FOURIER SERIES AND METRIC DIOPHANTINE APPROXIMATION: INHOMOGENEOUS DUFFIN-SCHAEFFER TYPE RESULTS

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Abstract. In this paper we discuss some results about inhomogeneous metric Diophantine approximation. By using a Fourier analytic method we see that in some sense the inhomogeneous Diophantine approximation is the same as its homogeneous counterpart. For example, when dealing with the Hausdorff dimension of well approximable numbers in unit interval, the inhomogeneity does not create any changes. As a by-product, this Fourier analytic method can also help us to get some partial result towards the (inhomogeneous) Duffin-Schaeffer conjecture.

1. Brief background of (inhomogeneous) Diophantine approximation

In this section we give a short survey about metric Diophantine approximation on unit interval. The central object to study is the following:

Definition 1.1. *Given any sequences*

$$f : \mathbb{N} \to [0, 1/2] \text{ and } \theta : \mathbb{N} \to [0, 1/2],$$

we define the well approximable numbers to be the following sets:

$$W_0(f, \theta) = \left\{ x \in [0, 1] : \left| x - \frac{q + \theta(p)}{p} \right| < \frac{f(p)}{p} \text{ for infinitely many pairs of numbers } p, q \right\}$$

$$W(f, \theta) = \left\{ x \in [0, 1] : \left| x - \frac{q + \theta(p)}{p} \right| < \frac{f(p)}{p} \text{ for infinitely many coprime pairs of numbers } p, q \right\}.$$

We refer the sequence f as approximation function and the sequence θ as inhomogeneous shift.

For homogeneous theory ($\forall n, \theta(n) = 0$) most of the materials here can be found in [BRV16, chapter 2]. For inhomogeneous theory there are many references. For linear forms we refer [BHH17], [LN12], [Bug03], [Lev98]. For more general setting including approximation on manifolds we refer [BV10], [Bad10], [BBV13]. In this paper we are dealing with the situation in unit interval. In this situation, we refer [Ram17] for more details on results and progresses inhomogeneous metric Diophantine approximation on real line.

A classical result in this setting is the following:

Theorem (Khintchine-Groshev)**.** *If f is non-increasing approximation function such that:*

$$\sum_{p=1}^{\infty} f(p) = \infty,$$

then for any $a \in [0, 1/2]$:

$$W_0(f, \mathbf{a}) \text{ has full Lebesgue measure.}$$

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Here for convenience the bold letter **a** denotes the constant sequence θ such that $\theta(n) = a$ for all n.

Later Duffin-Schaeffer [DS41] generalized Khintchine's result in homogeneous case. The result concerns approximation by rational numbers with lowest form and drops out the requirement of monotonicity.

Theorem (Duffin-Schaeffer)**.** *For any approximation function f, if the following additional condition is satisfied:*

$$\limsup_{n\to\infty} \frac{\sum_{p=1}^{n} \frac{f(p)\phi(p)}{p}}{\sum_{p=1}^{n} f(p)} > 0. \tag{1}$$

Then

$$\sum_{p=1}^{\infty} \frac{f(p)}{p}\phi(p) = \infty \implies W(f, \mathbf{0}) \text{ has full Lebesgue measure.}$$

In the same paper, Duffin and Schaeffer asked whether the condition (1) can be dropped. They made the following famous conjecture.

Conjecture (Duffin-Schaeffer)**.** *For any approximation function f:*

$$\sum_{p=1}^{\infty} \frac{f(p)}{p}\phi(p) = \infty \implies W(f, \mathbf{0}) \text{ has full Lebesgue measure.}$$

Remark.

A lot of work has been done since the birth of this conjecture. Various replacements of condition (1) have been found and we think that the mathoverflow webpage [Onl] gives a nice and brief overview. Most recently, Beresnevich, Harman, Haynes and Velani [BHHV13] proved that the statement of *Duffin-Schaeffer conjecture* is true if the function f satisfies the following extra divergent condition:

$$\sum_{p\geq 16} \frac{\phi(p)f(p)}{p\exp(c(\log\log p)(\log\log\log p))} = \infty. \tag{*}$$

For Hausdorff dimension of sets $W_0(.,.), W(.,.)$, the situation is much better known. For example we have the following results in [HPV12], [HS96]:

Theorem (Haynes-Pollington-Velani)**.** *For any approximation function f and any positive real number $s \in (0, 1]$:*

$$\sum_{p=1}^{\infty} \left(\frac{f(p)}{p}\right)^s \phi(p) = \infty \implies \dim_H W(f, \mathbf{0}) \geq s.$$

Theorem (Hinokuma-Shiga)**.** *For any approximation function f and real number $\alpha \in [1, \infty)$ we set*

$$C_\alpha(N) = \text{Cardinality of the set } \left\{p \leq N : f(p)/p \geq \frac{1}{p^\alpha}\right\}.$$

and

$$\delta(\alpha) = \sup\left\{\delta : \limsup_{N\to\infty} \frac{C_\alpha(N)}{N^\delta} > 0\right\}.$$

Then

$$\dim_H W_0(f, \mathbf{0}) = \min\{1, \sup_{\alpha\geq 1} \kappa(\alpha)\},$$

where $\kappa(\alpha)$ is the following number:

$$\kappa(\alpha)=\begin{cases}\frac{1+\delta(\alpha)}{\alpha} & \lim_{N\to\infty} C_\alpha(N)=\infty \\ 0 & \textit{otherwise}\end{cases}$$

For inhomogeneous case, when the approximation function is monotonic, we have the following result due to Levesley [Lev98]:

Theorem. *For any monotonically decreasing approximation function f and any positive real numbers $s\in(0,1], a\in[0,1/2]$:*

$$\sum_{p=1}^{\infty}\left(\frac{f(p)}{p}\right)^s p=\infty \implies \dim_H W(f,\mathbf{a})\geq s.$$

2. Results in this paper

The basic idea of proving those results is to use a so-called generalized Borel-Cantelli lemma (see theorem 4.1). The main task is to estimate the measure of intersection ($P(A_i\cap A_j)$ appeared in theorem 4.1). The difficulty for inhomogeneous case is that the measure of intersection is not simple enough to apply theorem 4.1. In this paper, we introduce a new idea involving Fourier analytic method to overcome some of the difficulties.

There are some results about inhomogeneous Duffin-Schaeffer type results. In [BHV17, conjecture 2.1], it was shown that if the inhomogeneous Duffin-Schaeffer theorem is true then the inhomogeneous Gallagher theorem is also true. Also in [Cho17, conjecture 1.7] a inhomogeneous Duffin-Schaeffer conjecture was formulated. We shall prove here a weaker result towards the inhomogeneous Duffin-Schaeffer theorem, see theorem 2.8.

We shall see that within the scale of Fourier analytic method, the homogeneous and inhomogeneous situations are more or less the same. In particular, we prove the following results:

Theorem 2.1. *[Main Result] For any approximation function f and inhomogeneous shift θ. We have the following equality:*

$$\dim_H W_0(f,\mathbf{0})=\dim_H W(f,\theta).$$

Remark 2.2. *Motto: for Hausdorff dimension, inhomogeneity and coprimeness are ignorable. It is interesting to ask whether the corresponding measure results hold. (see* [Ram17, Question 10]*)*

The above result follows by using the mass transference principle and the following by-product (for homogeneous case we got better condition (*)):

Theorem 2.3. *For any approximation function f and inhomogeneous shift θ:*

$$\limsup_{n\to\infty}\frac{\sum_{p=1}^{n}\phi(p)f(p)/p}{\log^2 n\log\log n\exp(3\log 2\log n/\log\log n)}=\infty \implies W(f,\theta) \textit{ has full Lebesgue measure.}$$

Remark 2.4. *The denominator here grows with a sub-polynomial but sup-logarithmic speed. It is enough for our purpose in this paper but it is far from optimal. It is likely that some better estimates than those in section 5 can be made and we can in fact find better denominator that grows more slowly. (If it can be made bounded then the Duffin-Schaeffer conjecture follows.)*

Remark 2.5. *In fact we can show that the expected number of solution for Lebesgue almost all $x\in[0,1]$ up to level n is of order $2\sum_{p=1}^{n}\phi(p)f(p)/p+o(\sum_{p=1}^{n}\phi(p)f(p)/p)$. See section 10 for more details.*

To prove the above theorem we show that the following Duffin-Schaeffer type result which is the main by-product of this paper.

Theorem 2.6 (Main by-product). *For any approximation function f and inhomogeneous shift θ:*

$$\limsup_{n\to\infty} \frac{\sum_{p=1}^{n}\phi(p)f(p)/p}{\sqrt{\sum_{p=1}^{n} f(p)d^3(p)\log^2 p}}=\infty \implies W(f,\theta) \text{ has full Lebesgue measure.}$$

Remark 2.7. *By Hardy-Ramanujan-Turán-Kubilius theorem on normal oder of logarithm of divisor function we see that if the aproximation function f is supported on a large subset of $\mathbb{N}$ on which $d(n)\leq \log^{1+\epsilon} n$ then we can provide an inhomogeneous Duffin-Schaeffer type result with extra logarithmic divergency (compare with the sup-logarithmic divergent condition (*)).*

For a positive number $\epsilon>0$, let $A\subset\mathbb{N}$ is such that:

$$a\in A \iff d(a)\leq \log^{1+\epsilon} a.$$

Note that A is of natural density 1.

Then for an approximation function suppoerted on A and any inhomogeneous shift θ:

$$f(n)\neq 0 \implies n\in A,$$

$$\limsup_{n\to\infty} \frac{\sum_{p=1}^{n}\phi(p)f(p)/p}{\log^{3+3.5\epsilon} n}=\infty \implies W(f,\theta) \text{ has full Lebesgue measure.}$$

Or in a more convenient form (with the help of argument in remark 2.13):

$$\sum_{p=1}^{\infty}\frac{f(p)}{\log^{3+4\epsilon} p}=\infty. \implies W(f,\theta) \text{ has full Lebesgue measure.}$$

The power $3+4\epsilon$ here is probably not optimal.

From Duffin-Schaeffer's theorem we see that if f is supported on the set of integers with bounded number of prime factors, for example primes then $\sum_p f(p)=\infty$ would imply the homogeneous well approximable numbers have full Lebesgue measure. For inhomogeneous case, we have almost the same result.

Theorem 2.8 (by-product 2+). *Let f be an approximation function supported on primes numbers then for all inhomogeneous shift θ:*

$$\sum_p f(p)=\infty \implies W(f,\theta) \text{ has full Lebesgue measure.}$$

More generally, let f be an approximation function supported on numbers with no more than K prime factors then:

$$\sum_p \frac{f(p)}{\log\log^{K-1+\epsilon} p}=\infty \implies W(f,\theta) \text{ has full Lebesgue measure.}$$

In addition to remark 2.7, theorem 2.8 We also have the following general result for approximation functions with small support. To compare with remark 2.7 which says that inhomogeneous Duffin-Schaeffer conjecture holds under extra logarithmic divergence if f is supported on a certain large set with complement set of size $\frac{N}{\log\log N}$ for large N. The following theorem saids that the inhomogeneous Duffin-Schaeffer conjecture holds if f is supported on a small enough for example of size $\frac{N}{\exp(4\log N/\log\log N)}$ for large enough N. There is a large gap between $\log\log N$ and $\exp(4\log N/\log\log N)$, some careful analysis on GCD sums will probably help us to somehow fill in the gap.

Theorem 2.9 (by-product 2++). *Let θ be any inhomogeneous shift. Let f be an approximation function and let A be the support of f:*

$$A = \{p : f(p) > 0\}.$$

Denote $\#(.)$ for cardinality, if

$$\limsup_{N\to\infty} \#(A \cap [1, N]) \frac{\log^3 N d^2(N)}{\phi(N)} < \infty,$$

then

$$\sum_p \frac{f(p)\phi(p)}{p} = \infty \implies W(f, \theta) \text{ has full Lebesgue measure.}$$

Another by-product related with extra conditions of the *Duffin-Schaeffer conjecture* is the following:

Theorem 2.10 (by-product 3). *For any approximation function f and inhomogeneous shift θ. Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(x) \to 0$ as $x \to 0$ and $h(x)/x$ is monotonic. If the following conditions holds:*

$$\limsup_{n\to\infty} \frac{\sum_{p=1}^{n} \phi(p) h(f(p)/p)}{\log^{2.5} n \left(\max_{p\in[1,n]} h(f(p)/p)^{1/2} p\right)} = \infty.$$

Then

$$\mathcal{H}^h (W(f, \theta)) = \mathcal{H}^h([0, 1]).$$

Here $\mathcal{H}^h(.)$ denotes the Hausdorff measure with dimension function h, more details can be found in [BV06, section 2] *and the reference there as well.*

Corollary 2.11. *For any approximation function f and inhomogeneous shift θ, if there exists a number $A > 3$ such that:*

$$f(p) = O\left(\frac{\log^A p}{p}\right),$$

$$\limsup_{n\to\infty} \frac{\sum_{p=1}^{n} \frac{f(p)}{p}\phi(p)}{\log^{\frac{A}{2}+2.5} n} > 0. \tag{2}$$

Then

$$\sum_{p=1}^{\infty} \frac{f(p)}{p}\phi(p) = \infty \implies W(f, \theta) \text{ has full lebesgue measure.}$$

proof of the corollary. We set $h(x) = x$ in theorem 2.10 and the conclusions are easy to see. □

Remark 2.12. *The above result can be compared with Vaaler* [Vaa78] *which says that if $f(p) = O(1/p)$, then the result of Duffin-Schaeffer conjecture holds without any extra conditions on the rate of divergence of $\sum_{p=1}^{\infty} \frac{f(p)}{p}\phi(p)$. The requirement $A > 3$ comes from the following consideration:*

$$f(p) = O\left(\frac{\log^A p}{p}\right) \implies \sum_{p=1}^{n} \frac{f(p)}{p}\phi(p) = O(\log^{A+1}(n)).$$

So for $A \leq 3$, condition (2) *can not be satisfied. However the number* 2.5 *appeared in the theorem is by no means the best choice with our method. Since we have only used some easy properties in lemma 5.1, 5.2 concerning arithmetic functions.*

Later we shall construct an example to show that the above condition is something slightly new.

Remark 2.13. *We can state another version of the above corollary by replacing condition (2) with the following condition:*

There exists an $\epsilon > 0$*,*

$$\sum_{p=2}^{\infty} \frac{f(p)}{\log^{A/2+2.5+\epsilon} p} = \infty.$$

Proof of remark 2.13 based on corollary 2.11. Consider the iterated exponential intervals:

$$I_k = \left[2^{2^k}, 2^{2^{k+1}}\right],$$

then there are infinitely many $k > 0$ such that:

$$\sum_{p \in I_k} \frac{f(p)}{\log^{A/2+2.5+\epsilon} p} > \frac{1}{k^2}$$

otherwise the following sum will not diverge:

$$\sum_{p=2}^{\infty} \frac{f(p)}{\log^{A/2+2.5+\epsilon} p}.$$

Then we see that:

$$\begin{aligned}
\sum_{p \in I_k} \frac{f(p)}{p}\phi(p) &= \sum_{p \in I_k} \frac{f(p)}{\log^{A/2+2.5+\epsilon} p} \frac{\phi(p)}{p} \log^{A/2+2.5+\epsilon} p \\
&\geq \min_{p \in I_k} \frac{\phi(p)}{p} \log^{A/2+2.5+\epsilon} p \sum_{p \in I_k} \frac{f(p)}{\log^{A/2+2.5+\epsilon} p} \\
&\geq C \frac{1}{\log\log 2^{2^{k+1}}} \log^{A/2+2.5+\epsilon} 2^{2^k} \frac{1}{k^2} \\
&\geq C' \frac{1}{k^3} 2^{(A/2+2.5+\epsilon)k} \\
&\geq C'' 2^{(A/2+2.5)(k+1)}
\end{aligned}$$

where C, C', C'' are constants which only depend on A, ϵ. The choice of constant C comes from the following fact involved with the Euler gamma γ:

$$\text{(FACT)} \qquad \liminf_{n \to \infty} \frac{\phi(n)}{n} \log\log n = e^{-\gamma}.$$

Then for all $k > 0$:

$$\sum_{p=2}^{2^{2^{k+1}}} \frac{f(p)}{p}\phi(p) \geq \sum_{p \in I_k} \frac{f(p)}{p}\phi(p) \geq C'' 2^{(A/2+2.5)(k+1)}.$$

By corollary 2.11 this remark follows. $\square$

Here we can provide a approximation function f without extra divergent condition (*), the Duffin-Schaeffer condition (1) as well as the Vaaler's condition $f(p) = O(1/p)$. This example will not contribute to later development. To begin with, we decompose the integer set into dyadic intervals:

$$D_k = [2^k, 2^{k+1}), k = 0, 1, \ldots$$

For each k we choose an integer $m(k)$ such that:

$$\liminf_{k\to\infty} m(k) \to \infty, \sum_{k=0}^{\infty} \frac{k^{2.4}}{m(k)!} = \infty.$$

Then in each $p \in D_k$ we assign the value $f(p) = \log^{10} p/p$ if p is a multiple of $m(k)!$. Otherwise we set $f(p) = 0$. It can happen that for some $D_k's$ there are no non zero values assigned. However, for large enough k, $|D_k|$ grows exponentially while $m(k)!$ can not grow exponentially for otherwise the above series would not diverge. Therefore $|D_k|$ will become much larger than $m(k)!$, and eventually a lot of non zero values will be assigned.

It is easy to see that for all large enough p:

$$\frac{\phi(p)f(p)}{p\exp(c(\log\log p)(\log\log\log p))} \leq \frac{1}{p\log^2 p}.$$

Therefore condition (*) is not satisfied.

Next, $f(p)$ is only non zero if p is a multiple of $n_p!$ for a suitable integer n_p and as $p \to \infty$, $n_p \to \infty$. Then we see that:

$$\frac{\phi(p)}{p} \leq \prod_{r \text{ prime}, r\leq n_p} \left(1 - \frac{1}{p}\right) \to 0, \text{ as } p \to \infty.$$

Therefore the Duffin-Schaeffer condition (1) is not satisfied.

Then for large enough k there are more than $0.5|D_k|/m(k)!$ numbers in D_k which are multiples of $m(k)!$, so we see that:

$$\sum_{p\in D_k, m(k)|p} \frac{\log^{10} p}{p\log^{7.6} p} \geq 0.5\frac{2^k}{m(k)!}\frac{k^{10}}{2^{k+1}(k+1)^{7.6}} \geq \frac{1}{2^{10}}\frac{k^{2.4}}{m(k)!},$$

here we used the fact that $k+1 \geq 2k$ for all $k > 1$. Then the conditions in corollary 2.11 (remark 2.13) are satisfied and we can use that theorem to show that the above chosen sequence $f(p)$ satisfies the conclusion of (inhomogeneous) *Duffin-Schaeffer conjecture* .

3. Notations

- In this paper we shall use $C, C', C'', C''', C''''...$ as positive constants.
- In this paper, we always use f to denote approximation functions and θ to denote inhomogeneous shifts. Without explicitly mentioning no extra condition is assume for f, θ apart from requirement that their range is $[0, 1/2]$.
- In this paper for any number $a \in \mathbb{R}$ we use $\mathbf{a}$ to denote the constant sequence whose terms are equal to a.
- In this paper we use $\dim_H$ for Hausdorff dimension and $\mathcal{H}^h$ for h-Hausdorff measure with dimension function h.
- In this paper we use the following notion of modified logarithmic function:

$$\log x = \begin{cases} \ln x & x \geq 3 \\ \ln 3 & x \in [1, 3] \\ \ln x & x \in (0, 1) \end{cases}$$

  The function is not continuous and there is no zero when x approaches 1 on the right.
- In this paper we shall use the following arithmetic functions:

1 : The Euler function. For $n \in \mathbb{N}$:
$\phi(n) =$ number of natural numbers smaller than n which are coprime to n.
2 : The greatest common divisor function. For $a, b \in \mathbb{N}$:
$(a, b) =$ the greatest common divisor of a, b.
3 : The divisor functions. For $n \in \mathbb{N}, \alpha \in \mathbb{R}$:
$d(n) =$ the number of divisors of n.
$d_\alpha(n) = \sum_{a|n} a^\alpha$.
4 : The Möbius function:
For $n \in \mathbb{N}$:

$$\mu(n) = \begin{cases} 1 & n \text{ is squarefree with even number of prime factors} \\ -1 & n \text{ is squarefree with odd number of prime factors} \\ 0 & n \text{ is not squarefree} \end{cases}$$

5 : The Ramanujan sum:
For $n, p \in \mathbb{N}$: $c_p(n) = \sum_{1 \leq a \leq p, (a,p)=1} e^{2\pi i \frac{an}{p}} = \mu\left(\frac{p}{(p,n)}\right) \frac{\phi(p)}{\phi\left(\frac{p}{(p,n)}\right)}$

- In this paper we use P for general probability measure on a probability space Ω and λ for Lebesgue measure on $[0, 1]$.
- For a sequence of sets $A_n \subset X$: $\limsup_{n\to\infty} A_n = \{x \in X : x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}$.

4. Results that will be used without proof

Given a function $f : [0, 1] \to \mathbb{R}$, which we will assume to be of some regularity, throughout this paper we will only consider Fourier series of step functions which are L^1 and L^2 functions. The Fourier series of f is given by:

$$\forall k \in \mathbb{Z}, \hat{f}(k) = \int_0^1 e^{2\pi i k x} f(x) dx.$$

We will need the following facts:

$$\hat{f}(0) = \|f\|_{L^1} \text{ whenever } f(x) \geq 0, \forall x \in [0, 1].$$

$$\widehat{fg}(0) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(-k).$$

The above results can be found in most text books on harmonic analysis for example in [Kat04, chapter 1, section 5.5]

We specify the version of Borel-Cantelli lemma (see [BRV16, lemma 2.2]) which will be used later.

Theorem 4.1. *Let A_p be a sequence of events in a probability space (Ω, P) such that:*

$$\sum_{p=1}^{\infty} P(A_p) = \infty,$$

then:

$$P(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \frac{(\sum_{p=1}^n P(A_p))^2}{\sum_{p,q=1}^n P(A_p \cap A_q)}.$$

Remark 4.2. *In fact for homogeneous metric Diophantine approximation to conclude the full measure result we only need to show:*

$$\limsup_{n\to\infty} \frac{\sum_{p=1}^{n} P(A_p))^2}{\sum_{p,q=1}^{n} P(A_p \cap A_q)} > 0.$$

This follows from a result of Gallagher [Gal61]*.*

We will also use the following version of mass transference principle in [BV06].

Theorem 4.3 (Beresnevich-Velani)**.** *Let $\{B_i\}_{i\in\mathbb{N}}$ be a countable collection of balls in $\mathbb{R}$ with $r(B_i) \to 0$ as $i \to 0$. Let h be a dimension function such that $h(x)/x$ is monotonic and suppose that for any ball in $\mathbb{R}$:*

$$\lambda(B \cap \limsup_{i\to\infty} B_i^h) = \lambda(B).$$

Then, for any ball B in $\mathbb{R}$:

$$\mathcal{H}^h(B \cap \limsup_{i\to\infty} B_i) = \mathcal{H}^h(B).$$

Here B^h denote the ball cocentered with B but radius diluted by h. That is to say, if the radius of B is r then the radius of B^h is $h(r)$.

5. Some asymptotic results on arithmetic functions

The estimates in this section are by no means optimal but they will be sufficient for later use. Improvement of any of the following estimates can lead to improvement of the theorems 2.3 and 2.10. In what follows, we will use some results concerning the Ramanujan sum which can be found in [HWHBS08] chapter 16:

$$c_p(k) = \sum_{(a,p)=1} e^{2\pi i \frac{a}{p} k} = \mu\left(\frac{p}{(p,k)}\right) \frac{\phi(p)}{\phi\left(\frac{p}{(p,k)}\right)}.$$

The following lemma will play a crucial role in later content:

Lemma 5.1. *There is a constant $C > 0$ such that for any integers $k, n > 0$:*

$$\frac{1}{d(k)\log n} \sum_{p=1}^{n} \frac{|c_p(k)|}{\phi(p)} < C.$$

Here $d(k)$ is the divisor function, that is, the number of divisors of the integer k.

Proof. By properties of the Ramanujan sum and the Euler totient function:

$$\begin{aligned}
\sum_{p=1}^{n} \frac{|c_p(k)|}{\phi(p)} &= \sum_{p=1}^{n} \frac{\left|\mu\left(\frac{p}{(p,k)}\right)\right|}{\phi\left(\frac{p}{(p,k)}\right)} \\
&= \sum_{p=1}^{n} \frac{\left|\mu\left(\frac{p}{(p,k)}\right)\right|}{\frac{p}{(p,k)} \prod_{r|\frac{p}{(p,k)}, r \text{ prime}} (1-\frac{1}{r})} \\
&= \sum_{p=1}^{n} \frac{\left|\mu\left(\frac{p}{(p,k)}\right)\right|}{\prod_{r|\frac{p}{(p,k)}, r \text{ prime}} (r-1)} \\
&= \sum_{l=1, l \text{ squarefree}}^{n} \prod_{r|l, r \text{ prime}} \frac{1}{r-1} \left|\left\{p \in [1,n] | l = \frac{p}{(p,k)}\right\}\right|.
\end{aligned}$$

The cardinality of the set can be bounded by:

$$\left|\left\{p \in [1,n] | l = \frac{p}{(p,k)}\right\}\right| \leq d(k),$$

because (p,k) must be a divisor of k, and for every such divisor $s|k$, the value of p (if exists) can be uniquely fixed by sl.

Then we see that:

$$\begin{aligned}\sum_{p=1}^{n} \frac{|c_p(k)|}{\phi(p)} &\leq d(k) \sum_{l=1, l \text{ squarefree}}^{n} \prod_{r|l, r \text{ prime}} \frac{1}{r-1} \\ &\leq d(k) \prod_{r \leq n, r \text{ prime}} \left(1 + \frac{1}{r-1}\right).\end{aligned}$$

Then this lemma follows from the following Grönwall's theorem [Grö13] :

Theorem (Grönwall)**.**

$$\lim_{n\to\infty} \frac{1}{\log n} \prod_{r \leq n, r \ prime} \left(1 + \frac{1}{r-1}\right) = e^{\gamma},$$

where γ is the Euler gamma $\gamma \approx 0.5772156$. $\square$

Another result that we will use is the following estimate of logarithmic divisor sum:

Lemma 5.2. *There exists a constant $C > 0$ such that for all integers $n > 1$:*

$$\sum_{k=1}^{n} \frac{d^2(k)}{k} < C \log^3 n.$$

Proof. First, observe that:

$$d^2(k) = \sum_{l|k} d(l^2).$$

Indeed for any integer with prime factorization $k = p_1^{a_1} \dots p_k^{a_n}$ we have that:

$$d(k) = \prod_{i=1}^{i=n} (a_i + 1).$$

Then we see that:

$$\begin{aligned}\sum_{l|k} d(l^2) &= \sum_{0 \leq b_i \leq a_i, i \in \{1,2\dots n\}} \prod_{i=1}^{i=n} (2b_i + 1) \\ &= \prod_{i=1}^{n} \left(\sum_{b_i=0}^{b_i=a_i} (2b_i + 1)\right) \\ &= \prod_{i=1}^{n} (a_i + 1)^2 = d^2(k).\end{aligned}$$

Then we have the following estimate:

$$\begin{aligned}
\sum_{k=1}^{n}\frac{d^2(k)}{k} &= \sum_{l=1}^{n} d(l^2)\sum_{k:l|k}^{k\le n}\frac{1}{k}\\
&\le \sum_{l=1}^{n}\frac{d(l^2)}{l}(\log n+1)\\
&\le (\log n+1)\sum_{m=1}^{n}\sum_{l:m|l^2}^{l\le n}\frac{1}{l}\\
&\le (\log n+1)\sum_{m=1}^{n}\sum_{l:m|l,l\in[1,n^2]}\frac{1}{l}\\
&\le (\log n+1)\sum_{m=1}^{n}\frac{1}{m}\left(\log n^2+1\right)\\
&\le (\log n+1)^2(2\log n+1)\le C\log^3 n,
\end{aligned}$$

for a suitable constant $C>0$. □

Lemma 5.3. *There exists a constant $C>0$ such that for q being any positive integer:*

$$\sum_{1\le p\le q} d(p)d(q)(p,q)\le Cd^3(q)q\log q.$$

Proof.

$$\begin{aligned}
\sum_{1\le p\le q} d(p)d(q)(p,q) &= d(q)\sum_{r|q}\sum_{p\le q,(p,q)=r} rd(p)\\
&= d(q)\sum_{r|q} r\sum_{a\le p/r,(a,q/r)=1} d(ar)\\
&\le d(q)\sum_{r|q} r\sum_{a\le p/r,(a,q/r)=1} d(a)d(r)\\
&\le Cd(q)\sum_{r|q} rd(r)\frac{q}{r}\log\frac{q}{r}\\
&\le Cqd(q)\log q\sum_{r|q} d(r)\\
&\le Cqd(q)\log q\sum_{r|q} d(r^2)\\
&= Cqd^3(q)\log q.
\end{aligned}$$

Here we used Dirichlet theorem for divisor summatory function (for the constant C) and the beginning part of the proof of lemma 5.2 □

6. Fourier series and Diophantine approximation: basic setting up

From the Borel-Cantelli lemmas 4.1 we see that it is important to show some properties of the measure of intersection of two level sets. Now we are going to set up the Fourier analysis method:

Let f, θ be as mentioned above, we denote

$$\epsilon_p = \frac{f(p)}{p}.$$

Then we define the function

$$g_p(x) : [0,1] \to \{0,1\}$$

by:

$$g_p(x) = 1 \iff \left| x - \frac{q+\theta(p)}{p} \right| < \frac{f(p)}{p}, \text{ for an integer } q \text{ with } (q,p) = 1.$$

Then it is clear that $g_p(x)$ is just the characteristic function on the set A_p:

$$A_p = \{x \in [0,1] | \exists 1 \leq q \leq p, (q,p) = 1, \left| x - \frac{q+\theta(p)}{p} \right| < \frac{f(p)}{p}\}.$$

In our case $f(p) \leq \frac{1}{2}$ and therefore A_P is a union of $\phi(p)$ many equal length disjoint intervals. The Lebegues measure of A_p is:

$$\|g_p\|_{L^1} = 2\epsilon_p \phi(p).$$

Now we see that $\lambda(A_p \cap A_q) = \|g_p g_q\|_{L^1}$. We need only to compute the case $p \neq q$ since otherwise the case is trivial.

Using Fourier series we can write the L^1-norm as:

$$\|g_p g_q\|_{L^1} = \sum_{k=-\infty}^{\infty} \hat{g_p}(k) \hat{g_q}(-k).$$

The above equality holds whenever the series is absolutely convergent. This happens whenever g_p, g_q are both L^1 functions. This is the case in our situation.

Now we need to evaluate the Fourier series of g_p, it is easy to see that g_p is just the characteristic function of

$$[-\epsilon_p, \epsilon_p] = \left[-\frac{f(p)}{p}, \frac{f(p)}{p} \right]$$

convolved with a sum of Dirac deltas:

$$\sum_{(a,p)=1} \delta(\frac{a+\theta(p)}{p}).$$

But we can compute the Fourier series directly for $k \neq 0$:

$$\begin{aligned}
\int_0^1 e^{2\pi i k x} g_p(x) dx &= \sum_{(a,p)=1} \int_{\frac{a+\theta(p)}{p} - \epsilon_p}^{\frac{a+\theta(p)}{p} + \epsilon_p} e^{2\pi i k x} dx \\
&= \sum_{(a,p)=1} \frac{1}{\pi k} \sin(2\pi \epsilon_p k) e^{2\pi i \frac{a+\theta(p)}{p} k} \\
&= \frac{1}{\pi k} \sin(2\pi \epsilon_p k) c_p(k) e^{2\pi i \frac{\theta(p)}{p} k},
\end{aligned}$$

where $c_p(k) = \sum_{(a,p)=1} e^{2\pi i \frac{a}{p} k}$ is the Ramanujan sum. For $k = 0$, $\hat{g}_p(0)$ is simply $\|g_p\|_{L^1}$.

Then we see that we can express $\lambda(A_p \cap A_q)$ with the following series:

(Main Formula)

$$\lambda(A_p \cap A_q) = 4\epsilon_p \epsilon_q \phi(p)\phi(q) + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin(2\pi\epsilon_p k) c_p(k) \sin(2\pi\epsilon_q k) c_q(k) \cos\left(2\pi i \left(\frac{\theta(p)}{p} - \frac{\theta(q)}{q}\right) k\right)}{k^2},$$

where we have used the fact that the Ramanujan sum is real and for all pairs of integers p,k:

$$c_p(k)=c_p(-k).$$

We see that inhomogeneous shifts θ create just an extra $\cos(.)$ term whose modulus is bounded by 1. In what follows we shall apply absolute value in order to estimate from above the sum in the (Main Formula) and therefore the inhomogeneous shifts do not create anything new to us.

7. PROOF OF THEOREM 2.3, 2.6, 2.8

By the Main Formula we see that:

$$\lambda(A_p\cap A_q)\leq 4\epsilon_p\epsilon_q\phi(p)\phi(q)+\frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{|\sin(2\pi\epsilon_p k)c_p(k)\sin(2\pi\epsilon_q k)c_q(k)|}{k^2}$$

The basic strategy is to split the sum of k up to a number M to be determined later:

$$\sum_{k=1}^{\infty}=\sum_{k=1}^{M}+\sum_{k=M+1}^{\infty}.$$

For the first part, we use the fact that $|\sin(x)|\leq\min\{|x|,1\}$ for all $x\in\mathbb{R}$:

$$\sum_{k=1}^{M}\frac{|\sin(2\pi\epsilon_p k)c_p(k)\sin(2\pi\epsilon_q k)c_q(k)|}{k^2}\leq\sum_{k=1}^{M}\frac{1}{k}\min\{\epsilon_p,\epsilon_q\}|c_p(k)||c_q(k)|.$$

Now we recall the formula for the Ramanujan sum:

$$c_p(k)=\mu\left(\frac{p}{(p,n)}\right)\frac{\phi(p)}{\phi\left(\frac{p}{(p,n)}\right)}.$$

Then we see that there exists an absolute constant $C>0$:

$$\begin{aligned}
\sum_{k=1}^{M}\frac{|\sin(2\pi\epsilon_p k)c_p(k)\sin(2\pi\epsilon_q k)c_q(k)|}{k^2} &\leq \sum_{k=1}^{M}\frac{1}{k}\min\{\epsilon_p,\epsilon_q\}|c_p(k)||c_q(k)|\\
&= \sum_{k=1}^{M}\frac{1}{k}\min\{\epsilon_p,\epsilon_q\}\left|\mu\left(\frac{p}{(p,k)}\right)\frac{\phi(p)}{\phi\left(\frac{p}{(p,k)}\right)}\mu\left(\frac{q}{(q,k)}\right)\frac{\phi(q)}{\phi\left(\frac{q}{(q,k)}\right)}\right|\\
&\leq \sum_{k=1}^{M}\frac{1}{k}\min\{\epsilon_p,\epsilon_q\}(p,k)(q,k)\left|\mu\left(\frac{p}{(p,k)}\right)\mu\left(\frac{q}{(q,k)}\right)\right|\\
&\leq \sum_{k=1}^{M}\frac{1}{k}\min\{\epsilon_p,\epsilon_q\}(p,k)(q,k)\\
&\leq C\log M d(p)d(q)(p,q)\min\{\epsilon_p,\epsilon_q\}.
\end{aligned}$$

Here we used the fact that $\phi(n)=n\prod_{r|n,r\text{ prime}}\frac{r-1}{r}$. For the last step we see that for any divisor s_p of p and r_q of q, we can sum those k such that:

$$(p,k)=s_p,(q,k)=r_q.$$

Such k must be multiple of $[s_p,r_q]$ and therefore we obtain:

$$\sum_{k:(p,k)=s_p,(q,k)=r_q}\frac{1}{k}(p,k)(q,k)\leq C\log M(s_p,r_q).$$

Then the previous estimate follows from summing over all divisors of p, q and use the fact that $(s_p, r_q) \leq (p, q)$.

For the second part $\sum_{k=M+1}^{\infty}$ we use the fact that $|\sin(x)| \leq 1$ and obtain for an absolute constant $C' > 0$:

$$\sum_{k=M}^{\infty} \frac{|\sin(2\pi\epsilon_p k)c_p(k)\sin(2\pi\epsilon_q k)c_q(k)|}{k^2} \leq \frac{C'}{M}d(p)d(q)(p,q).$$

We can now set $M = d(p)d(q)(p,q)p^4q^4$ assuming $\epsilon_p\epsilon_q \neq 0$ otherwise $\lambda(A_p \cap A_q) = 0$ and there is nothing to show. Then we see that $\log M \leq 10\log p + 10\log q$. In particular if $p, q \leq n$ then $\log M \leq 20\log n$.

We see that there exists a absolute constant $C'' > 0$:

$$\lambda(A_p \cap A_q) \leq 4\epsilon_p\epsilon_q\phi(p)\phi(q) + C''\min\{\epsilon_p, \epsilon_q\}d(p)d(q)(p,q)(10\log p + 10\log q) + C'\frac{1}{p^4q^4}$$

We can now use theorem 4.1 and lemma 5.3 to conclude the proof. Indeed we see that for a constant $C''' > 0$:

(*)

$$\sum_{p,q=1}^{n}\lambda(A_p\cap A_q) = (\sum_{p=1}^{n}\sum_{q\leq p}+\sum_{q=1}^{n}\sum_{p<q})\lambda(A_p\cap A_q) \leq (\sum_{p=1}^{n}2\epsilon_p\phi(p))^2+C'''\sum_{p=1}^{n}\epsilon_p p d^3(p)\log^2 p+100C'\zeta^2(4).$$

From here the theorem 2.6 follows. Next, it is easy to see the following result for an absolute constant C'''' and for all n:

$$\frac{n}{\phi(n)}d^3(n)\log^2 n \leq C''''\log^2 n\exp(3\log 2\log n/\log\log n)\log\log n.$$

We have used here the following result of divisor function:

$$\limsup_{n\to\infty}\frac{\log d(n)}{\log n/\log\log n} = \log 2.$$

From here the proof of theorem 2.3 concludes.

To see theorem 2.8, for prime numbers p, q we have $d(p) = 2$ and $(p, q) = 1$ unless $p = q$. Therefore we see that by copying previous results:

$$\lambda(A_p \cap A_q) \leq 4\epsilon_p\epsilon_q\phi(p)\phi(q) + C''\min\{\epsilon_p, \epsilon_q\}\log M + \frac{C'}{M}.$$

We can choose $M = p^{-4}q^{-4}$ again and for a suitable constant $C''''' > 0$:

$$\sum_{p,q=1}^{n}\lambda(A_p \cap A_q) \leq (\sum_{p=1}^{n}2\epsilon_p\phi(p))^2 + C'''''\sum_{p=1}^{n}\epsilon_p\pi(p)\log p \leq (\sum_{p=1}^{n}2\epsilon_p\phi(p))^2 + 2C'''''\sum_{p=1}^{n}\epsilon_p p.$$

Here $\pi(.)$ is the prime counting function and we have used the fact that $\pi(p)/p \leq 2\frac{1}{\log p}$ for all large enough p. From here the first part of theorem 2.8 follows. To see that second part we have to use the fact that for all large enough p:

$$\pi_K(p) \leq 2\frac{p}{\log p}\frac{\log\log^{K-1} p}{(K-1)!}.$$

This is a generalization of prime number theorem.

For general case, we can not assume anything on $(p,q)d(p)d(q)$. Let A be the support of f denote $A(p)=\#(A\cap[1,p])$ a small inspection of inequality (*) gives us the following bound:

$$\sum_{p,q=1}^{n}\lambda(A_p\cap A_q)\leq(\sum_{p=1}^{n}2\epsilon_p\phi(p))^2+C'''\sum_{p=1}^{n}\epsilon_p A(p)d^3(p)\log^2 p+100C'\zeta^2(4).$$

For large enough p we see that there exists a constant $C'''''>0$ such that:

$$A(p)\leq C''''''\frac{\phi(p)}{d^3(p)\log^2 p},$$

and then theorem 2.9 follows.

8. Proof of theorem 2.1

Here we shall use a pre-mentioned result:

Theorem (Hinokuma-Shiga)**.** *For any approximation function f and real number $\alpha\in[1,\infty)$ we set*

$$C_\alpha(N)=\textit{Cardinality of the set }\left\{p\leq N: f(p)/p\geq\frac{1}{p^\alpha}\right\}.$$

and

$$\delta(\alpha)=\sup\left\{\delta\in[0,1]:\limsup_{N\to\infty}\frac{C_\alpha(N)}{N^\delta}>0\right\}.$$

Then

$$\dim_H W_0(f,\mathbf{0})=\min\{1,\sup_{\alpha\geq 1}\kappa(\alpha)\},$$

where $\kappa(\alpha)$ is the following number:

$$\kappa(\alpha)=\begin{cases}\frac{1+\delta(\alpha)}{\alpha} & \lim_{N\to\infty}C_\alpha(N)=\infty\\ 0 & \textit{otherwise}\end{cases}$$

We now show that $\dim_H W(f,\theta)\geq\dim_H W_0(f,\mathbf{0})$. The other direction can be proved just by the same argument provided in [HS96]. Only for the lower bound there are some difficulties in estimating the size intersections $A_p\cap A_q$ by using direct number theoretic methods.

Now let f be any approximation function and θ be any inhomogeneous shift. As in the above theorem, for any α, we find the set $C_\alpha(N)$ and find the exponent $\delta(\alpha)$. Assume that $\kappa(\alpha)>0$ otherwise there is nothing to show.

Now for an arbitrarily small number $\sigma>0$ we use the dimension function $h(x)=x^{\frac{1-\sigma+\delta(\alpha)}{\alpha}}$ in mass transference principle theorem 4.3. Now shrinking some values of f if necessary, we can assume that $\epsilon_p=f(p)/p\geq 1/p^\alpha$ for a subset C_α of $\mathbb{N}$ such that (notice that $\delta(\alpha)$ can be 0):

$$\limsup_{N\to\infty}\#|C_\alpha\cap[1,N]|N^{-\delta(\alpha)+0.5\sigma}=\infty.$$

We see that $h(\epsilon_p)\geq\frac{1}{x^{1-\sigma+\delta(\alpha)}}$ and

$$\begin{aligned}&\limsup_{N\to\infty}\frac{\sum_{p=1}^{N}\phi(p)h(\epsilon_p)}{\log^2 N\log\log N\exp(3\log 2\log N/\log\log N)}\\ \geq\;&\limsup_{N\to\infty}\frac{\#|C_\alpha\cap[1,N]|\frac{1}{\log\log N}\frac{1}{N^{-\sigma+\delta(\alpha)}}}{\log^2 N\log\log N\exp(3\log 2\log N/\log\log N)}\\ \geq\;&\limsup_{N\to\infty}\frac{N^{0.5\sigma}}{\log^2 N\log\log^2 N\exp(3\log 2\log N/\log\log N)}=\infty.\end{aligned}$$

By theorem 4.3 we see that:

$$\mathcal{H}^{\frac{1-\sigma+\delta(\alpha)}{\alpha}}(W(f,\theta))=\infty.$$

This implies that for all $\sigma>0$:

$$\dim_H W(f,\theta)\geq\frac{1-\sigma+\delta(\alpha)}{\alpha}.$$

This implies further that:

$$\dim_H W(f,\theta)\geq\frac{1+\delta(\alpha)}{\alpha}.$$

Then combine with the theorem by Hinokuma-Shiga we see that:

$$\dim_H W(f,\theta)\geq\dim_H W_0(f,\mathbf{0}).$$

9. PROOFS OF THEOREM 2.10

We now try to estimate directly the following sum:

$$\sum_{p,q=1}^{n}\lambda(A_p\cap A_q).$$

Theorem 9.1. *Let f,θ,ϵ_p, as mentioned before. Then there is a constant $C>0$ such that:*

$$\sum_{p,q=1}^{n}\lambda(A_p\cap A_q)\leq C\left(\max_{p\in[1,n]}\epsilon_p^{0.5}\phi(p)\right)^2\log^5 n+(\sum_{p=1}^{n}2\epsilon_p\phi(p))^2$$

Proof. By (Main Formula) in section 6 we see that:

$$\sum_{p,q=1}^{n}\lambda(A_p\cap A_q)\leq(\sum_{p=1}^{n}2\epsilon_p\phi(p))^2+\frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}(\sum_{p=1}^{n}|\sin(2\pi\epsilon_p k)c_p(k)|)^2.$$

Because $|\sin(x)|\leq 1$ therefore for any $\alpha\in[0,1]$:

$$|\sin(x)|\leq|\sin(x)|^{\alpha}\leq|x|^{\alpha}.$$

The basic strategy is again to split the sum with respect to k, say:

$$\sum_{k=1}^{\infty}=\sum_{k=1}^{M}+\sum_{k=M+1}^{\infty},$$

for a later determined integer $M>0$. For convenience we make the following notation:

$$I=\sum_{k=1}^{M},$$

$$II=\sum_{k=M+1}^{\infty}.$$

Then for part I we use the estimate $|\sin(x)| \leq |\sin(x)|^{0.5} \leq |x|^{0.5}$:

$$\begin{aligned}
I &= \frac{2}{\pi^2}\sum_{k=1}^{M}\frac{1}{k^2}\left(\sum_{p=1}^{n}|\sin(2\pi\epsilon_p k)c_p(k)|\right)^2 \\
&\leq \frac{4}{\pi}\sum_{k=1}^{M}\frac{1}{k^2}\left(\sum_{p=1}^{n}\epsilon_p^{0.5}k^{0.5}|c_p(k)|\right)^2 \\
&\leq \frac{4}{\pi}\sum_{k=1}^{M}\frac{1}{k}\left(\sum_{p=1}^{n}\epsilon_p^{0.5}\phi(p)\frac{|c_p(k)|}{\phi(p)}\right)^2 \\
&\leq \frac{4}{\pi}\sum_{k=1}^{M}\frac{\left(\max_{p\in[1,n]}\epsilon_p^{0.5}\phi(p)\right)^2}{k}\left(\sum_{p=1}^{n}\frac{|c_p(k)|}{\phi(p)}\right)^2.
\end{aligned}$$

Then by lemma 5.1,5.2 we see that for a constant $C_1 > 0$:

$$I \leq C_1\left(\max_{p\in[1,n]}\epsilon_p^{0.5}\phi(p)\right)^2\log^2 n\log^3 M,$$

where $\log^2 n$ comes from lemma 5.1 and the $\log^3 M$ comes from lemma 5.2. Then for II we use the trivial bound $|\sin(x)| \leq 1$.

$$\begin{aligned}
II &= \frac{2}{\pi^2}\sum_{k=M+1}^{\infty}\frac{1}{k^2}\left(\sum_{p=1}^{n}\sin(2\pi\epsilon_p k)|c_p(k)|\right)^2 \\
&\leq \frac{2}{\pi^2}\sum_{k=M+1}^{\infty}\frac{1}{k^2}\left(\sum_{p=1}^{n}|c_p(k)|\right)^2 \\
&\leq \frac{2}{\pi^2}\sum_{k=M+1}^{\infty}\frac{1}{k^2}n^4 \leq C_2\frac{n^4}{M}
\end{aligned}$$

for another constant $C_2 > 0$. In above inequalities we used the fact that:

$$|c_p(k)| \leq \phi(p) \leq p.$$

With some careful analysis we can replace the n^4 with n^3, but there is no essential difference as we shall see.

Now we choose $M = n^5$. Then the following estimate holds for a suitable constant $C > 0$:

$$I + II \leq 125C_1\left(\max_{p\in[1,n]}\epsilon_p^{0.5}\phi(p)\right)^2\log^2 n\log^3 n + C_2\frac{1}{n} \leq C\left(\max_{p\in[1,n]}\epsilon_p^{0.5}\phi(p)\right)^2\log^2 n\log^3 n.$$

From here the result of this theorem follows. □

We can now prove theorem 2.10:

Proof of theorem 2.10 using theorem 9.1. By theorem 9.1 we see that for a constant $C > 0$:

$$\sum_{p,q=1}^{n}\lambda(A_p\cap A_q) \leq C\left(\max_{p\in[1,n]}\epsilon_p^{0.5}\phi(p)\right)^2\log^5 n + \left(\sum_{p=1}^{n}2\epsilon_p\phi(p)\right)^2$$

Then we see that:

$$\begin{aligned}\frac{(\sum_{p=1}^n 2\epsilon_p\phi(p))^2}{\sum_{p,q=1}^n \lambda(A_p\cap A_q)} &\geq \frac{(\sum_{p=1}^n 2\epsilon_p\phi(p))^2}{C\left(\max_{p\in[1,n]}\epsilon_p^{0.5}\phi(p)\right)^2\log^5 n+(\sum_{p=1}^n 2\epsilon_p\phi(p))^2}\\ &\geq \frac{1}{C\left(\max_{p\in[1,n]}\epsilon_p^{0.5}\phi(p)\right)^2\frac{\log^5 n}{(\sum_{p=1}^n 2\epsilon_p\phi(p))^2}+1},\end{aligned}$$

then we can apply the following condition for $h(x)=x$:

$$\limsup_{n\to\infty}\frac{\sum_{p=1}^n\phi(p)h(\epsilon_p)}{\log^{2.5} n\left(\max_{p\in[1,n]}h(\epsilon_p)^{1/2}p\right)}=\infty,$$

and we obtain that:

$$\begin{aligned}\limsup_{n\to\infty}\frac{(\sum_{p=1}^n 2\epsilon_p\phi(p))^2}{\sum_{p,q=1}^n \lambda(A_p\cap A_q)} &\geq \limsup_{n\to\infty}\frac{1}{C\frac{\log^{A+5} n}{(\sum_{p=1}^n 2\epsilon_p\phi(p))^2}+1}\\ &= \frac{1}{\liminf_{n\to\infty}C\left(\max_{p\in[1,n]}\epsilon_p^{0.5}\phi(p)\right)^2\frac{\log^5 n}{(\sum_{p=1}^n 2\epsilon_p\phi(p))^2}+1}\\ &= \frac{1}{0+1}=1.\end{aligned}$$

Then the conclusion of this theorem holds for special dimension function $h(x)=x$. For general dimension functions we can combine the special case and the mass transference principle theorem 4.3 this concludes the proof. □

10. Expected number of solutions

In this section we briefly discuss some results about the number of solutions for inhomogeneous Diophantine approximations.

Given an approximation function f and inhomogeneous shift θ. What is the generic growth of the number of solutions:

$$S(f,\theta,x,N)=\#\left|\left\{p,q\leq N,(p,q)=1:\left|x-\frac{q+\theta(p)}{p}\right|<\frac{f(p)}{p}\right\}\right|.$$

Recall in section 6 we constructed the functions $g_p(.)$ and we see that:

$$S(f,\theta,x,N)=\sum_{p=1}^N g_p(x).$$

Then it is easy to see that:

$$\int_0^1 S(f,\theta,x,N)dx=\sum_{p=1}^N 2\epsilon_p\phi(p)=E_N.$$

Now we estimate the variance:

$$\int_0^1|S(f,\theta,x,N)-E_N|^2dx=\int_0^1\sum_{p,q=1}^N g_p(x)g_q(x)dx-(E_N)^2.$$

By the Main Formula and argument in proof of theorem 2.3 we see directly that for some constant $C''' > 0$:

$$\int_0^1 |S(f,\theta,x,N) - E_N|^2 dx \leq C''' \sum_{p=1}^{N} \epsilon_p p d^3(p) \log^2 p,$$

Therefore we see that given any sequence of positive numbers $\beta : \mathbb{N} \to \mathbb{R}^+$:

$$K(f,\theta,x,N,\beta) := \lambda(x : |S(f,\theta,x,N) - E_N| \geq \beta_N) \leq C''' \frac{1}{\beta_N^2} \sum_{p=1}^{N} \epsilon_p p d^3(p) \log^2 p,$$

so if:

$$K(f,\theta,x,N,\beta) \to 0,$$

then there exist a subsequence N_i such that:

$$\sum_i K(f,\theta,x,N_i,\beta) < \infty$$

then for Lebesgue almost every x there are only finitely many N_i such that

$$|S(f,\theta,x,N_i) - E_{N_i}| > \beta_{N_i}.$$

Now we see that:

$$K(f,\theta,x,N,\beta) \leq C''' \frac{1}{\beta_N^2} E_N \log^2 N \log\log N \exp(3\log 2 \log N / \log\log N).$$

Now we denote:

$$A_N = \frac{E_N}{\log^2 N \log\log N \exp(3\log 2 \log N / \log\log N)}.$$

Suppose that $\limsup_{N\to\infty} A_N = \infty$, then we see that for $\beta_N = \sqrt{E_N}$:

$$K(f,\theta,x,N,\beta) \to 0.$$

This implies that for Lebesgue almost all $x \in [0,1]$ there are infinitely many integers $N > 0$ such that:

$$E_N + \sqrt{E_N} \geq S(f,\theta,x,N) \geq E_N - \sqrt{E_N}.$$

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