studied by Date, Jimbo, Kashiwara and Miwa [DJKM] where they investigated integrable systems arising from Landau-Lifshitz differential equation. Let

$$P(t) = \frac{t^4 - 2\beta t^2 + 1}{\beta^2 - 1}, \quad \beta \neq \pm 1.$$

Observe that in this case $P(t) = q(t)^2 - 1$ where $q(t) = \frac{t^2 - \beta}{\sqrt{\beta^2 - 1}}$.

For convenience, let

$$\lambda_0 = \frac{t^2 - \beta}{\sqrt{\beta^2 - 1}} + \sqrt{P},$$

$$\lambda_1 = \frac{t^2 + 1}{\sqrt{2(\beta + 1)}} + \sqrt{\frac{\beta - 1}{2}}\sqrt{P},$$

$$\lambda_2 = \frac{t^2 - 1}{\sqrt{2(\beta - 1)}} + \sqrt{\frac{\beta + 1}{2}}\sqrt{P}.$$

It is easy to verify that $\lambda_0, \lambda_1, \lambda_2 \in R^*(P)$. Actually, $\lambda_0\bar{\lambda}_0 = 1$, $\lambda_1\bar{\lambda}_1 = t^2$, $\lambda_2\bar{\lambda}_2 = t^2$, $\lambda_1\lambda_2 = t^2\lambda_0$. We have

Theorem 4. [CGLZ2, Theorem 13] *Let $P(t)$ be as above.*

  (a). *As a multiplicative group, $R^*(P)$ is generated by $\mathbb{C}^*$, t, λ_1, λ_2.*
  (b). *$R^*(P) \simeq \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.*

Next we will study properties of elements in the unit group $R^*(P)$.

Theorem 5. *Define polynomials u_n, v_{n-1} by $u_n + v_{n-1}\sqrt{P} = \lambda_0^n$ for $n \in \mathbb{Z}$. Then*

$$u_{n+2}(t) - 2q(t)u_{n+1}(t) + u_n(t) = 0, \quad v_{n+2}(t) - 2q(t)v_{n+1}(t) + v_n(t) = 0,$$

and

$$(3.1) \qquad -P\frac{dq}{dx}\frac{d^2y}{dx^2} + \left(P\frac{d^2q}{dx^2} - q\left(\frac{dq}{dx}\right)\right)\frac{dy}{dx} + n^2\left(\frac{dq}{dx}\right)^2 y = 0,$$

is the second order differential equation satisfied by the u_n.

The $v_n(t)$ satisfies

$$-P\frac{dq}{dx}\frac{d^2y}{dx^2} + \left(P\frac{d^2q}{dx^2} - 3q\left(\frac{dq}{dx}\right)\right)\frac{dy}{dx} + n(n+2)\left(\frac{dq}{dx}\right)^2 y = 0.$$

Proof. We have

$$\sum_{n\geqslant 0} u_n(t)z^n + \sum_{n\geqslant 0} v_{n-1}(t)\sqrt{P(t)}z^n = \sum_{n\geqslant 0}(q(t)+\sqrt{P})^n z^n$$

$$= \frac{1}{1-(q(t)+\sqrt{P(t)})z}$$

$$= \frac{1-q(t)z+\sqrt{P(t)}z}{1-2q(t)z+(q(t)^2-P(t))z^2}$$

$$= \frac{1-q(t)z+\sqrt{P(t)}z}{1-2q(t)z+z^2}$$

We have the generating series for the set of polynomials as

$$\sum_{n\geqslant 0} u_n(t)z^n = \frac{1-q(t)z}{1-2q(t)z+z^2}, \quad \sum_{n\geqslant 0} v_{n-1}(t)z^n = \frac{1}{1-2q(t)z+z^2}$$

The first equation gives us

$$\sum_{n\geqslant 0} u_n(t)z^n - \sum_{n\geqslant 0} 2q(t)u_n(t)z^{n+1} + \sum_{n\geqslant 0} u_n(t)z^{n+2} = 1-q(t)z,$$

or

$$\sum_{n\geqslant 2} u_n(t)z^n - \sum_{n\geqslant 1} 2q(t)u_n(t)z^{n+1} + \sum_{n\geqslant 0} u_n(t)z^{n+2} = 0,$$

or

$$\sum_{n\geqslant 0} u_{n+2}(t)t^{n+2} - \sum_{n\geqslant 0} 2q(t)u_{n+1}(t)z^{n+2} + \sum_{n\geqslant 0} u_n(t)z^{n+2} = 0$$

which in turn gives us the recurrence relation

$$(3.2) \qquad u_{n+2}(t) - 2q(t)u_{n+1}(t) + u_n(t) = 0,$$

with $u_0(t) = 1$ and $u_1(t) = q(t)$. The second equation gives us

$$\sum_{n\geqslant 0} v_{n-1}(t)z^n - \sum_{n\geqslant 0} 2q(t)v_{n-1}(t)z^{n+1} + \sum_{n\geqslant 0} v_{n-1}(t)z^{n+2} = 1,$$

or

$$\sum_{n\geqslant 2} v_{n-1}(t)z^n - \sum_{n\geqslant 1} 2q(t)v_{n-1}(t)z^{n+1} + \sum_{n\geqslant 0} v_{n-1}(t)z^{n+2} = 0,$$

or

$$\sum_{n\geqslant 0} v_{n+1}(t)z^{n+2} - \sum_{n\geqslant 0} 2q(t)v_n(t)z^{n+2} + \sum_{n\geqslant 0} v_{n-1}(t)z^{n+2} = 0$$

which in turn gives us the recurrence relation

$$(3.3) \qquad v_{n+2}(t) - 2q(t)v_{n+1}(t) + v_n(t) = 0,$$

with $v_0(t) = 1$ and $v_1(t) = 2q(t)$.

The Chebyshev polynomials $T_n(x)$ satisfy the recursion relation

$$T_{n+2}(x) - 2xT_{n+1}(x) + T_n(t) = 0,$$

with $T_0(x) = 1$ and $T_1(x) = x$. We note that this first family of Chebyshev polynomials satisfy the differential equation

$$(1 - t^2)y''(t) - ty'(t) + n^2y = 0.$$

Since this is similar to (3.2) we use the above differential equation to find one for the v_n and u_n as follows Now for $q = q(x)$ by the chain rule

$$\frac{d}{dx}y(q) = \frac{dy(q)}{dq}\frac{dq}{dx},$$

$$\frac{d^2}{dx^2}y(q) = \frac{d}{dq}\left(\frac{dy(q)}{dq}\frac{dq}{dx}\right)\frac{dq}{dx}$$

$$= \frac{d^2y(q)}{d^2q}\frac{dq}{dx}\frac{dq}{dx} + \frac{dy(q)}{dq}\frac{d}{dq}\left(\frac{dq}{dx}\right)\frac{dq}{dx}$$

$$= \frac{d^2y(q)}{d^2q}\left(\frac{dq}{dx}\right)^2 + \frac{dy(q)}{dq}\frac{d^2q}{dx^2}.$$

so that

$$\frac{dy(q)}{dq} = \frac{1}{\frac{dq}{dx}}\frac{d}{dx}y(q),$$

$$\frac{d^2y(q)}{d^2q} = \frac{1}{\left(\frac{dq}{dx}\right)^2}\left(\frac{d^2}{dx^2}y(q) - \frac{dy(q)}{dq}\frac{d^2q}{dx^2}\right).$$

Inserting these into the Chebyshev equation with respect to q, we get

$$\frac{1 - q^2}{\left(\frac{dq}{dx}\right)^2}\left(\frac{d^2}{dx^2}y(q) - \frac{dy(q)}{dq}\frac{d^2q}{dx^2}\right) - \frac{q}{\frac{dq}{dx}}\frac{d}{dx}y(q) + n^2y(q) = 0.$$

or

$$(1 - q^2)\left(\frac{d^2}{dx^2}y(q) - \frac{dy(q)}{dq}\frac{d^2q}{dx^2}\right) - q\frac{dq}{dx}\frac{d}{dx}y(q) + n^2y(q)\left(\frac{dq}{dx}\right)^2 = 0.$$

or

$$(3.4) \qquad -P\frac{dq}{dx}\frac{d^2y}{dx^2} + \left(P\frac{d^2q}{dx^2} - q\left(\frac{dq}{dx}\right)\right)\frac{dy}{dx} + n^2\left(\frac{dq}{dx}\right)^2 y = 0.$$

This is the second order differential equation satisfied by the u_n.

The second family of Chebyshev polynomials satisfy the differential equation

$$(1 - t^2)y''(t) - 3ty'(t) + n(n + 2)y = 0.$$

so that the polynomials v_n satisfy

$$(3.5) \qquad -P\frac{dq}{dx}\frac{d^2y}{dx^2} + \left(P\frac{d^2q}{dx^2} - 3q\left(\frac{dq}{dx}\right)\right)\frac{dy}{dx} + n(n + 2)\left(\frac{dq}{dx}\right)^2 y = 0.$$

$\square$

Theorem 6. *Define polynomials a_n, b_{n-1} by $a_n + b_{n-1}\sqrt{P} = \lambda_2^n$ for $n \in \mathbb{N}$. Then $a_n = b_0^{-1}(b_n - a_1 b_{n-1})$, and $b_n(t)$ satisfies*

$$
\begin{aligned}
0 = {} & t^2(t^2 - a_1^2)(a_1't - a_1)y'' \\
& + \Big(-a_1''t^3(t^2 - a_1^2) + \big(2(1-n)t^3 - 3a_1 a_1' t^2 + a_1^2(2n+1)t\big)(a_1't - a_1) \Big)y' \\
& + \Big(-a_1''t^2 \left(-nt^2 - a_1 a_1'(n+2)t + 2a_1^2(n+1) \right) \\
& + \Big(\big(n(n-1) + \big(n(a_1')^2 - a_1 a_1''\big)(n+2)\big)t^2 - na_1 a_1'(2n+1)t \Big)(a_1't - a_1) \Big)y.
\end{aligned}
$$

Proof. We now will apply the technique used in Arfken and Weber's book "Mathematical Methods for Physicists". First we have

$$
\begin{aligned}
\sum_{n \geqslant 0} a_n z^n + \sum_{n \geqslant 1} b_{n-1} z^n \sqrt{P} &= \sum_{n \geqslant 0} (a_1 + b_0\sqrt{P})^n z^n \\
&= \frac{1}{1 - (a_1 + b_0\sqrt{P})z} = \frac{1 - a_1 z + b_0\sqrt{p}z}{1 - 2a_1 z + t^2 z^2}
\end{aligned}
$$

as $a_1^2 - b_0^2 P = t^2$. The first five polynomials b_n are

$$
b_0 = \sqrt{\frac{\beta + 1}{2}}, \quad b_1 = 2a_1 b_0,
$$
$$
b_2 = b_0(4a_1^2 - t^2), \quad b_3 = 4a_1 b_0(2a_1^2 - t^2)
$$
$$
b_4 = b_0\left(16a_1^4 - 12a_1^2 t^2 + t^4\right).
$$

We differentiate the generating function

$$
\sum_{n \geqslant 0} w_n(t)z^n = \frac{b_0}{(1 - 2a_1 z + t^2 z^2)^\alpha},
$$

and then in the end set $\alpha = 1$ to get our desired differential equation. If one takes $\alpha = 1$ first and then differentiate, then it seems to us that it is not possible to get the second order differential equation that the b_n satisfy. However to prove that it is the desired differential equation we need to check by induction using the recursion relation that the differential equation is satisfied.

To that end we calculate

$$
\begin{aligned}
\sum_{n \geqslant 0} n w_n(t)z^{n-1} &= \frac{\alpha b_0(2a_1 - 2t^2 z)}{(1 - 2a_1 z + t^2 z^2)^{\alpha+1}} \\
&= \frac{\alpha b_0(2a_1 - 2t^2 z)}{(1 - 2a_1 z + t^2 z^2)} \sum_{n \geqslant 0} w_n(t)z^n
\end{aligned}
$$

so that

$$(1 - 2a_1 z + t^2 z^2) \sum_{n \geqslant 0} n w_n z^{n-1} = \frac{\alpha b_0 (2a_1 - 2t^2 z)}{(1 - 2a_1 z + t^2 z^2)^\alpha}$$

$$= \alpha(2a_1 - 2t^2 z) \sum_{n \geqslant 0} w_n z^n$$

The left hand side expands out to

$$(1 - 2a_1 z + t^2 z^2) \sum_{n \geqslant 0} n w_n z^{n-1}$$

$$= \sum_{n \geqslant 0} n w_n z^{n-1} - \sum_{n \geqslant 0} 2n a_1 w_n z^n + \sum_{n \geqslant 0} n t^2 w_n z^{n+1}$$

$$= w_1 + \sum_{n \geqslant 2} n w_n z^{n-1} - \sum_{n \geqslant 1} 2n a_1 w_n z^n + \sum_{n \geqslant 0} n t^2 w_n z^{n+1}$$

$$= w_1$$

$$+ \sum_{n \geqslant 0} \left((n+2) w_{n+2} - 2(n+1) a_1 w_{n+1} + n t^2 w_n \right) z^{n+1}.$$

The right hand side expands out to

$$\alpha(2a_1 - 2t^2 z) \sum_{n \geqslant 0} w_n z^n = \sum_{n \geqslant 0} 2\alpha a_1 w_n z^n - \sum_{n \geqslant 0} 2\alpha t^2 w_n z^{n+1}$$

$$= 2\alpha a_1 w_0 + \sum_{n \geqslant 1} 2\alpha a_1 w_n z^n - \sum_{n \geqslant 0} 2\alpha t^2 w_n z^{n+1}$$

$$= 2\alpha a_1 w_0 + \sum_{n \geqslant 0} 2\alpha \left(a_1 w_{n+1} - t^2 w_n \right) z^{n+1}.$$

Equating these two expansions we get

$$2\alpha a_1 w_0 = w_1,$$

and

$$(n+2) w_{n+2} - 2(n+1) a_1 w_{n+1} + n t^2 w_n = 2\alpha \left(a_1 w_{n+1} - t^2 w_n \right)$$

or

$$(n+2) w_{n+2} - 2a_1 (n+1+\alpha) w_{n+1} + (n+2\alpha) t^2 w_n = 0.$$

We rewrite this recursion relation as

$$(3.6) \qquad (n+1) w_{n+1} - 2a_1 (n+\alpha) w_n + (n-1+2\alpha) t^2 w_{n-1} = 0,$$

and then differentiate to get

$$(n+1) w'_{n+1} - 2a_1 (n+\alpha) w'_n - 2a'_1 (n+\alpha) w_n$$

$$+ (n-1+2\alpha) t^2 w'_{n-1} + 2(n-1+2\alpha) t w_{n-1} = 0,$$

which if we multiply by t and substitute in the recursion relation gives us

$$(3.7) \qquad \begin{aligned} 0 &= (n+1) t w'_{n+1} - 2a_1 (n+\alpha) t w'_n - 2(n+\alpha)(a'_1 t - 2a_1) w_n \\ &\quad + (n-1+2\alpha) t^3 w'_{n-1} - 2(n+1) w_{n+1}. \end{aligned}$$

On the other hand if we differentiate the generating function with respect to t we get

$$\frac{d}{dt}\frac{b_0}{(1 - 2a_1 z + t^2 z^2)^\alpha} = \frac{\alpha b_0(2a_1' z - 2tz^2)}{(1 - 2a_1 z + t^2 z^2)^{\alpha+1}}$$

$$= \frac{\alpha b_0(2a_1' z - 2tz^2)}{(1 - 2a_1 z + t^2 z^2)}\sum_{n \geqslant 0} w_n z^n$$

So we get

$$(1 - 2a_1 z + t^2 z^2)\sum_{n \geqslant 0} w_n' z^n = \alpha(2a_1' z - 2tz^2)\sum_{n \geqslant 0} w_n z^n$$

and the left hand side expands to

$$(1 - 2a_1 z + t^2 z^2)\sum_{n \geqslant 0} w_n' z^n = \sum_{n \geqslant 0} w_n' z^n - \sum_{n \geqslant 0} 2a_1 w_n' z^{n+1} + \sum_{n \geqslant 0} t^2 w_n' z^{n+2}$$

$$= w_0' + w_1' z - 2a_1 w_0' z$$

$$+ \sum_{n \geqslant 2} w_n' z^n - \sum_{n \geqslant 0} 2a_1 w_n' z^{n+1} + \sum_{n \geqslant 0} t^2 w_n' z^{n+2}$$

$$= w_0' + w_1' z - 2a_1 w_0' z$$

$$+ \sum_{n \geqslant 0} \left(w_{n+2}' - 2a_1 w_{n+1}' + t^2 w_n'\right) z^{n+2}$$

while the right hand side expands to

$$\alpha(2a_1' z - 2tz^2)\sum_{n \geqslant 0} w_n z^n = \sum_{n \geqslant 0} 2\alpha a_1' w_n z^{n+1} - \sum_{n \geqslant 0} 2\alpha t w_n z^{n+2}$$

$$= 2\alpha a_1' w_0 z + \sum_{n \geqslant 1} 2\alpha a_1' w_n z^{n+1} - \sum_{n \geqslant 0} 2\alpha t w_n z^{n+2}$$

$$= 2\alpha a_1' w_0 z + \sum_{n \geqslant 0} 2\alpha \left(a_1' w_{n+1} - t w_n\right) z^{n+2}$$

so that

$$2\alpha \left(a_1' w_{n+1} - t w_n\right) = w_{n+2}' - 2a_1 w_{n+1}' + t^2 w_n'.$$

We rewrite as

$$(3.8) \qquad 2\alpha \left(a_1' w_n - t w_{n-1}\right) = w_{n+1}' - 2a_1 w_n' + t^2 w_{n-1}'.$$

Which we multiply by $(n + \alpha)$

$$2\alpha(n + \alpha)\left(a_1' w_n - t w_{n-1}\right) - (n + \alpha)w_{n+1}'$$

$$+ 2(n + \alpha)a_1 w_n' - t^2(n + \alpha)w_{n-1}' = 0$$

and add to

$$(n + 1)w_{n+1}' - 2a_1(n + \alpha)w_n' - 2a_1'(n + \alpha)w_n$$

$$+ (n - 1 + 2\alpha)t^2 w_{n-1}' + 2(n - 1 + 2\alpha)t w_{n-1} = 0,$$

to get

$$(1 - \alpha)w_{n+1}' + 2(\alpha - 1)a_1'(n + \alpha)w_n + 2(1 - n - \alpha)(\alpha - 1)t w_{n-1}$$

$$+ t^2(\alpha - 1)w_{n-1}' = 0$$

Assuming $\alpha \neq 1$ we get

$$-w'_{n+1} + 2a'_1(n+\alpha)w_n + 2(1-n-\alpha)tw_{n-1} + t^2w'_{n-1} = 0$$

or

$$(3.9) \qquad w'_{n+1} - t^2w'_{n-1} = 2a'_1(n+\alpha)w_n + 2(1-n-\alpha)tw_{n-1}$$

Now recall the recursion relation

$$(3.10) \qquad (n+1)w_{n+1} - 2a_1(n+\alpha)w_n + (n-1+2\alpha)t^2w_{n-1} = 0,$$

We multiply (3.9) by $(n-1+2\alpha)t$ to get

$$0 = (n-1+2\alpha)tw'_{n+1} - (n-1+2\alpha)t^3w'_{n-1} - 2a'_1(n+\alpha)(n-1+2\alpha)tw_n$$
$$- 2(n-1+2\alpha)(1-n-\alpha)t^2w_{n-1}$$

and add to it $2(1-n-\alpha)$ times (3.10)

$$0 = (n-1+2\alpha)tw'_{n+1} - (n-1+2\alpha)t^3w'_{n-1} - 2a'_1(n+\alpha)(n-1+2\alpha)tw_n$$
$$- 2(n-1+2\alpha)(1-n-\alpha)t^2w_{n-1} + 2(1-n-\alpha)(n+1)w_{n+1}$$
$$- 4(1-n-\alpha)a_1(n+\alpha)w_n + 2(1-n-\alpha)(n-1+2\alpha)t^2w_{n-1}$$

$$= (n-1+2\alpha)tw'_{n+1} + (n-1+2\alpha)t^3w'_{n-1} - 2a'_1(n+\alpha)(n-1+2\alpha)tw_n$$
$$+ 2(1-n-\alpha)(n+1)w_{n+1} - 4(1-n-\alpha)a_1(n+\alpha)w_n$$

$$= (n-1+2\alpha)tw'_{n+1} - (n-1+2\alpha)t^3w'_{n-1} + 2(1-n-\alpha)(n+1)w_{n+1}$$
$$(3.11)$$
$$- 2(n+\alpha)\left(a'_1(n-1+2\alpha)t + 2a_1(1-n-\alpha)\right)w_n$$

to which we add (3.7)

$$0 = (n-1+2\alpha)tw'_{n+1} - (n-1+2\alpha)t^3w'_{n-1} + 2(1-n-\alpha)(n+1)w_{n+1}$$
$$- 2(n+\alpha)\left(a'_1(n-1+2\alpha)t + 2a_1(1-n-\alpha)\right)w_n$$
$$+ (n+1)tw'_{n+1} - 2a_1(n+\alpha)tw'_n - 2(n+\alpha)(a'_1t - 2a_1)w_n$$
$$+ (n-1+2\alpha)t^3w'_{n-1} - 2(n+1)w_{n+1}$$

$$= 2(\alpha+n)tw'_{n+1} - 2(n+\alpha)(n+1)w_{n+1}$$
$$- 2(n+\alpha)\left(a'_1(n+2\alpha)t - 2a_1(n+\alpha)\right)w_n$$
$$- 2a_1(n+\alpha)tw'_n$$

Thus if $\alpha \neq -n$, one has

$$(3.12) \qquad \begin{aligned} 0 &= tw'_{n+1} - a_1tw'_n - (n+1)w_{n+1} \\ &\quad - \left(a'_1(n+2\alpha)t - 2a_1(n+\alpha)\right)w_n \end{aligned}$$

Reconsider (3.7) from which we subtract $(n+1)t$ times (3.9)

$$
\begin{aligned}
0 = {} & (n+1)tw'_{n+1} - 2a_1(n+\alpha)tw'_n - 2(n+\alpha)(a'_1 t - 2a_1)w_n \\
& + (n-1+2\alpha)t^3 w'_{n-1} - 2(n+1)w_{n+1} \\
& - (n+1)tw'_{n+1} + (n+1)t^3 w'_{n-1} + 2a'_1(n+\alpha)(n+1)tw_n \\
& + 2(1-n-\alpha)(n+1)t^2 w_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
= {} & -2a_1(n+\alpha)tw'_n \\
& + (n-1+2\alpha)t^3 w'_{n-1} + (n+1)t^3 w'_{n-1} \\
& + 2a'_1(n+\alpha)(n+1)tw_n - 2(n+\alpha)(a'_1 t - 2a_1)w_n \\
& - 2(n+1)w_{n+1} + 2(1-n-\alpha)(n+1)t^2 w_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
= {} & -2a_1(n+\alpha)tw'_n \\
& + 2(n+\alpha)t^3 w'_{n-1} \\
& + 2(n+\alpha)(a'_1(n+1)t - (a'_1 t - 2a_1))w_n \\
& - 2(n+1)w_{n+1} + 2(1-n-\alpha)(n+1)t^2 w_{n-1}
\end{aligned}
$$

We then use (3.6) to eliminate w_{n+1}:

$$
\begin{aligned}
0 = {} & -2a_1(n+\alpha)tw'_n + 2(n+\alpha)t^3 w'_{n-1} \\
& + 2(n+\alpha)(a'_1(n+1)t - (a'_1 t - 2a_1))w_n \\
& - 4a_1(n+\alpha)w_n + 2(n-1+2\alpha)t^2 w_{n-1} + 2(1-n-\alpha)(n+1)t^2 w_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
= {} & -2a_1(n+\alpha)tw'_n + 2(n+\alpha)t^3 w'_{n-1} \\
& + 2(n+\alpha)(a'_1(n+1)t - (a'_1 t - 2a_1))w_n - 4a_1(n+\alpha)w_n \\
& + 2\left((n-1+2\alpha) + (1-n-\alpha)(n+1)\right)t^2 w_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
= {} & -2a_1(n+\alpha)tw'_n + 2(n+\alpha)t^3 w'_{n-1} \\
& + 2(n+\alpha)a'_1 ntw_n \\
& - 2(n+\alpha)(n-1)t^2 w_{n-1}
\end{aligned}
$$

so that if $\alpha \neq n$ we get

$$
0 = -2a_1 tw'_n + 2t^3 w'_{n-1} + 2a'_1 ntw_n - 2(n-1)t^2 w_{n-1}.
$$

Replacing n by $n+1$ in the above we get

$$
(3.13) \qquad 0 = -a_1 tw'_{n+1} + t^3 w'_n + a'_1(n+1)tw_{n+1} - nt^2 w_n
$$

We add to (3.13) to a_1 times (3.12)

$$
\begin{aligned}
0 = {} & -a_1 t w'_{n+1} + t^3 w'_n + a'_1(n+1)tw_{n+1} - nt^2 w_n \\
& + a_1 t w'_{n+1} - a_1^2 t w'_n - (n+1)a_1 w_{n+1} - a_1\left(a'_1(n+2\alpha)t - 2a_1(n+\alpha)\right)w_n
\end{aligned}
$$

$$
\begin{aligned}
= {} & t^3 w'_n + a'_1(n+1)tw_{n+1} - nt^2 w_n \\
& - a_1^2 t w'_n - (n+1)a_1 w_{n+1} - a_1\left(a'_1(n+2\alpha)t - 2a_1(n+\alpha)\right)w_n
\end{aligned}
$$

$$
\begin{aligned}
= {} & t(t^2 - a_1^2)w'_n + (n+1)(a'_1 t - a_1)w_{n+1} \\
& + \left(-nt^2 - a_1 a'_1(n+2\alpha)t + 2a_1^2(n+\alpha)\right)w_n
\end{aligned}
$$

(3.14)

If we differentiate this we get

$$
\begin{aligned}
0 = {} & t(t^2 - a_1^2)w''_n + (3t^2 - a_1^2 - 2ta_1 a'_1)w'_n \\
& + (n+1)a''_1 t w_{n+1} + (n+1)(a'_1 t - a_1)w'_{n+1} \\
& + (-2nt - (a'_1)^2(n+2\alpha)t - a_1 a''_1(n+2\alpha)t - a_1 a'_1(n+2\alpha) \\
& + 4a_1 a'_1(n+\alpha))w_n + \left(-nt^2 - a_1 a'_1(n+2\alpha)t + 2a_1^2(n+\alpha)\right)w'_n
\end{aligned}
$$

$$
\begin{aligned}
= {} & t(t^2 - a_1^2)w''_n \\
& + (n+1)a''_1 t w_{n+1} + (n+1)(a'_1 t - a_1)w'_{n+1} \\
& + \left(\left(-2n - \left((a'_1)^2 + a_1 a''_1\right)(n+2\alpha)\right)t + a_1 a'_1(3n+2\alpha)\right)w_n \\
& + \left((3-n)t^2 - a_1 a'_1(n+2\alpha+2)t + a_1^2(2n+2\alpha-1)\right)w'_n
\end{aligned}
$$

The next two steps are to eliminate the term with w'_{n+1} in it and then use (3.14) to remove the resulting equation with an w_{n+1} residing in it. To that end we multiply the above by t and use (3.13) and a'_1 times (3.12) to

remove the w'_{n+1}:

$$0 = t^2(t^2 - a_1^2)w''_n$$
$$+ (n+1)a''_1 t^2 w_{n+1} + (n+1)a'_1 t^2 w'_{n+1} - (n+1)a_1 t w'_{n+1}$$
$$+ \left((-2n - \left((a'_1)^2 + a_1 a''_1\right)(n+2\alpha)\right)t + a_1 a'_1(3n+2\alpha))\right) t w_n$$
$$+ \left((3-n)t^2 - a_1 a'_1(n+2\alpha+2)t + a_1^2(2n+2\alpha-1)\right) t w'_n$$

$$= t^2(t^2 - a_1^2)w''_n$$
$$+ (n+1)a''_1 t^2 w_{n+1}$$
$$+ (n+1)a'_1 t \left(a_1 t w'_n + (n+1)w_{n+1} + \left(a'_1(n+2\alpha)t - 2a_1(n+\alpha)\right) w_n\right)$$
$$- (n+1)\left(t^3 w'_n + a'_1(n+1)t w_{n+1} - n t^2 w_n\right)$$
$$+ \left((-2n - \left((a'_1)^2 + a_1 a''_1\right)(n+2\alpha)\right)t + a_1 a'_1(3n+2\alpha))\right) t w_n$$
$$+ \left((3-n)t^2 - a_1 a'_1(n+2\alpha+2)t + a_1^2(2n+2\alpha-1)\right) t w'_n$$

$$= t^2(t^2 - a_1^2)w''_n$$
$$+ (n+1)\Big(a''_1 t^2 w_{n+1}$$
$$\quad + a_1 a'_1 t^2 w'_n + (n+1)a'_1 t w_{n+1} + a'_1 \left(a'_1(n+2\alpha)t - 2a_1(n+\alpha)\right) t w_n$$
$$\quad - t^3 w'_n - a'_1(n+1)t w_{n+1} + n t^2 w_n\Big)$$
$$+ \left((-2n - \left((a'_1)^2 + a_1 a''_1\right)(n+2\alpha)\right)t + a_1 a'_1(3n+2\alpha))\right) t w_n$$
$$+ \left((3-n)t^2 - a_1 a'_1(n+2\alpha+2)t + a_1^2(2n+2\alpha-1)\right) t w'_n$$

$$= t^2(t^2 - a_1^2)w''_n$$
$$+ (n+1)\Big(a''_1 t^2 w_{n+1}$$
$$\quad + \left(a_1 a'_1 t^2 - t^3\right) w'_n + \left(n t^2 + a'_1 \left(a'_1(n+2\alpha)t^2 - 2a_1(n+\alpha)t\right)\right) w_n\Big)$$
$$+ \left((-2n - \left((a'_1)^2 + a_1 a''_1\right)(n+2\alpha)\right)t + a_1 a'_1(3n+2\alpha))\right) t w_n$$
$$+ \left((3-n)t^2 - a_1 a'_1(n+2\alpha+2)t + a_1^2(2n+2\alpha-1)\right) t w'_n$$

$$= t^2(t^2 - a_1^2)w''_n$$
$$+ (n+1)a''_1 t^2 w_{n+1}$$
$$\quad + (n+1)\left(a_1 a'_1 t^2 - t^3\right) w'_n + (n+1)\left(n t^2 + a'_1 \left(a'_1(n+2\alpha)t^2 - 2a_1(n+\alpha)t\right)\right) w_n$$
$$+ \left((-2n - \left((a'_1)^2 + a_1 a''_1\right)(n+2\alpha)\right)t + a_1 a'_1(3n+2\alpha))\right) t w_n$$
$$+ \left((3-n)t^2 - a_1 a'_1(n+2\alpha+2)t + a_1^2(2n+2\alpha-1)\right) t w'_n$$

$$0 = t^2(t^2 - a_1^2)w_n''$$
$$+ (n+1)a_1''t^2 w_{n+1}$$
$$+ \Big(\big((-2n - ((a_1')^2 + a_1 a_1'')(n+2\alpha)) t + a_1 a_1'(3n+2\alpha) \big) t$$
$$+ (n+1)\big(nt^2 + a_1'(a_1'(n+2\alpha)t^2 - 2a_1(n+\alpha)t) \big) \Big) w_n$$
$$+ \big((3-n)t^3 - a_1 a_1'(n+2\alpha+2)t^2 + a_1^2(2n+2\alpha-1)t + (n+1)(a_1 a_1' t^2 - t^3) \big) w_n'$$

$$= t^2(t^2 - a_1^2)w_n'' + (n+1)a_1''t^2 w_{n+1}$$
$$+ \Big(\big(n(n-1) + (n(a_1')^2 - a_1 a_1'')(n+2\alpha) \big) t^2 - na_1 a_1'(2n+2\alpha-1)t \Big) w_n$$
$$+ \big(2(1-n)t^3 - a_1 a_1'(2\alpha+1)t^2 + a_1^2(2n+2\alpha-1)t \big) w_n'$$

We multiply this by $a_1' t - a_1$ in order to use (3.14) to eliminate w_{n+1}:

$$0 = t^2(t^2 - a_1^2)w_n''(a_1' t - a_1) + (n+1)a_1''t^2(a_1' t - a_1)w_{n+1}$$
$$+ \Big(\big(n(n-1) + (n(a_1')^2 - a_1 a_1'')(n+2\alpha) \big) t^2 - na_1 a_1'(2n+2\alpha-1)t \Big)(a_1' t - a_1)w_n$$
$$+ \big(2(1-n)t^3 - a_1 a_1'(2\alpha+1)t^2 + a_1^2(2n+2\alpha-1)t \big)(a_1' t - a_1)w_n'$$

$$= t^2(t^2 - a_1^2)(a_1' t - a_1)w_n'' + a_1''t^2\Big(-t(t^2 - a_1^2)w_n'$$
$$- \big(-nt^2 - a_1 a_1'(n+2\alpha)t + 2a_1^2(n+\alpha) \big) w_n \Big)$$
$$+ \Big(\big(n(n-1) + (n(a_1')^2 - a_1 a_1'')(n+2\alpha) \big) t^2 - na_1 a_1'(2n+2\alpha-1)t \Big)(a_1' t - a_1)w_n$$
$$+ \big(2(1-n)t^3 - a_1 a_1'(2\alpha+1)t^2 + a_1^2(2n+2\alpha-1)t \big)(a_1' t - a_1)w_n'$$

$$= t^2(t^2 - a_1^2)(a_1' t - a_1)w_n'' - a_1''t^3(t^2 - a_1^2)w_n'$$
$$- a_1''t^2\big(-nt^2 - a_1 a_1'(n+2\alpha)t + 2a_1^2(n+\alpha) \big) w_n$$
$$+ \Big(\big(n(n-1) + (n(a_1')^2 - a_1 a_1'')(n+2\alpha) \big) t^2 - na_1 a_1'(2n+2\alpha-1)t \Big)(a_1' t - a_1)w_n$$
$$+ \big(2(1-n)t^3 - a_1 a_1'(2\alpha+1)t^2 + a_1^2(2n+2\alpha-1)t \big)(a_1' t - a_1)w_n'$$

$$= t^2(t^2 - a_1^2)(a_1' t - a_1)w_n''$$
$$+ \Big(-a_1''t^3(t^2 - a_1^2)$$
$$+ \big(2(1-n)t^3 - a_1 a_1'(2\alpha+1)t^2 + a_1^2(2n+2\alpha-1)t \big)(a_1' t - a_1) \Big) w_n'$$
$$+ \Big(-a_1''t^2\big(-nt^2 - a_1 a_1'(n+2\alpha)t + 2a_1^2(n+\alpha) \big)$$
$$+ \Big(\big(n(n-1) + (n(a_1')^2 - a_1 a_1'')(n+2\alpha) \big) t^2 - na_1 a_1'(2n+2\alpha-1)t \Big)(a_1' t - a_1) \Big) w_n$$

Recall

$$\sum_{n\geqslant 0} w_n(t)z^n = \frac{b_0}{(1-2a_1 z+t^2 z^2)^\alpha} = \sum_{n\geqslant 1} b_{n-1}z^{n-1} = \sum_{n\geqslant 0} b_n z^n$$

Setting $\alpha = 1$ we get the differential equation that the b_n satisfy:

(3.15)
$$0 = t^2(t^2 - a_1^2)(a_1' t - a_1)y''$$
$$+ \Big(- a_1'' t^3(t^2 - a_1^2) + \big(2(1-n)t^3 - 3a_1 a_1' t^2 + a_1^2(2n+1)t\big)(a_1' t - a_1)\Big)y'$$
$$+ \Big(- a_1'' t^2 \big(-nt^2 - a_1 a_1'(n+2)t + 2a_1^2(n+1)\big)$$
$$+ \Big(\big(n(n-1) + \big(n(a_1')^2 - a_1 a_1''\big)(n+2)\big)t^2 - na_1 a_1'(2n+1)t\Big)(a_1' t - a_1)\Big)y$$

where $a_1(t) := \dfrac{t^2 - 1}{\sqrt{2(\beta - 1)}}$. Note that $t^2 - a_1^2 = 2\dfrac{p}{\beta + 1}$.

After dividing this by t and simplifying, this differential equation becomes

(3.16)
$$0 = t\left(t^2 + 1\right)\left(t^4 - 2\beta t^2 + 1\right)y''$$
$$- \left((2n-3)t^6 + t^4(-4\beta n + 2n - 5) + t^2(4\beta - 4\beta n + 2n + 3) + 2n + 1\right)y'$$
$$- 2\left(2nt^5 + nt^3(\beta + (\beta+1)n + 5) + nt(-\beta + (\beta+1)n + 1)\right)y.$$

Thus we have written the above differential equation in the form $P^1(t)y'' + Q_n(t)y' + R_n(t)y = 0$, with

$$P^1(t) = t\left(t^2 + 1\right)\left(t^4 - 2\beta t^2 + 1\right),$$
$$Q_n(t) = -\left((2n-3)t^6 + t^4(-4\beta n + 2n - 5) + t^2(4\beta - 4\beta n + 2n + 3) + 2n + 1\right)$$
$$= -2(n-1)\left(t^2 + 1\right)\left(t^4 - 2\beta t^2 + 1\right) + t^6 + (4\beta + 3)t^4 - 5t^2 - 3,$$
$$R_n(t) = -2n\left(2t^5 + t^3(\beta + (\beta+1)n + 5) + t(-\beta + (\beta+1)n + 1)\right).$$

Note $Q_{n+1}(t) = Q_n(t) - 2P^1 t^{-1}$. The above derivation required that $\alpha \notin \mathbb{Z}$. We will show that the b_n satisfy the above differential equation (3.16) in the next proposition. $\square$

Proposition 7. *The polynomials $b_n(t)$ satisfy (3.16) for all $n \geqslant 1$.*

Proof. We prove this by induction on $n \geqslant 0$. We have the recursion relation (3.6) so that for $\alpha = 1$ we get for $n \geqslant 1$

(3.17)
$$b_{n+1}(t) - 2a_1(t)b_n(t) + t^2 b_{n-1}(t) = 0.$$

Then

(3.18) $b'_{n+1} = 2(a_1' b_n + a_1 b_n') - 2t b_{n-1} - t^2 b_{n-1}'$

(3.19) $b''_{n+1} = 2(a_1'' b_n + 2a_1' b_n' + a_1 b_n'') - 2b_{n-1} - 4t b_{n-1}' - t^2 b_{n-1}''.$

Thus

$$P^1(t)b''_{n+1} + Q_{n+1}b'_{n+1}$$
$$= P^1\left(2(a''_1 b_n + 2a'_1 b'_n + a_1 b''_n) - 2b_{n-1} - 4tb'_{n-1} - t^2 b''_{n-1}\right)$$
$$+ Q_{n+1}\left(2(a'_1 b_n + a_1 b'_n) - 2tb_{n-1} - t^2 b'_{n-1}\right)$$

$$= 2P^1 a''_1 b_n + 4P^1 a'_1 b'_n + 2a_1 P^1 b''_n - 2P^1 b_{n-1} - 4tP^1 b'_{n-1} - t^2 P^1 b''_{n-1}$$
$$+ 2Q_{n+1} a'_1 b_n + 2Q_{n+1} a_1 b'_n - 2tQ_{n+1} b_{n-1} - t^2 Q_{n+1} b'_{n-1}$$

$$= 2P^1 a''_1 b_n + 4P^1 a'_1 b'_n + 2a_1 P^1 b''_n - 2P^1 b_{n-1} - 4tP^1 b'_{n-1} - t^2 P^1 b''_{n-1}$$
$$+ 2Q_{n+1} a'_1 b_n + 2(Q_n - 2P^1 t^{-1})a_1 b'_n - 2tQ_{n+1} b_{n-1}$$
$$- t^2(Q_{n-1} - 4P^1 t^{-1})b'_{n-1}$$

$$= 2P^1 a''_1 b_n + 4P^1 a'_1 b'_n - 2P^1 b_{n-1} - 4tP^1 b'_{n-1}$$
$$+ 2Q_{n+1} a'_1 b_n - 4P^1 t^{-1} a_1 b'_n - 2tQ_{n+1} b_{n-1}$$
$$+ 4t^2 P^1 t^{-1} b'_{n-1} - 2R_n a_1 b_n + t^2 R_{n-1} b_{n-1}$$

$$= (2P^1 a''_1 + 2Q_{n+1} a'_1 - 2R_n a_1)b_n$$
$$+ 4P^1 a'_1 b'_n - 4P^1 t^{-1} a_1 b'_n$$
$$- (2P^1 + 2tQ_{n+1} - t^2 R_{n-1})b_{n-1}$$

$$= -R_{n+1} b_{n+1} - 4na'_1(1 + t^2)^2(t^2 - \beta)b_n$$
$$+ 4P^1 a'_1 b'_n - 4P^1 t^{-1} a_1 b'_n$$
$$+ 4(n + 1)t\left(t^2 + 1\right)^3 b_{n-1}$$

Thus the proof reduces to proving showing the terms after $-R_{n+1}b_{n+1}$ vanish for all $n \geqslant 1$. Set

$$\mathrm{Rem}(t, n) := -4na'_1(1 + t^2)^2(t^2 - \beta)b_n + 4P^1(a'_1 - t^{-1}a_1)b'_n$$
$$+ 4(n + 1)t\left(t^2 + 1\right)^3 b_{n-1}.$$

If one expands out

(3.20) $$\mathrm{Rem}(t, n + 1) - 2a(t)\mathrm{Rem}(t, n) + t^2\mathrm{Rem}(t, n - 1)$$

using (3.6) and (3.7), then (3.20) reduces to zero. Hence by induction we have proven that the b_n satisfy (3.16). □

In order to put this into Sturm-Louiville like form we first divide by $t\left(t^2 + 1\right)\left(t^4 - 2\beta t^2 + 1\right)$, and then multiply the differential equation above

by the integrating factor

$$\exp \int \frac{Q_n(t)}{P^1(t)} \, dt = \frac{\left(t^4 - 2\beta t^2 + 1\right)^{3/2}}{\left(t^2 + 1\right) t^{2n+1}}.$$

We obtain

$$(3.21) \qquad \frac{d}{dt}\left(\frac{\left(t^4 - 2\beta t^2 + 1\right)^{3/2}}{\left(t^2 + 1\right) t^{2n+1}} \frac{dy}{dt}\right) + \frac{\left(t^4 - 2\beta t^2 + 1\right)^{1/2} R_n(t)}{(t^2 + 1)^2 t^{2n+2}} y = 0.$$

If we set

$$\frac{d}{d\tau_n} := \frac{\left(t^4 - 2\beta t^2 + 1\right)^{3/2}}{\left(t^2 + 1\right) t^{2n+1}} \frac{d}{dt}$$

we get

$$\tau_n(t) = \int \frac{\left(t^2 + 1\right) t^{2n+1}}{\left(t^4 - 2\beta t^2 + 1\right)^{3/2}} \, dt,$$

which is an elliptic integral and can be expressed in the form of a series about $t = 0$. We will not write this out as it doesn't seem to simplify matters. Then (3.21) becomes

$$\frac{\left(t^2 + 1\right) t^{2n+1}}{\left(t^4 - 2\beta t^2 + 1\right)^{3/2}} \frac{d^2 y}{d\tau_n^2} + \frac{\left(t^4 - 2\beta t^2 + 1\right)^{1/2} R_n(t)}{(t^2 + 1)^2 t^{2n+2}} y = 0.$$

or

$$(3.22) \qquad \boxed{\frac{d^2 y}{d\tau_n^2} + \frac{\left(t^4(\tau_n) - 2\beta t^2(\tau_n) + 1\right)^2 R_n(t(\tau_n))}{(t^2(\tau_n) + 1)^3 t^{4n+3}(\tau_n)} y = 0.}$$

Theorem 8. *Define polynomials c_n, d_{n-1} by $c_n + d_n\sqrt{P} = \lambda_1^n$ for $n \in \mathbb{N}$. Then $d_n(t)$ satisfies*

$$P^2(t)y'' + Q_n^2(t)y' + R_n^2(t)y = 0$$

with

$P^2(t) = t(1 - t^2)(t^4 - 2\beta t^2 + 1),$
$Q_n^2(t) = (2n - 3)t^6 - t^4(4\beta n + 2n - 5) + t^2(-4\beta + 4\beta n + 2n + 3) - 2n - 1,$
$R_n^2(t) = 2nt\left(\beta - (\beta - 1)n + t^2(\beta + (\beta - 1)n - 5) + 2t^4 + 1\right)$

Proof. We next will derive second order linear differential equations satisfied by c_n and d_n where are defined by $c_n + d_n\sqrt{p(t)} = (c_1 + d_1\sqrt{p})^n$, for

$$(3.23) \qquad c_0 = 1, \quad d_0 = 0, \quad c_1 = \frac{t^2 + 1}{\sqrt{2(\beta + 1)}} \quad d_1 = \sqrt{\frac{\beta - 1}{2}}.$$

First we have

$$\sum_{n\geqslant 0} c_n z^n + \sum_{n\geqslant 0} d_n z^n \sqrt{p} = \sum_{n\geqslant 0} (c_1 + d_1\sqrt{p})^n z^n$$

$$= \sum_{n\geqslant 0} (\lambda_2(\beta, t))^n z^n$$

$$= \sum_{n\geqslant 0} (-\imath)^n (\lambda_1(-\beta, \imath t))^n z^n$$

$$= \sum_{n\geqslant 0} (-\imath)^n (a_1(-\beta, \imath t) + b_1(-\beta, \imath t)\sqrt{p})^n z^n$$

$$= \sum_{n\geqslant 0} (-\imath)^n (a_n(-\beta, \imath t) + b_n(-\beta, \imath t)\sqrt{p}) z^n$$

by the following formula

$$(3.24) \qquad -\imath\lambda_1(-\beta, \imath t) = -\imath\left(\frac{-t^2+1}{\sqrt{2(-\beta+1)}} + \sqrt{\frac{-\beta-1}{2}}\sqrt{p}\right)$$

$$= \frac{t^2-1}{\sqrt{2(\beta-1)}} + \sqrt{\frac{\beta+1}{2}}\sqrt{p}$$

$$= \lambda_2(\beta, t),$$

where $\imath = \sqrt{-1}$. Thus

$$c_n(\beta, t) = (-\imath)^n a_n(-\beta, \imath t), \quad d_n(\beta, t) = (-\imath)^n b_n(-\beta, \imath t).$$

We then set

$$P^2(t) := t(1 - t^2)(t^4 - 2\beta t^2 + 1)$$

$$Q_n^2(t) = (2n - 3)t^6 - t^4(4\beta n + 2n - 5) + t^2(-4\beta + 4\beta n + 2n + 3) - 2n - 1$$

$$R_n^2(t) = 2nt\left(\beta - (\beta - 1)n + t^2(\beta + (\beta - 1)n - 5) + 2t^4 + 1\right)$$

Consequently the $d_n(\beta, t)$ satisfy the second order linear differential equation

$$(3.25) \qquad P^2(t)y'' + Q_n^2(t)y' + R_n^2(t)y = 0.$$

$\square$

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References

[BJ05]   Mikhail Belolipetsky and Gareth A. Jones. Automorphism groups of Riemann surfaces of genus $p + 1$, where p is prime. *Glasg. Math. J.*, 47(2):379–393, 2005.

[Bre00]   Thomas Breuer. *Characters and automorphism groups of compact Riemann surfaces*, volume 280 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.

[CGLZ1] B. Cox, X. Guo, R. Lu, K. Zhao, B. Cox, X. Guo, R. Lu, K. Zhao, n-Point
         Virasoro algebras and their modules of densities, *Commun. Contemp. Math.* 16
         (2014), no. 3, 1350047.
[CGLZ2] B. Cox, X. Guo, R. Lu, K. Zhao, Simple superelliptic Lie algebras,
         arXiv:1412.7777.
[DJKM]   E. Date, M. Jimbo, M. Kashiwara,T. Miwa, Landau-Lifshitz equation: soli-
         tons, quasi periodic solutions and infinite-dimensional Lie algebras, J. Phys.
         A16(2)(1983), 221236.
[DS04]   Artūras Dubickas and Jörn Steuding. The polynomial Pell equation. *Elem.
         Math.*, 59(4):133–143, 2004.
[FBZ01] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*, vol-
         ume 88 of *Mathematical Surveys and Monographs*. American Mathematical So-
         ciety, Providence, RI, 2001.
[Jor86] D. A. Jordan. On the ideals of a Lie algebra of derivations. *J. London Math. Soc.
         (2)*, 33(1):33–39, 1986.
[KL91]   David Kazhdan and George Lusztig. Affine Lie algebras and quantum groups.
         Internat. Math. Res. Notices, (2):21–29, 1991.
[KN87a] Igor Moiseevich Krichever and S. P. Novikov. Algebras of Virasoro type, Rie-
         mann surfaces and strings in Minkowski space. *Funktsional. Anal. i Prilozhen.*,
         21(4):47–61, 96, 1987.
[KN87b] Igor Moiseevich Krichever and S. P. Novikov. Algebras of Virasoro type, Riemann
         surfaces and the structures of soliton theory. *Funktsional. Anal. i Prilozhen.*,
         21(2):46–63, 1987.
[KN89]   Igor Moiseevich Krichever and S. P. Novikov. Algebras of Virasoro type, the
         energy-momentum tensor, and operator expansions on Riemann surfaces. *Funk-
         tsional. Anal. i Prilozhen.*, 23(1):24–40, 1989.
[Sch03a] Martin Schlichenmaier. Higher genus affine algebras of Krichever-Novikov type.
         Mosc. Math. J., 3(4):1395–1427, 2003.
[Sch03b] Martin Schlichenmaier. Local cocycles and central extensions for multipoint al-
         gebras of Krichever-Novikov type. *J. Reine Angew. Math.*, 559:53–94, 2003.
[Sha03] Tanush Shaska. Determining the automorphism group of a hyperelliptic curve.
         In *Proceedings of the 2003 International Symposium on Symbolic and Algebraic
         Computation*, pages 248–254 (electronic). ACM, New York, 2003.
[She03] O. K. She˘inman. Second-order Casimirs for the affine Krichever-Novikov alge-
         bras $\widehat{\mathfrak{gl}}_{g,2}$ and $\widehat{\mathfrak{sl}}_{g,2}$. In *Fundamental mathematics today (Russian)*, pages 372–404.
         Nezavis. Mosk. Univ., Moscow, 2003.
[She05] O. K. Sheinman. Highest-weight representations of Krichever-Novikov algebras
         and integrable systems. *Uspekhi Mat. Nauk*, 60(2(362)):177–178, 2005.
[Skr88] S. M. Skryabin. Regular Lie rings of derivations. *Vestnik Moskov. Univ. Ser. I
         Mat. Mekh.*, (3):59–62, 1988.
[SS98]   M. Schlichenmaier and O. K. Scheinman. The Sugawara construction and
         Casimir operators for Krichever-Novikov algebras. *J. Math. Sci. (New York)*,
         92(2):3807–3834, 1998. Complex analysis and representation theory, 1.
[SS99]   M. Shlichenmaier and O. K. Sheinman. The Wess-Zumino-Witten-Novikov the-
         ory, Knizhnik-Zamolodchikov equations, and Krichever-Novikov algebras. *Us-
         pekhi Mat. Nauk*, 54(1(325)):213–250, 1999.

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