HOMOTOPY INVARIANCE OF NISNEVICH SHEAVES WITH MILNOR–WITT TRANSFERS

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Abstract. In [CF14], Calmès and Fasel introduce the category of finite Milnor–Witt correspondences, which constitutes a new type of correspondences closer to the setting of motivic homotopy theory than Suslin–Voevodsky's correspondences. A fundamental result of the theory of ordinary correspondences concerns homotopy invariance of sheaves with transfers. In the present paper, we address this question in the setting of Milnor–Witt correspondences. Employing techniques due to Druzhinin [Dru16], Garkusha–Panin [GP15] and Fasel–Østvær [FØ17], we show that homotopy invariance of presheaves with Milnor–Witt transfers is preserved under Nisnevich sheafification.

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1. Introduction

A stepping stone toward Voevodsky's construction of the category of motives $\mathbf{DM}(k)$ [Voe00b] is the notion of finite correspondences between smooth k-schemes. Such correspondences are in a certain sense multivalued functions taking only finitely many values. By considering finite correspondences instead of ordinary morphisms of schemes, one performs a linearization which allows for extra elbowroom and flexibility, and which in turn makes it possible to prove strong theorems. One of the "fundamental theorems" on the theory of correspondences concerns homotopy invariance, which is crucial for constructing the theory of motives.

Theorem 1.1 ([Voe00a, Theorem 5.6])**.** *For any homotopy invariant presheaf $\mathscr{F}$ on the category Cor_k of finite correspondences, the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ is also homotopy invariant.*

2010 *Mathematics Subject Classification.* 14F05, 14F20, 14F35, 14F42, 19E15.

Key words and phrases. Motives, Milnor–Witt K-theory, Chow–Witt groups, motivic homotopy theory.

In [CF14], Calmès and Fasel introduce a new type of correspondences called finite Milnor–Witt correspondences (or finite MW-correspondences for short). Milnor–Witt correspondences provide a setting that is closer to the motivic homotopy theoretic framework than Suslin–Voevodsky's correspondences; for example, the zero-line of sheaves of motivic homotopy groups of the sphere spectrum do not admit ordinary transfers, but they do admit MW-transfers [CF14]. Roughly speaking, a finite MW-correspondence amounts to an ordinary finite correspondence along with quadratic forms defined on the function field of each irreducible component of the support of the correspondence. We briefly recall some theory of MW-correspondences below. Our present goal is to prove a similar homotopy invariance result as Theorem 1.1 for sheaves with MW-transfers:

Theorem 1.2. *Let k be an infinite perfect field of characteristic different from 2. Then for any homotopy invariant presheaf $\mathscr{F}$ on the category $\widetilde{\mathrm{Cor}}_k$ of finite MW-correspondences, the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ is also homotopy invariant.*

We note that this result is already known by work of Déglise–Fasel [DF17, Theorem 3.2.9]. Their proof uses the fact that there is a functor $\mathrm{Fr}_*(k) \to \widetilde{\mathrm{Cor}}_k$ from the category of framed correspondences to MW-correspondences. As the analog of Theorem 1.2 is known for framed correspondences by work of Garkusha–Panin [GP15], it follows that the desired result holds also for $\widetilde{\mathrm{Cor}}_k$. The purpose of this paper is to rather give a more direct proof by using geometric input provided in [GP15, §13] to produce desired homotopies in $\widetilde{\mathrm{Cor}}_k$. Along the way we obtain results on MW-correspondences of independent interest. The proof strategy is due to Druzhinin [Dru16] and Garkusha–Panin [GP15], and uses techniques developed in [FØ17].
Recollections on Milnor–Witt correspondences. The Milnor–Witt K-groups $K_n^{MW}(k)$ of a perfect field k arose in the context of stable motivic homotopy groups of spheres. More precisely, Morel established isomorphisms [Mor04, Theorem 6.4.1]

$$\pi_{n,n}\mathbf{1} \cong K_{-n}^{MW}(k) \tag{1}$$

for all $n \in \mathbf{Z}$, where $\mathbf{1} \in \mathbf{SH}(k)$ denotes the sphere spectrum. The groups $K_n^{MW}(k)$ admit a description in terms of generators and relations:

Definition 1.3 (Hopkins–Morel)**.** Let k be a field. The *Milnor–Witt K-theory* $K_*^{MW}(k)$ of the field k is the graded associative $\mathbf{Z}$-algebra with one generator $[a]$ for each unit $a \in k^\times$, of degree $+1$, and one generator η of degree -1, subject to the following relations:

(1) $[a][1-a] = 0$ for any $a \in k^\times \setminus \{1\}$ (Steinberg relation).
(2) $\eta[a] = [a]\eta$ (η-commutativity).
(3) $[ab] = [a] + [b] + \eta[a][b]$ (twisted η-logarithmic relation).
(4) $(2 + \eta[-1])\eta = 0$ (hyperbolic relation).

We let $K_n^{MW}(k)$ denote the n-th graded piece of $K_*^{MW}(k)$. The product $[a_1]\cdots[a_n] \in K_n^{MW}(k)$ may also be denoted by $[a_1, \ldots, a_n]$.

Under the isomorphism (1) above, the element $[a] \in K_1^{MW}(k)$ corresponds to a map $[a] \in \pi_{-1,-1}\mathbf{1}$. A representative for $[a]$ is given by the pointed map

$$[a]\colon S^{0,0} \simeq \mathrm{Spec}(k)_+ \to (\mathbf{G}_m, 1) \simeq S^{1,1}$$

of motivic spheres sending the non-basepoint to the point $a \in \mathbf{G}_m$. On the other hand, the element $\eta \in K_{-1}^{MW}(k)$ corresponds to the motivic Hopf element $\eta \in \pi_{1,1}\mathbf{1}$ represented by the projection

$$\eta\colon S^{3,2} \simeq \mathbf{A}^2 \setminus 0 \to \mathbf{P}^1 \simeq S^{2,1}.$$

As the sphere spectrum is initial in the category of motivic ring spectra, the homotopy groups $\pi_{p,q}\mathsf{E}$ of a ring spectrum E inherits the relations of $\pi_{p,q}\mathbf{1}$ via the unit map $\mathbf{1} \to \mathsf{E}$. Thus Milnor–Witt K-theory is a fundamental object in motivic homotopy theory. In [CF14], Calmès and Fasel employ sheaves of Milnor–Witt K-theory to set up the theory of MW-correspondences. Based on the fact that the group $\mathrm{Cor}_k(X,Y)$ of finite correspondences from X to Y can be expressed as a colimit of Chow groups with support,

$$\mathrm{Cor}_k(X,Y) = \operatorname*{colim}_{T\in\mathcal{A}(X,Y)} H^{d_Y}_T(X\times Y, \mathbf{K}^M_{d_Y}) = \operatorname*{colim}_{T\in\mathcal{A}(X,Y)} \mathrm{CH}^{d_Y}_T(X\times Y),$$

Calmès and Fasel replace Milnor K-theory (or Chow groups) with (twisted) Milnor–Witt K-theory (or Chow–Witt groups), and define the group of finite MW-correspondences from X to Y as

$$\widetilde{\mathrm{Cor}}_k(X,Y) := \operatorname*{colim}_{T\in\mathcal{A}(X,Y)} H^{d_Y}_T(X\times Y, \mathbf{K}^{MW}_{d_Y}, p_Y^*\omega_{Y/k}) = \operatorname*{colim}_{T\in\mathcal{A}(X,Y)} \widetilde{\mathrm{CH}}^{d_Y}_T(X\times Y, p_Y^*\omega_{Y/k}),$$

where $p_Y\colon X\times Y\to Y$ is the projection. Here Y is assumed to be equidimensional of dimension d_Y, and $\mathcal{A}(X,Y)$ is the partially ordered set of closed subsets T of $X\times Y$ such that each irreducible component of T (with its reduced structure) is finite and surjective over X. Moreover, $\mathbf{K}^{MW}_n$ is the n-th unramified Milnor–Witt K-theory sheaf, as defined in [Mor12, §5]. We note that the Nisnevich cohomology groups $H^p(X,\mathbf{K}^{MW}_q,\mathscr{L})$ of the Milnor–Witt sheaf $\mathbf{K}^{MW}_q(\mathscr{L})$ twisted by a line bundle $\mathscr{L}$ can be computed using the Rost–Schmid complex [Mor12, Chapter 5], which provides a flabby resolution of $\mathbf{K}^{MW}_q(\mathscr{L})$. Recall that the p-th term of the Rost–Schmid complex is given by

$$C^p(X,\mathbf{K}^{MW}_q,\mathscr{L}) := \bigoplus_{x\in X^{(p)}} K^{MW}_{q-p}(k(x), \wedge^p(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \otimes_{k(x)} \mathscr{L}_x),$$

where $X^{(p)}$ denotes the set of codimension p-points of X. We let $\widetilde{\mathrm{Cor}}_k$ denote the category of finite MW-correspondences. The category $\widetilde{\mathrm{Cor}}_k$ is symmetric monoidal, and comes equipped with an embedding $\mathrm{Sm}_k\to\widetilde{\mathrm{Cor}}_k$ from smooth k-schemes, as well as a forgetful functor $\widetilde{\mathrm{Cor}}_k\to\mathrm{Cor}_k$ to Suslin–Voevodsky's correspondences; see [CF14] for details.

Let $\widetilde{\mathrm{PSh}}(k)$ denote the category of presheaves with MW-transfers, i.e., additive presheaves of abelian groups $\mathscr{F}\colon \widetilde{\mathrm{Cor}}_k^{\mathrm{op}}\to\mathrm{Ab}$. As noted in [CF14], there are more presheaves on $\widetilde{\mathrm{Cor}}_k$ than on Cor_k. One example is of course provided by the sheaves $\mathbf{K}^{MW}_*$, which admit MW-transfers but not ordinary transfers [CF14]. Among the various presheaves with MW-transfers, the homotopy invariant ones will be of most interest to us.

Definition 1.4. A presheaf $\mathscr{F}\in\widetilde{\mathrm{PSh}}(k)$ with MW-transfers is *homotopy invariant* if for each $X\in\mathrm{Sm}_k$, the projection $p\colon X\times\mathbf{A}^1\to X$ induces an isomorphism $p^*\colon\mathscr{F}(X)\xrightarrow{\cong}\mathscr{F}(X\times\mathbf{A}^1)$. Equivalently, the zero section $i_0\colon X\to X\times\mathbf{A}^1$ induces an isomorphism $i_0^*\colon\mathscr{F}(X\times\mathbf{A}^1)\xrightarrow{\cong}\mathscr{F}(X)$.

Let us also mention that by [DF17, Proposition 1.2.10], the Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ associated to a presheaf $\mathscr{F}\in\widetilde{\mathrm{PSh}}(k)$ comes equipped with a unique MW-transfer structure. This result follows essentially from [DF17, Lemma 1.2.6], which states that if $p\colon U\to X$ is a Nisnevich covering of a smooth k-scheme X, and if $\widetilde{\mathrm{c}}(X)$ denotes the representable presheaf $\widetilde{\mathrm{c}}(X)(Y) := \widetilde{\mathrm{Cor}}_k(Y,X)$, then the Čech-complex $\widetilde{\mathrm{c}}(U_X^\bullet)\to\widetilde{\mathrm{c}}(X)\to 0$ is exact on the associated Nisnevich sheaves.
Outline. In Section 2 we establish some notation, and collect a few lemmas needed later on. In Section 3 we review how Cartier divisors give rise to finite MW-correspondences, following [FØ17]. This gives a procedure to construct desired homotopies in the later sections.

In Section 4 we prove the first main ingredient in the proof of Theorem 1.2, which is a Zariski excision result for MW-presheaves. More precisely, in Theorem 4.1 we show that if $V\subseteq U\subseteq\mathbf{A}^1$ are two Zariski open neighborhoods of a closed point $x\in\mathbf{A}^1$, then the inclusion $i\colon V\to U$ induces an isomorphism

$$i^*\colon \frac{\mathscr{F}(U\setminus x)}{\mathscr{F}(U)} \xrightarrow{\cong} \frac{\mathscr{F}(V\setminus x)}{\mathscr{F}(V)}$$

for any homotopy invariant $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$. The proof of Zariski excision consists of producing left and right inverses in $\widetilde{\mathrm{Cor}}_k$ of i up to homotopy. This is done in Sections 6 and 7.

The main result of Section 8 is Theorem 8.1, which informally states the following: Let $X \in \mathrm{Sm}_k$, and pick a closed point $x \in X$ along with a closed subscheme $Z \subseteq X$ containing the point x. Then, up to $\mathbf{A}^1$-homotopy, we are able to "move the point x away from Z" using MW-correspondences. See Section 8 for more details.

In Section 9 we prove the last main ingredient of the proof of Theorem 1.2, namely a Nisnevich excision result. The situation is as follows: Given an elementary distinguished Nisnevich square

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow \Pi \\ V & \longrightarrow & X \end{array}$$

with X and X' affine k-smooth, let $S := (X \setminus V)_{\mathrm{red}}$ and $S' := (X' \setminus V')_{\mathrm{red}}$. Suppose that $x \in S$ and $x' \in S'$ are two points satisfying $\Pi(x') = x$, and put $U := \operatorname{Spec} \mathcal{O}_{X,x}$ and $U' := \operatorname{Spec} \mathcal{O}_{X',x'}$. Then the map Π induces an isomorphism

$$\Pi^* \colon \frac{\mathscr{F}(U \setminus S)}{\mathscr{F}(U)} \xrightarrow{\cong} \frac{\mathscr{F}(U' \setminus S')}{\mathscr{F}(U')}$$

for any homotopy invariant $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$. Again the proof consists of producing left and right inverses to Π up to homotopy, which is done in Sections 10 and 11.

Finally, in Section 12 we will see how homotopy invariance of the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$ follows from the above results.

Conventions. Throughout we will assume that k is an infinite perfect field of characteristic different from 2. We let Sm_k denote the category of smooth separated schemes of finite type over k. All undecorated fiber products mean fiber product over k. Throughout, the symbols i_0 and i_1 will denote the rational points $i_0, i_1 \colon \operatorname{Spec} k \to \mathbf{A}^1$ given by 0 and 1, respectively.

We will frequently abuse notation and write simply $f \in \widetilde{\mathrm{Cor}}_k(X,Y)$ for $\widetilde{\gamma}_f$, where $\widetilde{\gamma}_f$ is the image of a morphism of schemes $f \colon X \to Y$ under the embedding $\widetilde{\gamma} \colon \mathrm{Sm}_k \to \widetilde{\mathrm{Cor}}_k$ of [CF14, §4.3]. We let $\sim_{\mathbf{A}^1}$ denote $\mathbf{A}^1$-homotopy equivalence. Following Calmès–Fasel [CF14], if $p_Y \colon X \times Y \to Y$ is the projection, we may write ω_Y as shorthand for $p_Y^* \omega_{Y/k}$ if no confusion is likely to arise. Note that ω_Y is then canonically isomorphic to $\omega_{X \times Y/X}$. In general, given a morphism of schemes $f \colon X \to Y$ we write $\omega_f := \omega_{X/k} \otimes f^* \omega_{Y/k}^\vee$.

Acknowledgments. I am very grateful to Ivan Panin for sharing with me his knowledge on homotopy invariance, to Jean Fasel for valuable comments and suggestions, and to Paul Arne Østvær for proofreading and for suggesting to me the problem. I thank Institut Mittag-Leffler for the kind hospitality during spring 2017.

2. Pairs of Milnor–Witt correspondences

We will frequently encounter the situation of a pair $U \subseteq X$ of schemes, and we will be led to study the associated quotient $\mathscr{F}(U)/\operatorname{im}(\mathscr{F}(X) \to \mathscr{F}(U))$ for a given presheaf $\mathscr{F}$ on $\widetilde{\mathrm{Cor}}_k$. It is therefore notationally convenient to introduce a category $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ of pairs of MW-correspondences.

Following [GP15] we let SmOp_k denote the category whose objects are pairs (X,U) with $X \in \mathrm{Sm}_k$ and U an open subscheme of X, and whose morphisms are maps $f \colon (X,U) \to (Y,V)$, where $f \colon X \to Y$ is a morphism of schemes such that $f(U) \subseteq V$. Below we extend this notion of morphisms of pairs to MW-correspondences.

Definition 2.1 ([GP15, Definition 2.3])**.** Let $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ denote the category whose objects are those of SmOp_k and whose morphisms are defined as follows. For $(X,U), (Y,V) \in \mathrm{SmOp}_k$, with open immersions $j_X\colon U \to X$ and $j_Y\colon V \to Y$, let

$$\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X,U),(Y,V)) := \ker\left(\widetilde{\mathrm{Cor}}_k(X,Y) \oplus \widetilde{\mathrm{Cor}}_k(U,V) \xrightarrow{j_X^* - (j_Y)_*} \widetilde{\mathrm{Cor}}_k(U,Y)\right).$$

Thus a morphism in $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ is a pair (α,β), where $\alpha \in \widetilde{\mathrm{Cor}}_k(X,Y)$ and $\beta \in \widetilde{\mathrm{Cor}}_k(U,V)$, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle j_X}\uparrow & & \uparrow{\scriptstyle j_Y} \\ U & \xrightarrow{\beta} & V \end{array}$$

commutes in $\widetilde{\mathrm{Cor}}_k$. Composition in $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ is defined by $(\alpha,\beta)\circ(\gamma,\delta) := (\alpha\circ\gamma, \beta\circ\delta)$.

The category SmOp_k has Sm_k as a full subcategory, the embedding $\mathrm{Sm}_k \to \mathrm{SmOp}_k$ being defined by $X \mapsto (X, \varnothing)$.

Proposition 2.2 ([GP15, Construction 2.8])**.** *Suppose that $\mathscr{F}$ is a presheaf on $\widetilde{\mathrm{Cor}}_k$. For any $(X,U) \in \mathrm{SmOp}_k$, let $\mathscr{F}(X,U) := \mathscr{F}(U)/\operatorname{im}(\mathscr{F}(X) \to \mathscr{F}(U))$. Then for any $(\alpha,\beta) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X,U),(Y,V))$, $\mathscr{F}$ induces a morphism*

$$(\alpha,\beta)^*\colon \mathscr{F}(Y,V) \to \mathscr{F}(X,U).$$

Definition 2.3 ([GP15, Definition 2.3])**.** Define the homotopy category $h\widetilde{\mathrm{Cor}}_k$ of $\widetilde{\mathrm{Cor}}_k$ as follows. The objects of $h\widetilde{\mathrm{Cor}}_k$ are the same as those of $\widetilde{\mathrm{Cor}}_k$, and the morphisms are given by

$$h\widetilde{\mathrm{Cor}}_k(X,Y) := \widetilde{\mathrm{Cor}}_k(X,Y)/\sim_{\mathbf{A}^1} = \operatorname{coker}\left(\widetilde{\mathrm{Cor}}_k(\mathbf{A}^1\times X, Y) \xrightarrow{i_0^* - i_1^*} \widetilde{\mathrm{Cor}}_k(X,Y)\right).$$

Similarly, let $h\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ denote the category whose objects are those of $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$, and whose morphisms are given by

$$h\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X,U),(Y,V)) := \operatorname{coker}\left(\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}(\mathbf{A}^1\times(X,U),(Y,V)) \xrightarrow{i_0^* - i_1^*} \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X,U),(Y,V))\right).$$

Here $\mathbf{A}^1 \times (X,U)$ is shorthand for $(\mathbf{A}^1\times X, \mathbf{A}^1 \times U)$. If $\alpha \in \widetilde{\mathrm{Cor}}_k(X,Y)$ is a finite MW-correspondence, we write $\overline{\alpha}$ for the image of α in $h\widetilde{\mathrm{Cor}}_k(X,Y)$. Similarly, if (α,β) is a morphism in $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ from (X,U) to (Y,V), write $\overline{(\alpha,\beta)}$ for the image of α in $h\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((X,U),(Y,V))$. Note that a presheaf on $\widetilde{\mathrm{Cor}}_k$ is homotopy invariant if and only if it factors through $h\widetilde{\mathrm{Cor}}_k$.

Next we record a few observations that will come in handy later on:

Lemma 2.4. *Suppose that α is a finite MW-correspondence from X to Y, with Y equidimensional of dimension d_Y. Let $T_1,\dots,T_n$ be the connected components of the support T of α. Then for each $i = 1,\dots,n$ there are uniquely determined finite MW-correspondences $\alpha_i \in \widetilde{\mathrm{CH}}_{T_i}^{d_Y}(X\times Y, \omega_Y)$ such that $\alpha = \sum_i \alpha_i$.*

Proof. Since $\alpha \in \bigoplus_{x\in(X\times Y)^{d_Y}} K_0^{MW}(k(x), \wedge^{d_Y}(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \otimes (\omega_Y)_x)$, we may write $\alpha = \sum_i \alpha_i$ where α_i is supported on T_i. To conclude we must show that $\alpha_i \in \widetilde{\mathrm{CH}}_{T_i}^{d_Y}(X\times Y,\omega_Y)$, i.e., that $\partial(\alpha_i) = 0$ for all i. Now $\partial_x(\alpha_i) = 0$ for all $x \in X\times Y$ except perhaps for $x \in T_i$. But since T_i is disjoint from the other T_j's and $\partial(\alpha) = 0$ by assumption, we must have $\partial_x(\alpha_i) = 0$ also for $x \in T_i$. □

Lemma 2.5. *Let $X \in \mathrm{Sm}_k$, and let T_1 and T_2 be disjoint closed subschemes of X. Then for any $n \in \mathbf{Z}$ and any line bundle $\mathscr{L}$ on X we have $\widetilde{\mathrm{CH}}^n_{T_1\amalg T_2}(X,\mathscr{L}) \cong \widetilde{\mathrm{CH}}^n_{T_1}(X,\mathscr{L}) \oplus \widetilde{\mathrm{CH}}^n_{T_2}(X,\mathscr{L})$.*

Lemma 2.6. *Let X be a smooth scheme, let $q \in \mathbf{Z}$ be an integer, and let $\mathscr{L}$ be a line bundle over X. Let $j\colon U \to X$ be an open subscheme, and suppose that $T \subseteq U$ is a subset which is closed in both U and X. Then the map*

$$j^*\colon H^p_T(X, \mathbf{K}^{MW}_q, \mathscr{L}) \to H^p_T(U, \mathbf{K}^{MW}_q, j^*\mathscr{L})$$

is an isomorphism for each $p \in \mathbf{Z}$, with inverse j_*.

Proof. The map j^* is an isomorphism by étale excision [CF14, Lemma 3.5]. The composition j^*j_* is the identity map on the Rost–Schmid complex $C^*_T(U, \mathbf{K}^{MW}_q, j^*\mathscr{L})$ supported on T, which implies the claim. □

Corollary 2.7. *Let $X, Y \in \mathrm{Sm}_k$, and let $j\colon V \to Y$ be an open subscheme. Suppose that $\alpha \in \widetilde{\mathrm{Cor}}_k(X, Y)$ is a finite MW-correspondence such that $\operatorname{supp}\alpha \subseteq X \times V$. Then there is a unique finite MW-correspondence $\beta \in \widetilde{\mathrm{Cor}}_k(X, V)$ such that $j \circ \beta = \alpha$. In fact, we have $\beta = (1 \times j)^*\alpha$.*

Proof. Let $T := \operatorname{supp}\alpha$, so that by Lemma 2.6 we have mutually inverse isomorphisms

$$(1 \times j)^*\colon \widetilde{\mathrm{CH}}^{d_Y}_T(X \times Y, \omega_Y) \rightleftarrows \widetilde{\mathrm{CH}}^{d_Y}_T(X \times V, \omega_V) :(1 \times j)_*$$

with $\alpha \in \widetilde{\mathrm{CH}}^{d_Y}_T(X \times Y, \omega_Y)$. Thus, if $\beta := (1 \times j)^*(\alpha)$ then $(1 \times j)_*\beta = \alpha$. We conclude the equality $(1 \times j)_*\beta = j \circ \beta$ from [CF14, Example 4.14]. □

Lemma 2.8. *Suppose that $j_X\colon U \to X$ and $j_Y\colon V \to Y$ are open subschemes of smooth connected k-schemes X, Y. Assume further that $\alpha \in \widetilde{\mathrm{Cor}}_k(X, Y)$ is a finite MW-correspondence such that the support $T := \operatorname{supp}\alpha$ satisfies $T \cap (U \times Y) \subseteq U \times V$. Let $\alpha' := (j_X \times j_Y)^*(\alpha)$. Then we have $(\alpha, \alpha') \in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k((X, U), (Y, V))$.*

Proof. First we show that $\alpha' \in \widetilde{\mathrm{Cor}}_k(U, V)$. By contravariant functoriality of Chow–Witt groups we may write $\alpha' = (1 \times j_Y)^*(j_X \times 1)^*(\alpha)$. Now $(j_X \times 1)^*(\alpha) = \alpha \circ j_X \in \widetilde{\mathrm{Cor}}_k(U, Y)$ by [CF14, Example 4.13]. By [CF14, Lemmas 4.7, 4.8], $\operatorname{supp}(j_X \times 1)^*(\alpha) = T \cap (U \times Y)$ is finite and surjective over U. Since $T \cap (U \times Y) \subseteq U \times V$, we have

$$\alpha' \in \widetilde{\mathrm{CH}}^{d_Y}_{T \cap (U \times Y)}(U \times V, (1 \times j_Y)^*\omega_Y),$$

where $d_Y := \dim Y$. As j_Y is étale we have $(1 \times j_Y)^*\omega_Y \cong \omega_V$; hence α' is a finite MW-correspondence from U to V.

Next we show that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle j_X}\uparrow & & \uparrow{\scriptstyle j_Y} \\ U & \xrightarrow{\alpha'} & V \end{array}$$

commutes in $\widetilde{\mathrm{Cor}}_k$. As $T \cap (U \times Y) = T \cap (U \times V)$, the morphism $(j_X \times 1)^*$ factors as

$$\begin{array}{ccc} \widetilde{\mathrm{CH}}^{d_Y}_T(X \times Y, \omega_Y) & \xrightarrow{(j_X \times j_Y)^*} & \widetilde{\mathrm{CH}}^{d_Y}_{T\cap(U\times V)}(U \times V, \omega_V) \\ & \searrow{\scriptstyle (j_X\times 1)^*} & \downarrow{\scriptstyle (1\times j_Y)_*} \\ & & \widetilde{\mathrm{CH}}^{d_Y}_{T\cap(U\times Y)}(U \times Y, \omega_Y). \end{array}$$

Hence

$$j_Y \circ \alpha' = (1 \times j_Y)_*(j_X \times j_Y)^*(\alpha) = (j_X \times 1)^*(\alpha) = \alpha \circ j_X$$

by [CF14, Examples 4.13, 4.14]. □

3. Milnor–Witt correspondences from Cartier divisors

Let us recall from [FØ17, §2] how a Cartier divisor gives rise to a finite MW-correspondence. Suppose that $X \in \mathrm{Sm}_k$ is a smooth integral k-scheme, and let $D = \{(U_i, f_i)\}$ be a Cartier divisor on X, with support $|D|$. We can associate a cohomology class

$$\widetilde{\mathrm{div}}(D) \in H^1_{|D|}(X, \mathbf{K}_1^{MW}, \mathcal{O}_X(D)) = \widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}_X(D))$$

to D as follows. If $x \in X^{(1)}$ is a codimension 1-point on X, choose i such that $x \in U_i$. Consider the element

$$[f_i] \otimes f_i^{-1} \in K_1^{MW}(k(X), \mathcal{O}_X(D) \otimes k(X)).$$

Definition 3.1 ([FØ17, Definition 2.1.1])**.** In the above setting, define

$$\widetilde{\mathrm{ord}}_x(D) := \partial_x([f_i] \otimes f_i^{-1}) \in K_0^{MW}(k(x), (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \otimes_{k(x)} \mathcal{O}_X(D)_x),$$

and

$$\widetilde{\mathrm{ord}}(D) := \sum_{x \in X^{(1)} \cap |D|} \widetilde{\mathrm{ord}}_x(D) \in C^1(X, \mathbf{K}_1^{MW}, \mathcal{O}_X(D)).$$

By [FØ17, Lemma 2.1.2], the definition of $\widetilde{\mathrm{ord}}_x(D)$ does not depend on the choice of U_i, and by [FØ17, Lemma 2.1.3] we have $\partial(\widetilde{\mathrm{ord}}(D)) = 0$. Therefore the element $\widetilde{\mathrm{ord}}(D)$ defines a cohomology class in $\widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}_X(D))$, which we denote by $\widetilde{\mathrm{div}}(D)$.

Lemma 3.2. *Let $X \in \mathrm{Sm}_k$ be a smooth integral k-scheme and suppose that D and D' are two Cartier divisors on X such that*

- *the supports of D and D' are disjoint, and*
- *there are trivializations $\chi\colon \mathcal{O}(D) \cong \mathcal{O}_X$ and $\chi'\colon \mathcal{O}(D') \cong \mathcal{O}_X$.*

Then χ and χ' induce an isomorphism

$$\widetilde{\mathrm{CH}}^1_{|D+D'|}(X, \mathcal{O}(D+D')) \cong \widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}(D)) \oplus \widetilde{\mathrm{CH}}^1_{|D'|}(X, \mathcal{O}(D')).$$

Under this isomorphism we have the identification $\widetilde{\mathrm{div}}(D+D') = \widetilde{\mathrm{div}}(D) + \widetilde{\mathrm{div}}(D')$.

Proof. Since $\mathcal{O}(D+D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')$, χ and χ' furnish a trivialization $\chi \otimes \chi'\colon \mathcal{O}(D+D') \cong \mathcal{O}_X$. As $|D+D'| = |D| \amalg |D'|$, we thus obtain isomorphisms

$$\begin{aligned} \widetilde{\mathrm{CH}}^1_{|D+D'|}(X, \mathcal{O}(D+D')) &\cong \widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}(D+D')) \oplus \widetilde{\mathrm{CH}}^1_{|D'|}(X, \mathcal{O}(D+D')) \\ &\cong \widetilde{\mathrm{CH}}^1_{|D|}(X) \oplus \widetilde{\mathrm{CH}}^1_{|D'|}(X) \\ &\cong \widetilde{\mathrm{CH}}^1_{|D|}(X, \mathcal{O}(D)) \oplus \widetilde{\mathrm{CH}}^1_{|D'|}(X, \mathcal{O}(D')). \end{aligned}$$

To show the last claim, let D and D' be given by the data $\{(U_i, f_i)\}$ respectively $\{(U_i, f_i')\}$, so that $D + D' = \{(U_i, f_i f_i')\}$. Let $x \in X^{(1)} \cap |D|$, and choose an i such that $x \in U_i$. Since the vanishing loci of f_i and f_i' are disjoint we may assume that $f_i' \in \Gamma(U_i, \mathcal{O}_X^\times)$, shrinking U_i if necessary. Hence $\partial_x([f_i']) = 0$, and we obtain

$$\begin{aligned} \partial_x([f_i f_i'] \otimes (f_i f_i')^{-1}) &= \partial_x(([f_i'] + \langle f_i' \rangle [f_i]) \otimes (f_i f_i')^{-1}) \\ &= \langle f_i' \rangle \langle (f_i')^{-1} \rangle \partial_x([f_i] \otimes f_i^{-1}) \\ &= \partial_x([f_i] \otimes f_i^{-1}). \end{aligned}$$

Thus $\partial_x([f_i f_i'] \otimes (f_i f_i')^{-1}) = \widetilde{\mathrm{ord}}_x(D)$. A similar argument shows that $\partial_x([f_i f_i'] \otimes (f_i f_i')^{-1}) = \widetilde{\mathrm{ord}}_x(D')$ for all $x \in X^{(1)} \cap |D'|$, and the result follows. ∎

If we require a condition on the line bundle $\mathcal{O}(D)$ and on the support of D, the class $\widetilde{\operatorname{div}}(D)$ does indeed give rise to a finite MW-correspondence:

Lemma 3.3. *Let X and Y be smooth connected k-schemes with $\dim Y = 1$. Let D be a Cartier divisor on $X \times Y$. Suppose that there is an isomorphism $\chi\colon \mathcal{O}_{X\times Y}(D) \cong \omega_Y$ and that the support $|D|$ of D is finite and surjective over X. Then $\widetilde{\operatorname{div}}(D)$ and χ define a finite MW-correspondence from X to Y, which we denote by $\widetilde{\operatorname{div}}(D,\chi)$.*

Proof. We let $\widetilde{\operatorname{div}}(D,\chi)$ be the image of $\widetilde{\operatorname{div}}(D)$ under the isomorphism

$$\widetilde{\mathrm{CH}}^1_{|D|}(X\times Y, \mathcal{O}_{X\times Y}(D)) \xrightarrow{\cong} \widetilde{\mathrm{CH}}^1_{|D|}(X\times Y, \omega_Y)$$

induced by χ. By assumption, $|D|$ is an admissible subset, hence $\widetilde{\operatorname{div}}(D,\chi) \in \widetilde{\mathrm{Cor}}_k(X,Y)$. □

Lemma 3.4. *Assume the hypotheses of Lemma 3.3, and let $f\colon X' \to X$ be a morphism of smooth schemes. Then the composite MW-correspondence $\widetilde{\operatorname{div}}(D,\chi)\circ f \in \widetilde{\mathrm{Cor}}_k(X',Y)$ is given by*

$$\widetilde{\operatorname{div}}(D,\chi)\circ f = \widetilde{\operatorname{div}}((f\times 1)^*D, (f\times 1)^*\chi).$$

Proof. As $\widetilde{\operatorname{div}}(D,\chi)\circ f = (f\times 1)^*\widetilde{\operatorname{div}}(D,\chi)$, the claim follows from the fact that $(f\times 1)^*$ commutes with the boundary map ∂ in the Rost–Schmid complex. □

For later reference, let us also state the version of Corollary 2.7 for Cartier-divisors:

Lemma 3.5. *Assume the hypotheses of Lemma 3.3. Suppose moreover that $j\colon V \to Y$ is an open subscheme of Y such that the support $|D|$ is contained in $X \times V$. Then there exists a unique finite MW-correspondence $\beta \in \widetilde{\mathrm{Cor}}_k(X,V)$ such that $j\circ\beta = \widetilde{\operatorname{div}}(D,\chi)$. In fact, β is given by*

$$\beta = \widetilde{\operatorname{div}}((1\times j)^*D, (1\times j)^*\chi).$$

Proof. By the same argument as in the proof of Lemma 3.4 we have $(1\times j)^*\widetilde{\operatorname{div}}(D,\chi) = \widetilde{\operatorname{div}}((1\times j)^*D, (1\times j)^*\chi)$. Hence the claim follows from Corollary 2.7. □

The above lemmas give a procedure to construct a morphism of pairs from a Cartier divisor:

Lemma 3.6. *Assume the hypotheses of Lemma 3.3, and let $j_X\colon U \to X$ and $j_Y\colon V \to Y$ be open subschemes. Let $D' := D|_{U\times Y}$ be the restriction of D to $U \times Y$. Suppose that $|D'| \subseteq U \times V$. Then*

$$(\widetilde{\operatorname{div}}(D,\chi), \widetilde{\operatorname{div}}((j_X\times j_Y)^*D, (j_X\times j_Y)^*\chi)) \in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k((X,U),(Y,V)).$$

Proof. By Lemma 3.4, $\widetilde{\operatorname{div}}((j_X\times j_Y)^*D) = (j_X\times j_Y)^*\widetilde{\operatorname{div}}(D)$, hence the claim follows from Lemma 2.8. □

4. Zariski excision

The aim of this section is to prove the following excision result:

Theorem 4.1. *Let $x \in \mathbf{A}^1$ be a closed point and suppose that $V \subseteq U \subseteq \mathbf{A}^1$ are two Zariski open neighborhoods of x. Let $i\colon V \hookrightarrow U$ denote the inclusion, and let $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$ be a homotopy invariant presheaf with MW-transfers. Then the induced map*

$$i^*\colon \frac{\mathscr{F}(U\setminus x)}{\mathscr{F}(U)} \to \frac{\mathscr{F}(V\setminus x)}{\mathscr{F}(V)}$$

is an isomorphism.

The proof of Zariski excision proceeds in three steps. First we prove:

Theorem 4.2 (Injectivity on the affine line)**.** *With the notation in Theorem 4.1, there exists a finite MW-correspondence $\Phi \in \widetilde{\mathrm{Cor}}_k(U,V)$ such that*

$$\bar{i} \circ \overline{\Phi} = \mathrm{id}_U$$

in $h\widetilde{\mathrm{Cor}}_k$.

Theorem 4.2 implies that $\Phi^* \circ i^* = \mathrm{id}_{\mathscr{F}(U)}$ for any homotopy invariant $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$, i.e., that i^* is injective. Letting $V = U \setminus x$, this means that $\mathscr{F}(U)$ is a subgroup of $\mathscr{F}(U \setminus x)$, justifying the notation of Theorem 4.1.

The next step is then to extend Theorem 4.2 to the category $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ of pairs, which establishes injectivity of the map $i^* \colon \mathscr{F}(U \setminus x)/\mathscr{F}(U) \to \mathscr{F}(V \setminus x)/\mathscr{F}(V)$:

Theorem 4.3 (Injectivity of Zariski excision)**.** *With the notation in Theorem 4.1, there exists a finite MW-correspondence $\Phi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus x), (V, V \setminus x))$ such that*

$$\bar{i} \circ \overline{\Phi} = \mathrm{id}_U$$

in $h\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$.

In the final step we establish surjectivity of i^*:

Theorem 4.4 (Surjectivity of Zariski excision)**.** *With the notation in Theorem 4.1, there exist finite MW-correspondences $\Psi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U\setminus x), (V, V\setminus x))$ and $\Theta \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((V, V\setminus x), (V\setminus x, V\setminus x))$ such that*

$$\overline{\Psi} \circ \bar{i} - \overline{j_V} \circ \overline{\Theta} = \mathrm{id}_V$$

in $h\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$, where $j_V \colon V \setminus x \to V$ is the inclusion.

We note that Theorem 4.1 is a consequence of Theorems 4.3 and 4.4:

Proof of Theorem 4.1. As Φ is a morphism of pairs by Theorem 4.3, Proposition 2.2 tells us that Φ induces a morphism on the quotient $\Phi^* \colon \mathscr{F}(V \setminus x)/\mathscr{F}(V) \to \mathscr{F}(U \setminus x)/\mathscr{F}(U)$. Moreover, $\Phi^* \circ i^* = \mathrm{id}$ by Theorem 4.3, hence i^* is injective.

On the other hand, as Θ has image contained in $V \setminus x$ by Theorem 4.4, it follows that $j_V \circ \Theta$ induces the trivial map on the quotient. Hence $i^* \circ \Psi^* = \mathrm{id} \colon \mathscr{F}(V \setminus x)/\mathscr{F}(V) \to \mathscr{F}(V \setminus x)/\mathscr{F}(V)$, so that i^* is surjective. □

It is therefore enough to prove Theorems 4.2, 4.3, and 4.4.

5. Injectivity on the affine line

We continue with the same notation as in Theorem 4.1. Thus $V \subseteq U \subseteq \mathbf{A}^1$ are two Zariski open neighborhoods of a closed point $x \in \mathbf{A}^1$, with inclusion $i \colon V \to U$. In order to produce the desired MW-correspondence $\Phi \in \widetilde{\mathrm{Cor}}_k(U,V)$ of Theorem 4.2, we will need to consider certain "thick diagonals" $\Delta_m \in \widetilde{\mathrm{Cor}}_k(U,U)$, constructed as follows.

Let $U \times U \subseteq \mathbf{A}^2$ have coordinates X and Y, respectively, and let $\Delta := \Delta(U) \subseteq U \times U$ denote the diagonal. For each $m \geq 1$, let f_m denote the polynomial $f_m(X,Y) := (Y - X)^m \in k[U \times U]$. As f_m is monic in Y, it follows that the divisor

$$D_m := \mathcal{V}(f_m) := \{f_m = 0\} \subseteq U \times U$$

is finite and surjective over U (indeed, the elements $1, Y, \ldots, Y^{m-1}$ generate $k[U \times U]/(f_m)$ as a $k[U]$-module). Moreover, as D_m is a principal Cartier divisor on $U \times U$, there is a trivialization $\mathcal{O}(D_m) \cong \mathcal{O}_{U \times U}$ given by $f_m^{-1} \mapsto 1$. We further obtain an isomorphism $\chi_m \colon \mathcal{O}(D_m) \cong \omega_U$ by $f_m^{-1} \mapsto dY$. By Lemma 3.3, it follows that the divisor D_m gives rise to a finite MW-correspondence from U to U.

Definition 5.1. For each $m \geq 1$, let $\Delta_m := \widetilde{\mathrm{div}}(D_m, \chi_m) \in \widetilde{\mathrm{Cor}}_k(U, U)$ be the finite MW-correspondence defined by the data D_m and χ_m above.

Remark 5.2. By the definition of $\widetilde{\mathrm{div}}(D_m, \chi_m)$, we see that Δ_m is given by the total residue

$$\Delta_m = \partial([f_m] \otimes dY) \in \widetilde{\mathrm{CH}}^1_\Delta(U \times U, \omega_U)$$

of the element $[f_m] \otimes dY \in K_1^{MW}(k(U \times U), \omega_U)$. Thus the support of the MW-correspondence Δ_m is the diagonal $\Delta = D_1 \subseteq U \times U$.

Lemma 5.3. *For any $m \geq 0$ we have $\Delta_{m+1} - \Delta_m = \langle -1 \rangle^m \cdot \Delta_1 \in \widetilde{\mathrm{Cor}}_k(U, U)$, with $\Delta_1 = \mathrm{id}_U$.*

Proof. Since Δ_m is supported on the diagonal $\Delta \subseteq U \times U$, it suffices to compute the residue $\partial_y([f_m] \otimes dY)$ at the codimension 1-point $y \in (U \times U)^{(1)}$ corresponding to the diagonal.

Recall from [Mor12, Lemma 3.14] that for any integer $n \geq 0$ we have $[a^n] = n_\epsilon[a]$ in K_1^{MW}, where $n_\epsilon = \sum_{i=1}^n \langle (-1)^{i-1} \rangle$. We thus get

$$\partial_y([f_m] \otimes dY) = m_\epsilon \otimes \overline{(Y - X)} dY \in K_0^{MW}(k(y), (\mathfrak{m}_y/\mathfrak{m}_y^2)^\vee \otimes (\omega_U)_y).$$

For $m = 1$, this reads $\Delta_1 = \langle 1 \rangle \otimes \overline{(Y - X)} dY = \mathrm{id}_U$. In the general case we obtain

$$\Delta_{m+1} - \Delta_m = ((m+1)_\epsilon - m_\epsilon) \otimes \overline{(Y - X)} dY = \langle (-1)^m \rangle \cdot \mathrm{id}_U,$$

using that $\Delta_1 = \mathrm{id}_U \in \widetilde{\mathrm{Cor}}_k(U, U)$. □

Our next objective is to prove the following:

Lemma 5.4. *For $m \gg 0$ there exists a finite MW-correspondence $\Phi_m \colon U \to V$ such that $i \circ \Phi_m = \Delta_m$ in $h\widetilde{\mathrm{Cor}}_k(U, U)$.*

Having established these properties of Δ_m and Φ_m, we will set $\Phi := \Phi_{m+1} - \Phi_m$ and show that we then have $i \circ \Phi \sim_{\mathbf{A}^1} \mathrm{id}_U$ provided m is an even integer $\gg 0$. To define Φ_m, we will need to assure the existence of polynomials with certain specified properties.

Lemma 5.5 ([GP15, §5])**.** *Let $A := \mathbf{A}^1 \setminus U$ and $B = U \setminus V$. For $m \gg 0$, there exists a polynomial $G_m \in k[U][Y] = k[U \times \mathbf{A}^1]$, monic and of degree m in Y, satisfying the following properties:*

1. $G_m(Y)|_{U \times B} = 1$.
2. $G_m(Y)|_{U \times A} = (Y - X)^m|_{U \times A}$.
3. $G_m(Y)|_{U \times x} = (Y - X)^m|_{U \times x}$.

Lemma 5.6. *Let D_{G_m} be the divisor on $U \times U$ defined by G_m, and let $\phi_m \colon \mathcal{O}(D_{G_m}) \cong \omega_U$ be the isomorphism determined by choosing the generator dY for ω_U. Then*

$$\widetilde{\mathrm{div}}((1 \times i)^* D_{G_m}, (1 \times i)^* \phi_m) \in \widetilde{\mathrm{Cor}}_k(U, V).$$

Proof. Since G_m is monic in Y, the support $\mathcal{V}(G_m)$ of D_{G_m} is finite and surjective over U. Using the trivializations of $\mathcal{O}(D_{G_m})$ and of ω_U, Lemma 3.3 implies that $\widetilde{\mathrm{div}}(D_{G_m}, \phi_m) \in \widetilde{\mathrm{Cor}}_k(U, U)$. Now, the polynomial G_m satisfies the following:

- $G_m|_{U \times A} \in k[U \times A]^\times$. This follows from the fact that $U \times A = U \times (\mathbf{A}^1 \setminus U)$ contains no diagonal points.
- $G_m|_{U \times B} \in k[U \times B]^\times$. This is obvious, as $G_m|_{U \times B} = 1$.

The above properties imply that $\mathcal{V}(G_m) \subseteq U \times V$. Hence the claim follows from Lemma 3.5. □

Definition 5.7. For $m \gg 0$, we define $\Phi_m := \widetilde{\mathrm{div}}((1 \times i)^* D_{G_m}, (1 \times i)^* \phi_m) \in \widetilde{\mathrm{Cor}}_k(U, V)$.

We now aim to define a homotopy $\mathscr{H}_m\colon i\circ\Phi_m \sim_{\mathbf{A}^1} \Delta_m$. Consider the product $\mathbf{A}^1\times U\times\mathbf{A}^1$, where θ is the coordinate of the first copy of $\mathbf{A}^1$, U has coordinate X and the last $\mathbf{A}^1$ has coordinate Y. Let $H_\theta\in k[\mathbf{A}^1\times U\times\mathbf{A}^1]$ be the polynomial

$$H_\theta(Y) := \theta G_m + (1-\theta)(Y-X)^m.$$

Since $U\times A$ contains no diagonal points, the restriction $G_m(Y)|_{U\times A} = (Y-X)^m|_{U\times A}$ does not vanish on $U\times A$. It follows that

$$H_\theta(Y)|_{\mathbf{A}^1\times U\times A} = (Y-X)^m|_{\mathbf{A}^1\times U\times A}\in k[\mathbf{A}^1\times U\times A]^\times.$$

Hence $\mathcal{V}(H_\theta)\subseteq\mathbf{A}^1\times U\times U$. Let D_{H_θ} be the principal Cartier divisor on $\mathbf{A}^1\times U\times U$ defined by H_θ, and let $\psi\colon\mathcal{O}(D_{H_\theta})\cong\omega_U$ be the isomorphism given by choosing the generator dY for ω_U.

Lemma 5.8. *Let* $\mathscr{H}_m := \widetilde{\mathrm{div}}(D_{H_\theta},\psi)$. *Then* $\mathscr{H}_m\in\widetilde{\mathrm{Cor}}_k(\mathbf{A}^1\times U,U)$.

Proof. As G_m is monic and of degree m in Y, it follows that the linear combination H_θ of G_m and $(Y-X)^m$ is also monic and of degree m in Y. Therefore $k[\mathbf{A}^1\times U][Y]/(H_\theta)$ is a free $k[\mathbf{A}^1\times U]$-module on generators $1,Y,\dots,Y^{m-1}$, and so the support $\mathcal{V}(H_\theta)$ of D_{H_θ} is finite and surjective over $\mathbf{A}^1\times U$. The result then follows from Lemma 3.3. □

Lemma 5.9. *Let* $\mathscr{H}_m|_0 := \mathscr{H}_m\circ i_0, \mathscr{H}_m|_1 := \mathscr{H}_m\circ i_1\in\widetilde{\mathrm{Cor}}_k(U,U)$ *denote the respective precompositions of* $\mathscr{H}_m\in\widetilde{\mathrm{Cor}}_k(\mathbf{A}^1\times U,U)$ *with the rational points* $i_0,i_1\colon U\to\mathbf{A}^1\times U$. *Then* $\mathscr{H}_m|_0=\Delta_m$ *and* $\mathscr{H}_m|_1 = i\circ\Phi_m$.

Proof. By Lemma 3.4 we have

$$\mathscr{H}_0 = \widetilde{\mathrm{div}}((i_0\times 1)^*D_{H_\theta},(i_0\times 1)^*\psi) = \widetilde{\mathrm{div}}(D_m,\chi_m) = \Delta_m.$$

On the other hand,

$$\mathscr{H}_1 = \widetilde{\mathrm{div}}((i_1\times 1)^*D_{H_\theta},(i_0\times 1)^*\psi) = \widetilde{\mathrm{div}}(D_{G_m},\phi_m) = i\circ\Phi_m$$

by Lemma 3.5. □

We are now ready to prove the injectivity of the induced morphism $i^*\colon\mathscr{F}(U)\to\mathscr{F}(V)$, for any homotopy invariant $\mathscr{F}\in\widetilde{\mathrm{PSh}}(k)$.

Proof of Theorem 4.2. Let $m\gg 0$ be an integer large enough so that the polynomial G_m of Lemma 5.5 exists. If $\Phi:=\Phi_{2m+1}-\Phi_{2m}$, we then have $i\circ\Phi\sim_{\mathbf{A}^1}(\Delta_{2m+1}-\Delta_{2m}) = \langle(-1)^{2m}\rangle\,\mathrm{id}_U = \mathrm{id}_U$ by Lemma 5.3. As $\mathscr{F}$ is homotopy invariant, this yields $\Phi^*\circ i^* = \mathrm{id}_{\mathscr{F}(U)}$, hence i^* is injective. □

6. Injectivity of Zariski excision

We wish to extend Theorem 4.2 to the category of pairs $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$—in other words to produce a morphism $(\Phi_m,\Phi_m^x)\in\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U,U\setminus x),(V,V\setminus x))$ and a homotopy $(\mathscr{H}_m,\mathscr{H}_m^x)\in\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}(\mathbf{A}^1\times(U,U\setminus x),(U,U\setminus x))$ from Δ_m to $(i,i|_{V\setminus x})\circ(\Phi_m,\Phi_m^x)$. This establishes Theorem 4.3.

Let j_U and j_V denote the respective open immersions $j_U\colon U\setminus x\to U$ and $j_V\colon V\setminus x\to V$.

Lemma 6.1. *Let* $\Phi_m^x := \widetilde{\mathrm{div}}((j_U\times j_V)^*D_{G_m},(j_U\times j_V)^*\phi_m)$. *Then* (Φ_m,Φ_m^x) *constitutes a morphism in* $\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$ *from* $(U,U\setminus x)$ *to* $(V,V\setminus x)$.

Proof. By Lemma 3.6, it suffices to show that the support of $(j_U\times j_V)^*D_{G_m}$ is contained in $(U\setminus x)\times(V\setminus x)$. As we already know that $\mathcal{V}(G_m)\cap((U\setminus x)\times\mathbf{A}^1)\subseteq(U\setminus x)\times V$, it is enough to check that G_m does not vanish on $(U\setminus x)\times x$. By condition (3) of Lemma 5.5, $G_m(Y)|_{U\times x} = (Y-X)^m|_{U\times x}$. As $(U\setminus x)\times x$ contains no diagonal points, it therefore follows that $G_m|_{(U\setminus x)\times x}\in k[(U\setminus x)\times x]^\times$. Hence $\mathcal{V}(G_m)\cap((U\setminus x)\times\mathbf{A}^1)\subseteq(U\setminus x)\times(V\setminus x)$. □

Lemma 6.2. *Let* $\mathscr{H}_\theta^x := \widetilde{\operatorname{div}}(((1\times j_U)\times j_U)^* D_{H_\theta}, ((1\times j_U)\times j_U)^*\psi)$. *Then*

$$(\mathscr{H}_\theta, \mathscr{H}_\theta^x) \in \widetilde{\operatorname{Cor}}_k^{\operatorname{pr}}(\mathbf{A}^1 \times (U, U\setminus x), (U, U\setminus x)).$$

Proof. In light of Lemma 3.6, it remains to check that

$$\mathcal{V}(H_\theta) \cap (\mathbf{A}^1 \times (U\setminus x) \times \mathbf{A}^1) \subseteq \mathbf{A}^1 \times (U\setminus x) \times (U\setminus x).$$

It is sufficient to show that H_θ does not vanish on $\mathbf{A}^1 \times (U\setminus x)\times x$. But

$$H_\theta(Y)|_{\mathbf{A}^1\times(U\setminus x)\times x} = \theta\cdot(Y-X)^m + (1-\theta)\cdot(Y-X)^m = (Y-X)^m|_{\mathbf{A}^1\times(U\setminus x)\times x},$$

and $(Y-X)^m|_{\mathbf{A}^1\times(U\setminus x)\times x} \in k[\mathbf{A}^1\times(U\setminus x)\times x]^\times$ as $(U\setminus x)\times x$ contains no diagonal points. Whence the claim. ☐

Proof of Theorem 4.3. By a similar argument as in the proof of Lemma 5.9, $(\mathscr{H}_\theta, \mathscr{H}_\theta^x)$ is a homotopy from Δ_m to $(i, i|_{V\setminus x})\circ(\Phi_m, \Phi_m^x)$. Thus the same proof as that of Theorem 4.2 applies. ☐

7. Surjectivity of Zariski excision

We proceed to prove Theorem 4.4. To begin with, we interpolate polynomials in a similar fashion as Lemma 5.5:

Lemma 7.1 ([GP15, §5])**.** *For* $m \gg 0$ *there exists a polynomial* $G_m(Y) \in k[U][Y] = k[U\times\mathbf{A}^1]$, *monic and of degree* m *in* Y, *satisfying the following properties:*

(1′) $G_m(Y)|_{U\times B} = 1$.
(2′) $G_m(Y)|_{U\times A} = (Y-X)|_{U\times A}$.
(3′) $G_m(Y)|_{U\times x} = (Y-X)|_{U\times x}$.

Lemma 7.2 ([GP15, §5])**.** *For* $m \gg 0$ *there exists a polynomial* $F_{m-1}(Y) \in k[V][Y] = k[V\times\mathbf{A}^1]$, *monic and of degree* $m-1$ *in* Y, *satisfying the following properties:*

(1″) $F_{m-1}(Y)|_{V\times B} = (Y-X)^{-1} \in k[V\times B]^\times$.
(2″) $F_{m-1}(Y)|_{V\times A} = 1$.
(3″) $F_{m-1}(Y)|_{\Delta(V)} = 1$.

Remark 7.3. As $B = U\setminus V$, the set $V\times B$ does not contain any diagonal points. Hence the function $Y-X$ is invertible on $V\times B$, so (1″) makes sense.

Definition 7.4. Set $E_m := (Y-X)\cdot F_{m-1} \in k[V][Y]$ and $H_\theta := \theta G_m + (1-\theta)E_m \in k[\mathbf{A}^1\times V][Y]$, where θ is the coordinate of $\mathbf{A}^1$.

Observe that the divisor $\mathcal{V}(E_m)$ satisfies $\mathcal{V}(E_m) = \mathcal{V}(Y-X)\cup\mathcal{V}(F_{m-1}) = \Delta(V)\cup\mathcal{V}(F_{m-1})$. In fact, by (3″), this union is a disjoint union. Moreover, using the definition of F_{m-1} we see that E_m enjoys the following properties:

(1_E) $E_m(Y)|_{V\times B} = 1 = G_m(Y)|_{V\times B}$.
(2_E) $E_m(Y)|_{V\times A} = (Y-X)|_{V\times A} = G_m(Y)|_{V\times A}$.
(3_E) $E_m(Y)|_{V\times x} = (Y-X)|_{V\times x} = G_m(Y)|_{V\times x}$.

The last property (3_E) implies

($3'_E$) $E_m(Y)|_{(V\setminus x)\times x} = G_m(Y)|_{(V\setminus x)\times x} \in k[(V\setminus x)\times x]^\times$.

Remark 7.5. The above polynomials as well as those in Section 5 are all constructed using variants of the Chinese remainder theorem, allowing us to find polynomials with specified behavior at given subschemes. The requirement that the desired polynomial be monic can be thought of as specifying its behavior at infinity. For example, the Chinese remainder theorem establishes a surjection $k[U\times\mathbf{A}^1] \to k[U\times A]\oplus k[U\times B]$, from which we can deduce Lemma 5.5.

Let us first construct the MW-correspondence $\Psi \in \widetilde{\mathrm{Cor}}_k(U, V)$ using the polynomial G_m of Lemma 7.1 for $m \gg 0$. By Lemma 7.1, $\mathcal{V}(G_m) \subseteq U \times V$, and we may consider the principal divisor D_{G_m} on $U \times V$ defined by G_m. Let $\psi\colon \mathcal{O}(D_{G_m}) \cong \omega_V$ be the isomorphism determined by choosing the generator dY for ω_V.

Lemma 7.6. *Put* $\Psi := \widetilde{\mathrm{div}}(D_{G_m}, \psi)$ *and* $\Psi^x := \widetilde{\mathrm{div}}((j_U \times j_V)^* D_{G_m}, (j_U \times j_V)^* \psi)$*. Then*

$$(\Psi, \Psi^x) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus x), (V, V \setminus x)).$$

Proof. Since G_m is monic in Y, $\mathcal{V}(G_m)$ is finite and surjective over U. Thus Lemma 3.3 assures that Ψ is a finite MW-correspondence from U to V. Moreover, as $G_m(Y)|_{U \times x} = (Y - X)|_{U \times x}$, it follows that $G_m|_{(U \setminus x) \times x}$ is invertible on $(U \setminus x) \times x$. Hence there is an inclusion

$$\mathcal{V}(G_m) \cap ((U \setminus x) \times V) \subseteq (U \setminus x) \times (V \setminus x).$$

By Lemma 3.6 it follows that (Ψ, Ψ^x) is a morphism of pairs from $(U, U \setminus x)$ to $(V, V \setminus x)$. □

In order to define the desired homotopy, we proceed in a familiar fashion. By (1_E) and (2_E), H_θ is invertible on $\mathbf{A}^1 \times V \times B$ and $\mathbf{A}^1 \times V \times A$. Hence $\mathcal{V}(H_\theta) \subseteq \mathbf{A}^1 \times V \times V$, and we may consider the divisor D_{H_θ} on $\mathbf{A}^1 \times V \times V$. We let $\chi\colon \mathcal{O}(D_{H_\theta}) \cong \omega_V$ be the isomorphism given by choosing the generator dY for ω_V.

Lemma 7.7. *Let* $\mathscr{H}_\theta := \widetilde{\mathrm{div}}(D_{H_\theta}, \chi)$ *and* $\mathscr{H}_\theta^x := \widetilde{\mathrm{div}}(((1 \times j_V) \times j_V)^* D_{H_\theta}, ((1 \times j_V) \times j_V)^* \chi)$*. Then*

$$(\mathscr{H}_\theta, \mathscr{H}_\theta^x) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}(\mathbf{A}^1 \times (V, V \setminus x), (V, V \setminus x)).$$

Proof. To see that $\mathscr{H}_\theta$ is a finite MW-correspondence from $\mathbf{A}^1 \times V$ to V, note that both G_m and E_m are monic and of the same degree in Y. Therefore the linear combination H_θ of G_m and E_m is also monic in Y, and from this it follows that the support $\mathcal{V}(H_\theta)$ of D_{H_θ} is finite and surjective over $\mathbf{A}^1 \times V$. Hence $\mathscr{H}_\theta \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times V, V)$ by Lemma 3.3.

Turning to $\mathscr{H}_\theta^x$, we must show that

$$\mathcal{V}(H_\theta) \cap (\mathbf{A}^1 \times (V \setminus x) \times V) \subseteq \mathbf{A}^1 \times (V \setminus x) \times (V \setminus x).$$

We already know that H_θ is invertible on $\mathbf{A}^1 \times (V \setminus x) \times A$ and on $\mathbf{A}^1 \times (V \setminus x) \times B$. It remains to check the set $\mathbf{A}^1 \times (V \setminus x) \times x$. But by (3_E) and $(3'_E)$ we have

$$E_m(Y)|_{(V \setminus x) \times x} = G_m(Y)|_{(V \setminus x) \times x} = (Y - X)|_{(V \setminus x) \times x},$$

which is invertible as $(V \setminus x) \times x$ does not intersect the diagonal. Therefore the linear combination H_θ of E_m and G_m is also invertible on $(V \setminus x) \times x$, and the claim follows. Using Lemma 3.6, this shows that $(\mathscr{H}_\theta, \mathscr{H}_\theta^x)$ constitutes a morphism of pairs from $\mathbf{A}^1 \times (V, V \setminus x)$ to $(V, V \setminus x)$. □

Let us compute the start-, and endpoints $\mathscr{H}_0$, $\mathscr{H}_1$ of the homotopy $\mathscr{H}_\theta$—that is, the precomposition of $\mathscr{H}_\theta$ with the rational points $i_0, i_1\colon V \to \mathbf{A}^1 \times V$.

Lemma 7.8. *We have* $\mathscr{H}_0 = \mathrm{id}_V + j_V \circ \Theta$ *where* $\Theta \in \widetilde{\mathrm{Cor}}_k(V, V \setminus x)$*. On the other hand,* $\mathscr{H}_1 = \Psi \circ i$*, where* $i\colon V \hookrightarrow U$ *is the inclusion.*

Proof. By Lemma 3.4 we have $\mathscr{H}_1 = \widetilde{\mathrm{div}}((i_1 \times 1)^* D_{H_\theta}, (i_1 \times 1)^* \chi) = \Psi \circ i$. As for $\mathscr{H}_0$, we have

$$\mathscr{H}_0 = \widetilde{\mathrm{div}}((i_0 \times 1)^* D_{H_\theta}, (i_0 \times 1)^* \chi) = \widetilde{\mathrm{div}}(D_{E_m}, (i_0 \times 1)^* \chi),$$

where D_{E_m} is the principal Cartier divisor on $V \times V$ defined by the polynomial E_m. Let $D_{F_{m-1}}$ be the principal divisor on $V \times V$ defined by F_{m-1}. As $\mathcal{V}(E_m) = \Delta(V) \amalg \mathcal{V}(F_{m-1})$, Lemma 3.2 tells us that

$$\mathscr{H}_0 = \Delta_1 + \widetilde{\mathrm{div}}(D_{F_{m-1}}, (i_0 \times 1)^* \chi).$$

Here Δ_1 is the divisor defined in Definition 5.1, satisfying $\Delta_1 = \mathrm{id}_V$. As $\mathcal{V}(F_{m-1}) \subseteq V \times (V \setminus x)$, Lemma 3.5 assures that there is a unique element $\Theta \in \widetilde{\mathrm{CH}}^1_{\mathcal{V}(F_{m-1})}(V \times (V \setminus x), \omega_{V\setminus x})$ such that $j_V \circ \Theta = \widetilde{\mathrm{div}}(D_{E_m}, (i_0 \times 1)^*\chi)$. By Lemma 7.2, $\mathcal{V}(F_{m-1})$ is finite and surjective over $V \setminus x$, and hence $\Theta \in \widetilde{\mathrm{Cor}}_k(V, V \setminus x)$. □

Proof of Theorem 4.4. The content of Theorem 4.4 is a rephrasing of Lemma 7.8. □

8. Injectivity for local schemes

The goal of this section is to prove the following theorem.

Theorem 8.1. *Let X be a smooth k-scheme and $x \in X$ a closed point. Let $U := \operatorname{Spec} \mathcal{O}_{X,x}$ and let $\mathrm{can}\colon U \to X$ be the canonical inclusion. Let $i\colon Z \to X$ be a closed subscheme with $x \in Z$ and let $j\colon X \setminus Z \to X$ be the open complement. Then there exists a finite MW-correspondence $\Phi \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$ such that the diagram*

$$\begin{array}{ccc} & & X \setminus Z \\ & \overset{\Phi}{\nearrow} & \big\downarrow j \\ U & \xrightarrow{\mathrm{can}} & X \end{array}$$

commutes in $h\widetilde{\mathrm{Cor}}_k$.

For homotopy invariant presheaves on $\widetilde{\mathrm{Cor}}_k$ we immediately obtain:

Corollary 8.2. *Suppose that $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$ is a homotopy invariant presheaf with MW-transfers. If $s \in \mathscr{F}(X)$ is a section such that $s|_{X\setminus Z} = 0$, then $s|_U = 0$.*

Let $X^\circ \subseteq X$ be an open neighborhood of the point x, and let $Z^\circ := Z \cap X^\circ$. As noted in [GP15, §7], it is enough to solve the problem for the triple U, X° and $X^\circ \setminus Z^\circ$. In particular, we may assume that X is irreducible and that the canonical sheaf $\omega_{X/k}$ is trivial. In fact, we will shrink X so that we are in the situation of a relative curve over a quasi-projective scheme. The advantage of this approach is that it turns problems regarding subschemes of high codimension into problems regarding divisors, which is a much more flexible setting. For the shrinking process we refer to the following theorem, which is originally due to M. Artin.

Theorem 8.3 ([PSV09, Proposition 1])**.** *Let X, Z and $x \in Z$ be as in Theorem 8.1. Then there is a Zariski open neighborhood $X^\circ \subseteq X$ of the point x, an open subscheme B of $\mathbf{P}^{\dim X-1}$ and a commutative diagram*

$$\begin{array}{ccccc} X^\circ & \longrightarrow & \overline{X}^\circ & \longleftarrow & X^\circ_\infty \\ & {\scriptstyle p}\searrow & \big\downarrow {\scriptstyle \overline{p}} & \swarrow{\scriptstyle p_\infty} & \\ & & B & & \end{array}$$

satisfying the following properties:

1. *$\overline{p}$ is a smooth projective morphism, whose fibers are irreducible projective curves.*
2. *$X^\circ_\infty := \overline{X}^\circ \setminus X^\circ$, and $p_\infty\colon X^\circ_\infty \to B$ is finite étale.*
3. *$p|_Z$ is finite and $Z \cap X_\infty = \varnothing$.*

The morphism $p\colon X^\circ \to B$ is called an almost elementary fibration.

Following [GP15, §7], we may shrink X such that there exists an almost elementary fibration $p\colon X \to B$ and such that $\omega_{X/k}$ and $\omega_{B/k}$ are trivial, i.e., $\omega_{X/k} \cong \mathcal{O}_X$ and $\omega_{B/k} \cong \mathcal{O}_B$. Let $\mathscr{X} := X \times_B U$ and $\mathscr{Z} := Z \times_B U$. Let also $p_X\colon \mathscr{X} \to X$ and $p_U\colon \mathscr{X} \to U$ be the projections onto X and U, respectively, and let d_X denote the dimension of X. Finally, let Δ denote the morphism $\Delta := (\mathrm{can}, \mathrm{id})\colon U \to \mathscr{X}$.

Lemma 8.4 ([GP15, Lemma 7.1]). *There exists a finite surjective morphism*

$$H_\theta = (h_\theta, p_U)\colon \mathscr{X} \to \mathbf{A}^1 \times U$$

over U, such that if we let $\mathscr{D}_1 := H_\theta^{-1}(1 \times U)$ and $\mathscr{D}_0 := H_\theta^{-1}(0 \times U)$ denote the scheme-theoretic preimages, then the following hold:

(1) $\mathscr{D}_1 \subseteq \mathscr{X} \setminus \mathscr{Z}$.
(2) $\mathscr{D}_0 = \Delta(U) \amalg \mathscr{D}_0'$ *with* $\mathscr{D}_0' \subseteq \mathscr{X} \setminus \mathscr{Z}$.

We will use Lemma 8.4 to produce the desired MW-correspondence Φ. The aim is to define Φ as the image $(H_\theta \times 1)_*(p_X)$ of the projection $p_X \in \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathscr{X} \times X, \omega_X)$ under the pushforward map

$$(H_\theta \times 1)_*\colon \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathscr{X} \times X, \omega_{H_\theta \times 1} \otimes \omega_X) \to \widetilde{\mathrm{CH}}^{d_X}_{(H_\theta \times 1)(\Gamma_{p_X})}(\mathbf{A}^1 \times U \times X, \omega_X).$$

To this end, we need a trivialization of $\omega_{H_\theta \times 1} = \omega_{\mathscr{X} \times X/k} \otimes (H_\theta \times 1)^* \omega^\vee_{\mathbf{A} \times U \times X/k}$. As U is local we have $\omega_{U/k} \cong \mathcal{O}_U$. Keeping in mind the discussion preceding Lemma 8.4, it follows that the relative bundle $\omega_{H_\theta \times 1}$ is also trivial. Thus we may choose an isomorphism $\chi\colon \omega_{H_\theta \times 1} \cong \mathcal{O}_X$.

Definition 8.5. Let $p_X \in \widetilde{\mathrm{Cor}}_k(\mathscr{X}, X)$ denote the projection. Using the trivialization χ above, we let $\mathscr{H}_\theta \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times U, X)$ denote the image of $p_X \in \widetilde{\mathrm{Cor}}_k(\mathscr{X}, X)$ under the composition

$$\widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathscr{X} \times X, \omega_X) \cong \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathscr{X} \times X, \omega_{H_\theta \times 1} \otimes \omega_X) \xrightarrow{(H_\theta \times 1)_*} \widetilde{\mathrm{CH}}^{d_X}_{(H_\theta \times 1)(\Gamma_{p_X})}(\mathbf{A}^1 \times U \times X, \omega_X).$$

Lemma 8.6. *The morphism $H_\theta \times 1$ maps $\Gamma_{p_X} \cong \mathscr{X}$ isomorphically onto its image. Let $\mathscr{H}_0 := \mathscr{H}_\theta \circ i_0$ and $\mathscr{H}_1 := \mathscr{H}_\theta \circ i_1$. Identifying $\mathscr{X}$ with its image in $\mathbf{A}^1 \times U \times X$, we then have* $\operatorname{supp} \mathscr{H}_\theta = \mathscr{X}$, $\operatorname{supp} \mathscr{H}_0 = \mathscr{D}_0$, *and* $\operatorname{supp} \mathscr{H}_1 = \mathscr{D}_1$.

Proof. If $y = ((x, u), x), y' = ((x', u'), x') \in \Gamma_{p_X}$ is such that

$$(H_\theta \times 1)(y) = (h_\theta(x, u), u, x) = (H_\theta \times 1)(y') = (h_\theta(x', u'), x', u'),$$

it follows that $x = x'$ and $u = u'$, hence $y = y'$. Thus we can consider $\mathscr{X}$ as a subscheme of $\mathbf{A}^1 \times U \times X$ by $(x, u) \mapsto (h_\theta(x, u), u, x)$. Now, the MW-correspondence p_X is supported on Γ_{p_X}, hence $\operatorname{supp} \mathscr{H}_\theta = (H_\theta \times 1)(\Gamma_{p_X}) \cong \mathscr{X}$. We turn to the restrictions $\mathscr{H}_0$ and $\mathscr{H}_1$ of the homotopy $\mathscr{H}_\theta$. By [CF14, Example 4.13] we have $\mathscr{H}_\theta \circ i_\epsilon = (i_\epsilon \times 1)^*(\mathscr{H}_\theta)$, where $\epsilon = 0, 1$. It follows that $\operatorname{supp} \mathscr{H}_\epsilon = (i_\epsilon \times 1)^{-1}((H_\theta \times 1)(\Gamma_{p_X}))$, and this closed subset is determined by those points $(x, u) \in \mathscr{X}$ satisfying $h_\theta(x, u) = \epsilon$. In other words, $\operatorname{supp} \mathscr{H}_\epsilon = \mathscr{D}_\epsilon$. ☐

Lemma 8.7. *The finite MW-correspondence $\mathscr{H}_\theta$ is a homotopy from $\mathscr{H}_0 = \mathrm{can} + j \circ \Phi_0'$ to $\mathscr{H}_1 = j \circ \Phi_1$, where $\Phi_0', \Phi_1 \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$.*

Proof. By Lemmas 8.4 and 8.6 we have $\operatorname{supp} \mathscr{H}_0 = \Delta(U) \amalg \mathscr{D}_0'$, where $\mathscr{D}_0' \subseteq \mathscr{X} \setminus \mathscr{Z}$. By Lemma 2.4 we may therefore write $\mathscr{H}_0 = \alpha + \beta$ where $\alpha \in \widetilde{\mathrm{Cor}}_k(U, X)$ is supported on $\Delta(U)$ and $\beta \in \widetilde{\mathrm{Cor}}_k(U, X)$ is supported on $\mathscr{D}_0'$. Since $\operatorname{supp} \beta = \mathscr{D}_0' \subseteq \mathscr{X} \setminus \mathscr{Z}$, Corollary 2.7 assures that there exists a unique finite MW-correspondence $\Phi_0' \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$ such that $j \circ \Phi_0' = \beta$. Hence $\mathscr{H}_0$ is of the form $\mathscr{H}_0 = \alpha + j \circ \Phi_0'$ for $\Phi_0' \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$. The same reasoning shows that, since $\operatorname{supp} \mathscr{H}_1 = \mathscr{D}_1 \subseteq \mathscr{X} \setminus \mathscr{Z}$, there is a unique MW-correspondence $\Phi_1 \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$ such that $\mathscr{H}_1 = j \circ \Phi_1$.

It therefore only remains to understand the finite MW-correspondence $\alpha \in \widetilde{\mathrm{CH}}^{d_X}_{\Delta(U)}(U \times X, \omega_X)$. Recall that, by definition,

$$\mathscr{H}_0 = (i_0 \times 1)^*(H_\theta \times 1)_*(\Gamma_{p_X})_*(\langle 1 \rangle).$$

Let $i_{\Delta(U)}$ and $i_{\mathscr{D}_0}$ denote the respective inclusions $i_{\Delta(U)}\colon \Delta(U) \subseteq \mathscr{X}$ and $i_{\mathscr{D}_0}\colon \mathscr{D}_0 \subseteq \mathscr{X}$. The base change formula [CF14, Proposition 3.2] applied to the pullback square

$$\begin{array}{ccc} (\Delta(U) \amalg \mathscr{D}_0') \times X & \xrightarrow{i_{\mathscr{D}_0}\times 1} & \mathscr{X} \times X \\ {\scriptstyle H_\theta|_{\mathscr{D}_0}\times 1}\big\downarrow & & \big\downarrow{\scriptstyle H_\theta \times 1} \\ U \times X & \xrightarrow{i_0 \times 1} & \mathbf{A}^1 \times U \times X \end{array}$$

reveals that $\alpha = (H_\theta|_{\Delta(U)} \times 1)_*(i_{\Delta(U)} \times 1)^*(\Gamma_{p_X})_*(\langle 1\rangle)$. Using that $\Delta\colon U \to \mathscr{X}$ is an isomorphism onto its image and that $H_\theta|_{\Delta(U)}\colon \Delta(U) \to U$ is an isomorphism, we may write $\alpha = ((\Delta \times 1)^*(\Gamma_{p_X})_*(\langle 1\rangle)$. Next, consider the pullback diagram

$$\begin{array}{ccc} U & \xrightarrow{\Delta} & \mathscr{X} \\ {\scriptstyle \Gamma_{\mathrm{can}}}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma_{p_X}} \\ U \times X & \xrightarrow{\Delta \times 1} & \mathscr{X} \times X. \end{array}$$

Using base change once more, we obtain $\alpha = (\Gamma_{\mathrm{can}})_*(\langle 1\rangle) = \widetilde{\gamma}_{\mathrm{can}}$. □

Remark 8.8. The homotopy $\mathscr{H}_\theta$ does indeed depend on the choice of trivialization of $\omega_{H_\theta \times 1}$. However, in the case of the finite MW-correspondence α in the proof of Lemma 8.7 above, recall that $\alpha = (H_\theta|_{\Delta(U)} \times 1)_*(i_{\Delta(U)} \times 1)^*(\Gamma_{p_X})_*(\langle 1\rangle)$. Now, the relative bundle of $H_\theta|_{\Delta(U)}\colon \Delta(U) \to U$ is *canonically* trivial, so the pushforward $(H_\theta|_{\Delta(U)} \times 1)_*$ does not depend on the choice of trivialization. Any other choice of a trivialization of $\omega_{H_\theta \times 1}$ yielding another homotopy $\mathscr{H}_\theta'$ would then satisfy $\mathscr{H}_0' = \mathrm{can} + j \circ \Phi_0''$ for some $\Phi_0'' \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$, which is sufficient for our purposes.

Proof of Theorem 8.1. In the notation of Lemma 8.7, let $\Phi := \Phi_1 - \Phi_0' \in \widetilde{\mathrm{Cor}}_k(U, X \setminus Z)$. As $\mathscr{H}_\theta$ provides a homotopy from $\mathrm{can} + j \circ \Phi_0'$ to $j \circ \Phi_1$, it follows that $\mathrm{can} \sim_{\mathbf{A}^1} j \circ (\Phi_1 - \Phi_0') = j \circ \Phi$. □

9. Nisnevich excision

The setting of this section is as follows. Suppose that $X, X' \in \mathrm{Sm}_k$ are smooth affine k-schemes such that there is an elementary distinguished Nisnevich square

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow{\scriptstyle \Pi} \\ V & \longrightarrow & X. \end{array}$$

Define the closed subschemes $S := (X \setminus V)_{\mathrm{red}} \subseteq X$ and $S' := (X' \setminus V')_{\mathrm{red}} \subseteq X'$. Let $x \in S$ and $x' \in S'$ be two points satisfying $\Pi(x') = x$. Moreover, we set $U := \operatorname{Spec} \mathcal{O}_{X,x}$ and $U' := \operatorname{Spec} \mathcal{O}_{X',x'}$. Let $\mathrm{can}\colon U \to X$ and $\mathrm{can}'\colon U' \to X'$ be the canonical inclusions and let $\pi := \Pi|_{U'}\colon U' \to U$. We can summarize the situation with the following diagram:

$$\begin{array}{ccccc} V' & \longrightarrow & X' & \xleftarrow{\mathrm{can}'} & U' \\ \big\downarrow & & \big\downarrow{\scriptstyle \Pi} & & \big\downarrow{\scriptstyle \pi} \\ V & \longrightarrow & X & \xleftarrow[\mathrm{can}]{} & U. \end{array}$$

The main result of this section is the following excision theorem for Nisnevich squares.

Theorem 9.1 (Nisnevich excision)**.** *Let $\mathscr{F}$ be a homotopy invariant presheaf on $\widetilde{\mathrm{Cor}}_k$. Then for any elementary distinguished Nisnevich square as above, the induced morphism*

$$\pi^*\colon \frac{\mathscr{F}(U \setminus S)}{\mathscr{F}(U)} \to \frac{\mathscr{F}(U' \setminus S')}{\mathscr{F}(U')}$$

is an isomorphism.

The proof of Theorem 9.1 relies on the two following results, establishing respectively injectivity and surjectivity of π^*:

Theorem 9.2 (Injectivity of Nisnevich excision)**.** *With the notations above, there exist finite MW-correspondences* $\Phi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X', X' \setminus S'))$ *and* $\Theta \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X \setminus S, X \setminus S))$ *such that*

$$\overline{\Pi} \circ \overline{\Phi} - \overline{j}_X \circ \overline{\Theta} = \overline{\mathrm{can}}$$

in $h\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X, X \setminus S))$*. Here* $j_X \colon (X \setminus S, X \setminus S) \to (X, X \setminus S)$ *is the inclusion.*

Theorem 9.3 (Surjectivity of Nisnevich excision)**.** *With the notations above, assume that* S *is smooth over* k*. Then there exist finite MW-correspondences* $\Psi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X', X' \setminus S'))$ *and* $\Xi \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U', U' \setminus S'), (X' \setminus S', X' \setminus S'))$ *such that*

$$\overline{\Psi} \circ \overline{\pi} - \overline{j}_{X'} \circ \overline{\Xi} = \overline{\mathrm{can}'}$$

in $h\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U', U' \setminus S'), (X', X' \setminus S'))$*. Here* $j_{X'} \colon (X' \setminus S', X' \setminus S') \to (X', X' \setminus S')$ *is the inclusion.*

Assuming Theorems 9.2 and 9.3, Theorem 9.1 now follows:

Proof of Theorem 9.1. Let $\mathscr{F}$ be a homotopy invariant presheaf with MW-transfers. As the correspondence Θ of Theorem 9.2 has image contained in $X \setminus S$, $j_X \circ \Theta$ induces the trivial map

$$(j_X \circ \Theta)^* = 0 \colon \frac{\mathscr{F}(X \setminus S)}{\mathscr{F}(X)} \to \frac{\mathscr{F}(U \setminus S)}{\mathscr{F}(U)}.$$

Hence $\Phi^* \circ \Pi^* = \mathrm{can}^*$. Similarly, $\Xi^* = 0$ and hence $\pi^* \circ \Psi^* = (\mathrm{can}')^*$. □

We proceed to prove Theorems 9.2 and 9.3.

10. Injectivity of Nisnevich excision

In this section we aim to prove Theorem 9.2. As preparation, we need to perform a shrinking process similar to that in Section 8. By [GP15, Lemma 8.4], there is an open subscheme $X^\circ \subseteq X$ along with an almost elementary fibration $q \colon X^\circ \to B$ such that $\omega_{B/k} \cong \mathcal{O}_B$ and $\omega_{X^\circ/k} \cong \mathcal{O}_{X^\circ}$. By [GP15, §8] we may replace X by X° and X' by $\Pi^{-1}(X^\circ)$. We regard X' as a B-scheme via the map $q \circ \Pi$.

Let Δ denote the morphism $\Delta := (\mathrm{id}, \mathrm{can}) \colon U \to U \times_B X$, and let p_X and $p_{\mathbf{A}^1 \times U}$ denote the projections from $\mathbf{A}^1 \times U \times_B X$ onto X respectively $\mathbf{A}^1 \times U$.

Proposition 10.1 ([GP15, Proposition 8.9])**.** *Let* θ *be the coordinate of* $\mathbf{A}^1$*. There exists a function* $h_\theta \in k[\mathbf{A}^1 \times U \times_B X]$ *such that the following properties hold for the functions* h_θ*,* $h_0 := h_\theta|_{0 \times U \times_B X}$ *and* $h_1 := h_\theta|_{1 \times U \times_B X}$*:*

- (*a*) *The morphism* $H_\theta := (p_{\mathbf{A}^1 \times U}, h_\theta) \colon \mathbf{A}^1 \times U \times_B X \to \mathbf{A}^1 \times U \times \mathbf{A}^1$ *is finite and surjective. Letting* $Z_\theta := h_\theta^{-1}(0) \subseteq \mathbf{A}^1 \times U \times_B X$*, it follows that* Z_θ *is finite, surjective and flat over* $\mathbf{A}^1 \times U$*.*
- (*b*) *Let* $Z_0 := h_0^{-1}(0) \subseteq U \times_B X$*. Then there is the equality of schemes* $Z_0 = \Delta(U) \amalg G$*, where* $G \subseteq U \times_B (X \setminus S)$*.*
- (*c*) *The closed subscheme* $\mathcal{V}((\mathrm{id}_U \times \Pi)^*(h_1)) \subseteq U \times_B X'$ *is a disjoint union of two closed subschemes* $Z'_1 \amalg Z'_2$*. Moreover, the map* $(\mathrm{id}_U \times \Pi)|_{Z'_1}$ *identifies* Z'_1 *with* $Z_1 := h_1^{-1}(0)$*.*

$$\begin{array}{ccccc} U \times_B X' & \xrightarrow{1 \times \Pi} & U \times_B X & \xrightarrow{h_1} & \mathbf{A}^1 \\ \uparrow & & \uparrow & & \\ Z'_1 & \xrightarrow{\cong} & Z_1 = \mathcal{V}(h_1) & & \end{array}$$

(*d*) *We have* $Z_\theta \cap (\mathbf{A}^1 \times (U \setminus x) \times_B X) \subseteq \mathbf{A}^1 \times (U \setminus x) \times_B (X \setminus x)$.

Corollary 10.2 ([GP15, Remark 8.10])**.** *We have the following inclusions:*

(1) $Z_\theta \cap (\mathbf{A}^1 \times (U \setminus S) \times_B X) \subseteq \mathbf{A}^1 \times (U \setminus S) \times_B X \setminus S$.
(2) $Z_0 \cap ((U \setminus S) \times_B X) \subseteq (U \setminus S) \times_B (X \setminus S)$.
(3) $Z_1 \cap ((U \setminus S) \times_B X) \subseteq (U \setminus S) \times_B (X \setminus S)$.
(4) $Z_1' \cap ((U \setminus S) \times_B X') \subseteq (U \setminus S) \times_B (X' \setminus S')$.

Definition 10.3. Choose a trivialization χ of $\omega_{H_\theta \times 1}$. We define $\mathscr{H}_\theta \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times U, X)$ as the image of the projection $p_X \in \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathbf{A}^1 \times U \times_B X \times X, \omega_X)$ under the composition

$$\begin{array}{ccc} \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathbf{A}^1 \times U \times_B X \times X, \omega_X) & \xrightarrow{\cong} & \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathbf{A}^1 \times U \times_B X \times X, \omega_{H_\theta \times 1} \otimes \omega_X) \\ & \swarrow_{(H_\theta \times 1)_*} & \\ \widetilde{\mathrm{CH}}^{d_X}_{(H_\theta \times 1)(\Gamma_{p_X})}(\mathbf{A}^1 \times U \times \mathbf{A}^1 \times X, \omega_X) & \xrightarrow{(1 \times i_0 \times 1)^*} & \widetilde{\mathrm{CH}}^{d_X}_{T}(\mathbf{A}^1 \times U \times X, \omega_X), \end{array}$$

where $d_X := \dim X$, $T := (1 \times i_0 \times 1)^{-1}((H_\theta \times 1)(\Gamma_{p_X}))$, and where the first isomorphism is induced by χ.

Lemma 10.4. *The finite MW-correspondence* $\mathscr{H}_\theta$ *is supported on* Z_θ*. Moreover, for* $\epsilon = 0, 1$ *we have* $\operatorname{supp} \mathscr{H}_\epsilon = Z_\epsilon$ *(where* $\mathscr{H}_\epsilon := \mathscr{H}_\theta \circ i_\epsilon$*).*

Proof. Let T denote the support of $\mathscr{H}_\theta$. As indicated in Definition 10.3 we have $T = (1 \times i_0 \times 1)^{-1}((H_\theta \times 1)(\Gamma_{p_X}))$. By the same argument as in Lemma 8.6, $(H_\theta \times 1)$ injects Γ_{p_X} onto its image, hence $(H_\theta \times 1)(\Gamma_{p_X}) \cong \mathbf{A}^1 \times U \times_B X$. Thus T consists of those points $(t, u, x) \in \mathbf{A}^1 \times U \times_B X$ such that $h_\theta(t, u, x) = 0$, i.e., $T = Z_\theta$.

Turning to the support of $\mathscr{H}_\epsilon$, note that $\mathscr{H}_\epsilon$ is the image of p_X under the composition

$$\widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_X}}(\mathbf{A}^1 \times U \times_B X \times X, \omega_X) \xrightarrow{(i_\epsilon \times 1)^* \circ (1 \times i_0 \times 1)^* \circ (H_\theta \times 1)_*} \widetilde{\mathrm{CH}}^{d_X}_{\operatorname{supp} \mathscr{H}_\epsilon}(\epsilon \times U \times X, \omega_X).$$

By the same reasoning as above, pulling back along $i_\epsilon \times 1$ amounts to substituting $\theta = \epsilon$ in h_θ, which yields the desired result. □

Lemma 10.5. *There are finite MW-correspondences* $\Phi \in \widetilde{\mathrm{Cor}}_k(U, X')$ *and* $\Theta \in \widetilde{\mathrm{Cor}}_k(U, X \setminus S)$ *such that* $\mathscr{H}_0 = \mathrm{can} + j_X \circ \Theta$ *and* $\mathscr{H}_1 = \Pi \circ \Phi$.

Proof. By Proposition 10.1 (*b*), we can write $\mathscr{H}_0 = \alpha + \Theta'$, where $\Theta' \in \widetilde{\mathrm{Cor}}_k(U, X)$ is supported on G and $\alpha \in \widetilde{\mathrm{Cor}}_k(U, X)$ is supported on $\Delta(U)$. Using Proposition 10.1 (*b*), Lemma 2.6 assures that there is a unique finite MW-correspondence $\Theta \in \widetilde{\mathrm{Cor}}_k(U, X \setminus S)$ such that $\Theta' = j_X \circ \Theta$. We proceed similarly for $\mathscr{H}_1$: By Proposition 10.1 (*c*), the pullback

$$(1 \times \Pi)^*(\mathscr{H}_1) \in \widetilde{\mathrm{CH}}^{d_X}_{(1 \times \Pi)^{-1}(Z_1)}(U \times X', \omega_{X'})$$

is supported on $Z_1' \amalg Z_2'$, and $(1 \times \Pi)|_{Z_1'}$ is an isomorphism from Z_1' onto Z_1. It follows that we have an isomorphism

$$(1 \times \Pi)_* \colon \widetilde{\mathrm{CH}}^{d_X}_{Z_1'}(U \times X', \omega_{X'}) \xrightarrow{\cong} \widetilde{\mathrm{CH}}^{d_X}_{Z_1}(U \times X, \omega_X).$$

Hence $\Phi := (1 \times \Pi)_*^{-1}(\mathscr{H}_1) = (1 \times \Pi)^*(\mathscr{H}_1) \in \widetilde{\mathrm{Cor}}_k(U, X')$ satisfies $\Pi \circ \Phi = \mathscr{H}_1$.

It remains to show that $\alpha = \mathrm{can}$, the proof of which being similar as in the proof of Lemma 8.7. As

$$(1 \times i_0 \times 1) \circ (i_0 \times 1) = (i_0 \times 1 \times i_0 \times 1) \colon U \times X \to \mathbf{A}^1 \times U \times \mathbf{A}^1 \times X,$$

we can write $\mathscr{H}_0 = (i_0 \times 1 \times i_0 \times 1)^*(H_\theta \times 1)_*(\Gamma_{p_X})_*(\langle 1\rangle)$. Using the base change formula twice as in Lemma 8.7, we find that

$$\alpha = (H_\theta|_{\Delta(U)} \times 1)_*(i_{\Delta(U)} \times 1)^*(\Gamma_{p_X})_*(\langle 1\rangle) = (\Gamma_{\mathrm{can}})_*(\langle 1\rangle) = \widetilde{\gamma}_{\mathrm{can}},$$

where $i_{\Delta(U)}\colon \Delta(U) \to U \times_B X$ is the inclusion. □

Lemma 10.6. *Let $j_U\colon U \setminus S \to U$, $j_X\colon X \setminus S \to X$ and $j_{X'}\colon X' \setminus S' \to X'$ denote the inclusions, and set:*

$$\mathscr{H}_\theta^S := (1 \times j_U \times j_X)^*(\mathscr{H}_\theta) \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times (U \setminus S), X \setminus S).$$
$$\Phi^S := (j_U \times j_{X'})^*(\Phi) \in \widetilde{\mathrm{Cor}}_k(U \setminus S, X' \setminus S').$$
$$\Theta^S := (j_U \times 1)^*(\Theta) \in \widetilde{\mathrm{Cor}}_k(U \setminus S, X \setminus S).$$

Then we have:

$$(\mathscr{H}_\theta, \mathscr{H}_\theta^S) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}(\mathbf{A}^1 \times (U, U \setminus S), (X, X \setminus S)).$$
$$(\Phi, \Phi^S) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X', X' \setminus S')).$$
$$(\Theta, \Theta^S) \in \widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}((U, U \setminus S), (X \setminus S, X \setminus S)).$$

Proof. In light of Corollary 10.2, this follows from Lemma 2.8. □

Proof of Theorem 9.2. Combining Lemmas 10.5 and 10.6 we obtain the identity

$$\overline{\Pi} \circ \overline{\Phi} - \overline{j}_X \circ \overline{\Theta} = \overline{\mathrm{can}}$$

in $h\widetilde{\mathrm{Cor}}_k^{\mathrm{pr}}$. □

11. Surjectivity of Nisnevich excision

We proceed to prove Theorem 9.3. In this section, the closed subscheme $S \subseteq X$ is assumed to be smooth over k. Performing a similar shrinking process as in Section 10, we may assume that there is an almost elementary fibration $q\colon X \to B$ such that $\omega_{B/k} \cong \mathcal{O}_B$ and $\omega_{X/k} \cong \mathcal{O}_X$. Since Π is étale, it follows that $\omega_{X'/k} \cong \mathcal{O}_{X'}$.

Let $\Delta' := (\mathrm{id}, \mathrm{can}')\colon U' \to U' \times_B X'$, and let $p_{X'}$ and $p_{\mathbf{A}^1 \times U'}$ denote the projections from $\mathbf{A}^1 \times U' \times_B X'$ to X' respectively $\mathbf{A}^1 \times U'$. First we recall the following fact from [GP15]:

Proposition 11.1 ([GP15, Proposition 10.5])**.** *Let $\mathbf{A}^1$ have coordinate θ. There exist functions $F \in k[U \times X']$ and $h'_\theta \in k[\mathbf{A}^1 \times U' \times_B X']$ such that the following properties hold for the functions F, h'_θ, $h'_0 := h'_\theta|_{0 \times U' \times_B X'}$, and $h'_1 := h'_\theta|_{1 \times U' \times_B X'}$:*

- (*a*) *The morphism $H'_\theta := (p_{\mathbf{A}^1 \times U'}, h'_\theta)\colon \mathbf{A}^1 \times U' \times_B X' \to \mathbf{A}^1 \times U' \times \mathbf{A}^1$ is finite and surjective. Letting $Z'_\theta := (h'_\theta)^{-1}(0) \subseteq \mathbf{A}^1 \times U' \times_B X'$, it follows that Z'_θ is finite, surjective and flat over $\mathbf{A}^1 \times U'$.*
- (*b*) *Let $Z'_0 := (h'_0)^{-1}(0)$. Then there is the equality of schemes $Z'_0 = \Delta'(U') \amalg G'$, where $G' \subseteq U' \times_B (X' \setminus S')$.*
- (*c*) *$h'_1 = (\pi \times \mathrm{id}_{X'})^*(F)$. We write $Z'_1 := (h'_1)^{-1}(0)$.*
- (*d*) *$Z'_\theta \cap (\mathbf{A}^1 \times (U' \setminus S') \times_B X') \subseteq \mathbf{A}^1 \times (U' \setminus S') \times_B (X' \setminus S')$.*
- (*e*) *The morphism $(\mathrm{pr}_U, F)\colon U \times X' \to U \times \mathbf{A}^1$ is finite and surjective. Letting $Z_1 := F^{-1}(0)$, it follows that Z_1 is finite and surjective over U.*
- (*f*) *$Z_1 \cap ((U \setminus S) \times X') \subseteq (U \setminus S) \times (X' \setminus S')$.*

Corollary 11.2 ([GP15, Remark 10.6])**.** *We have the following inclusions:*

1. *$Z'_\theta \cap (\mathbf{A}^1 \times (U' \setminus S') \times_B X') \subseteq \mathbf{A}^1 \times (U' \setminus S') \times_B (X' \setminus S')$.*
2. *$Z'_0 \cap ((U' \setminus S') \times_B X') \subseteq (U' \setminus S') \times_B (X' \setminus S')$.*

(3) $Z_1' \cap ((U' \setminus S') \times_B X') \subseteq (U' \setminus S') \times_B (X' \setminus S')$.
(4) $Z_1 \cap ((U \setminus S) \times X') \subseteq (U \setminus S) \times (X' \setminus S')$.

Definition 11.3. Choose a trivialization χ' of $\omega_{H'_\theta \times 1}$. We define $\mathscr{H}'_\theta \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times U', X')$ as the image of the projection $p_{X'} \in \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_{X'}}}(\mathbf{A}^1 \times U' \times_B X' \times X', \omega_{X'})$ under the composition

$$\widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_{X'}}}(\mathbf{A}^1 \times U' \times_B X' \times X', \omega_{X'}) \xrightarrow{\cong} \widetilde{\mathrm{CH}}^{d_X}_{\Gamma_{p_{X'}}}(\mathbf{A}^1 \times U' \times_B X' \times X', \omega_{H'_\theta \times 1} \otimes \omega_{X'})$$
$$\xrightarrow{(H'_\theta \times 1)_*} \widetilde{\mathrm{CH}}^{d_X}_{(H'_\theta \times 1)(\Gamma_{p_{X'}})}(\mathbf{A}^1 \times U' \times \mathbf{A}^1 \times X', \omega_{X'}) \xrightarrow{(1 \times i_0 \times 1)^*} \widetilde{\mathrm{CH}}^{d_X}_{T'}(\mathbf{A}^1 \times U' \times X', \omega_{X'}),$$

where $T' := (1 \times i_0 \times 1)^{-1}((H'_\theta \times 1)(\Gamma_{p_{X'}}))$, and where the first isomorphism is induced by χ'.

The same argument as in Lemma 10.4 readily yields:

Lemma 11.4. *The finite MW-correspondence $\mathscr{H}'_\theta$ is supported on Z'_θ. Moreover, for $\epsilon = 0, 1$ we have* $\operatorname{supp} \mathscr{H}'_\epsilon = Z'_\epsilon$ *(where, as usual, $\mathscr{H}'_\epsilon := \mathscr{H}'_\theta \circ i_\epsilon$).*

Lemma 11.5. *There are finite MW-correspondences $\Psi \in \widetilde{\mathrm{Cor}}_k(U, X')$ and $\Xi \in \widetilde{\mathrm{Cor}}_k(U', X' \setminus S')$ such that $\mathscr{H}'_0 = \mathrm{can}' + j_{X'} \circ \Xi$ and $\mathscr{H}'_1 = \Psi \circ \pi$.*

Proof. The claim about $\mathscr{H}'_0$ follows from an identical argument as in the proof of Lemma 10.5 by using Proposition 11.1 (b), so let us turn our attention to $\mathscr{H}'_1$. By Proposition 11.1 (c), the morphism $\pi \times 1$ identifies Z'_1 with Z_1. By étale excision [CF14, Lemma 3.5], $\pi \times 1$ induces an isomorphism

$$(\pi \times 1)^* \colon \widetilde{\mathrm{CH}}^{d_X}_{Z_1}(U \times X', \omega_{X'}) \xrightarrow{\cong} \widetilde{\mathrm{CH}}^{d_X}_{Z'_1}(U' \times X', \omega_{X'}).$$

Arguing similarly to the proof of Lemma 10.5, it follows that there exists a unique element $\Psi \in \widetilde{\mathrm{CH}}^{d_X}_{Z'_1}(U' \times X', \omega_{X'}) \subseteq \widetilde{\mathrm{Cor}}_k(U', X')$ such that $\mathscr{H}'_1 = \Psi \circ \pi$. □

Finally we check that the finite MW-correspondences constructed above are in fact morphisms of pairs:

Lemma 11.6. *Let $j_{U'} \colon U' \setminus S' \to U'$ denote the inclusion, and define:*

$$(\mathscr{H}'_\theta)^{S'} := (1 \times j_{U'} \times j_{X'})^*(\mathscr{H}'_\theta) \in \widetilde{\mathrm{Cor}}_k(\mathbf{A}^1 \times (U' \setminus S'), X' \setminus S').$$
$$\Psi^{S'} := (j_U \times j_{X'})^*(\Psi) \in \widetilde{\mathrm{Cor}}_k(U \setminus S, X' \setminus S').$$
$$\Xi^{S'} := (j_{U'} \times 1)^*(\Xi) \in \widetilde{\mathrm{Cor}}_k(U' \setminus S', X' \setminus S').$$

Then

$$(\mathscr{H}'_\theta, (\mathscr{H}'_\theta)^{S'}) \in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k(\mathbf{A}^1 \times (U', U' \setminus S'), (X', X' \setminus S')).$$
$$(\Psi, \Psi^{S'}) \in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k((U, U \setminus S), (X', X' \setminus S')).$$
$$(\Xi, \Xi^{S'}) \in \widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k((U', U' \setminus S'), (X' \setminus S', X' \setminus S')).$$

Proof. By Corollary 11.2, the supports of the given MW-correspondences satisfy the hypothesis of Lemma 2.8. □

Proof of Theorem 9.3. Lemma 11.5 and Lemma 11.6 yields the desired equality

$$\overline{\Psi} \circ \overline{\pi} - \overline{j}_{X'} \circ \overline{\Xi} = \overline{\mathrm{can}'}$$

in $h\widetilde{\mathrm{Cor}}^{\mathrm{pr}}_k$. □

12. Homotopy invariance

In this section we show, following [GP15, Proof of Theorem 2.1] and [Dru14], how homotopy invariance of the sheaves $\mathscr{F}_{\mathrm{Zar}}$ and $\mathscr{F}_{\mathrm{Nis}}$ follows from the excision theorems along with injectivity for local schemes. Throughout this section, $\mathscr{F}$ will denote a homotopy invariant presheaf with MW-transfers and $X \in \mathrm{Sm}_k$ will denote a smooth irreducible k-scheme with generic point $\eta\colon \operatorname{Spec} k(X) \to X$.

Homotopy invariance of $\mathscr{F}_{\mathrm{Zar}}$. Below we will use Zariski excision along with injectivity for local schemes to show homotopy invariance of the Zariski sheaf $\mathscr{F}_{\mathrm{Zar}}$ associated to $\mathscr{F}$. Let $x \in X$ be a closed point of X. We write $\mathscr{F}(\operatorname{Spec} \mathcal{O}_{X,x})$ or $\mathscr{F}(\mathcal{O}_{X,x})$ for the stalk $\mathscr{F}_x$ of $\mathscr{F}$ at x in the Zariski topology.

Lemma 12.1. *The natural map $\mathscr{F}(\mathcal{O}_{X,x}) \to \mathscr{F}(k(X))$ is injective.*

Proof. For $U := \operatorname{Spec} \mathcal{O}_{X,x}$ we have $\mathscr{F}(U) = \operatorname{colim}_{V \ni x} \mathscr{F}(V)$, and $\mathscr{F}(k(X)) = \operatorname{colim}_{W \neq \varnothing} \mathscr{F}(W)$. Let $s_x \in \mathscr{F}(\mathcal{O}_{X,x})$ be a germ mapping to 0 in $\mathscr{F}(k(X))$. This means that there is some nonempty open $W \subseteq X$ such that $s|_W = 0$. If $x \in W$ then $s_x = 0$ in $\mathscr{F}(\mathcal{O}_{X,x})$ and we are done. So suppose that $x \notin W$, and let Z denote the closed complement of W in X. Then $s|_{X \setminus Z} = 0$, and thus Corollary 8.2 applies, yielding $s_x = 0$ in $\mathscr{F}(\mathcal{O}_{X,x})$. □

Corollary 12.2. *The map $\eta^*\colon \mathscr{F}_{\mathrm{Zar}}(X) \to \mathscr{F}_{\mathrm{Zar}}(k(X))$ is injective.*

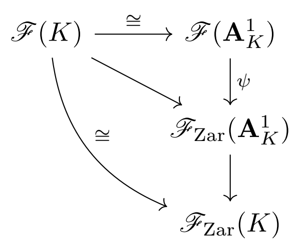
Proof. Suppose that $s \in \mathscr{F}_{\mathrm{Zar}}(X)$ maps to 0 in $\mathscr{F}_{\mathrm{Zar}}(\operatorname{Spec} k(X))$. By Lemma 12.1, the germs $s_x \in \mathscr{F}_x$ of s vanish at all closed points of X, which yields $s = 0$. □

Corollary 12.3. *For any nonempty open subscheme $i\colon V \subseteq X$, the map $i^*\colon \mathscr{F}_{\mathrm{Zar}}(X) \to \mathscr{F}_{\mathrm{Zar}}(V)$ is injective.*

Proof. We know that $k(X) = k(V)$, hence Corollary 12.2 assures that there are injections $\mathscr{F}_{\mathrm{Zar}}(X) \hookrightarrow \mathscr{F}(k(X))$ and $\mathscr{F}_{\mathrm{Zar}}(V) \hookrightarrow \mathscr{F}(k(X))$ induced by the generic point. Since $\mathscr{F}_{\mathrm{Zar}}(X) \hookrightarrow \mathscr{F}(k(X))$ factors through $\mathscr{F}_{\mathrm{Zar}}(V)$, the result follows. □

Lemma 12.4. *The sheafification map $\psi\colon \mathscr{F}(\mathbf{A}^1_{k(X)}) \to \mathscr{F}_{\mathrm{Zar}}(\mathbf{A}^1_{k(X)})$ is an isomorphism.*

Proof. Write $K := k(X)$ for the function field of X. Since stalks remain the same after sheafification, the commutative diagram



shows that ψ is injective. It remains to show surjectivity.

Let $s \in \mathscr{F}_{\mathrm{Zar}}(\mathbf{A}^1_K)$ be a section, mapping to the germ $s_\eta \in \mathscr{F}_{\mathrm{Zar}}(K) = \mathscr{F}(K)$ at the generic point η under the morphism $\mathscr{F}_{\mathrm{Zar}}(\mathbf{A}^1_K) \to \mathscr{F}(K)$. As $\mathscr{F}(K) = \operatorname{colim}_V \mathscr{F}(V)$, we can find a nonempty open $V \subseteq \mathbf{A}^1_K$ and a section $s' \in \mathscr{F}(V)$ such that $s|_V = s'|_V$ as sections of $\mathscr{F}_{\mathrm{Zar}}$. Thus the germs of s and s' coincide at any $v \in V$. The idea from here is to extend the section $s' \in \mathscr{F}(V)$ to a global section of the presheaf $\mathscr{F}$.

We may assume that $V = \mathbf{A}^1_K \setminus x$, where x is a closed point. Indeed, the general case follows by induction since V is then the complement of finitely many closed points. For $U_x := \operatorname{Spec} \mathcal{O}_{\mathbf{A}^1_K, x}$, the commutative diagram

$$\begin{array}{ccc} U_x \setminus x & \hookrightarrow & \mathbf{A}^1_K \setminus x \\ \downarrow & & \downarrow \\ U_x & \hookrightarrow & \mathbf{A}^1_K \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} \dfrac{\mathscr{F}(V)}{\mathscr{F}(\mathbf{A}^1_K)} & \xrightarrow{\cong} & \dfrac{\mathscr{F}(U_x \setminus x)}{\mathscr{F}(U_x)} \\ \uparrow & & \uparrow \\ \mathscr{F}(V) & \longrightarrow & \mathscr{F}(U_x \setminus x) = \mathscr{F}(K) \\ \uparrow & & \uparrow \\ \mathscr{F}(\mathbf{A}^1_K) & \longrightarrow & \mathscr{F}(U_x). \end{array}$$

Here the upper horizontal arrow is an isomorphism by Zariski excision. Moreover, note that $U_x \setminus x$ is the generic point, and hence that $\mathscr{F}(U_x \setminus x) = \mathscr{F}(K)$ is the stalk at the generic point of X. Thus we have the equality

$$\frac{\mathscr{F}(U_x \setminus x)}{\mathscr{F}(U_x)} = \frac{\mathscr{F}_{\mathrm{Zar}}(U_x \setminus x)}{\mathscr{F}_{\mathrm{Zar}}(U_x)}.$$

We want to lift $s' \in \mathscr{F}(V)$ to $\mathscr{F}(\mathbf{A}^1_K)$, which is possible if and only if s' maps to 0 in the cokernel of the map $\mathscr{F}(\mathbf{A}^1) \to \mathscr{F}(V)$. But $s' \mapsto s_\eta$ under the map $\mathscr{F}(V) \to \mathscr{F}(U_x \setminus x)$ by the choice of s'. Moreover, $s_\eta \in \mathscr{F}(K)$ is the image of the germ $s_x \in \mathscr{F}(U_x)$ of s at x. Hence s_η vanishes in $\mathscr{F}(U_x \setminus x)/\mathscr{F}(U_x)$. By the excision isomorphism we conclude that s' vanishes in $\mathscr{F}(V)/\mathscr{F}(\mathbf{A}^1_K)$, and hence that there is a section $s'' \in \mathscr{F}(\mathbf{A}^1_K)$ such that $s''|_V = s'$.

Finally, we need to check that $s'' \in \mathscr{F}(\mathbf{A}^1_K)$ maps to s under the morphism $\mathscr{F}(\mathbf{A}^1_K) \to \mathscr{F}_{\mathrm{Zar}}(\mathbf{A}^1_K)$. It suffices to show that the germs of s'' and s coincide at every point of $\mathbf{A}^1_K$. For the points $v \in V$ we know that $s''_v = s_v$ as $s|_V = s'|_V$ and $s''|_V = s'$. It remains to check the point x, i.e., that $s''_x = s_x$ in $\mathscr{F}(U_x)$. Consider the commutative diagram

$$\begin{array}{ccc} \mathscr{F}(V) & \longrightarrow & \mathscr{F}(U_x \setminus x) \\ \uparrow & & \uparrow \\ \mathscr{F}(\mathbf{A}^1_K) & \longrightarrow & \mathscr{F}(U_x). \end{array}$$

Under the composite $\mathscr{F}(\mathbf{A}^1_K) \to \mathscr{F}(V) \to \mathscr{F}(U_x \setminus x)$, $s'' \in \mathscr{F}(\mathbf{A}^1_K)$ maps to s' and to s'_η, while under the other composite, s'' maps to s''_x and to $s_\eta = s'_\eta$. But there is another section of $\mathscr{F}(U_x)$ mapping to $s_\eta \in \mathscr{F}(U_x \setminus x)$, namely s_x. By Lemma 12.1, the map $\mathscr{F}(U_x) \to \mathscr{F}(U_x \setminus x) = \mathscr{F}(K)$ is injective, hence $s''_x = s_x$. □

Theorem 12.5. *If $\mathscr{F} \in \widetilde{\mathrm{PSh}}(k)$ is a homotopy invariant presheaf with MW-transfers, then $\mathscr{F}_{\mathrm{Zar}}$ is homotopy invariant.*

Proof. Let i_0 be the zero section $i_0 \colon X \to X \times \mathbf{A}^1$. We then have $p \circ i_0 = \mathrm{id}_X$, where $p \colon X \times \mathbf{A}^1 \to X$ is the projection. Hence the induced map $i_0^* \colon \mathscr{F}_{\mathrm{Zar}}(X \times \mathbf{A}^1) \to \mathscr{F}_{\mathrm{Zar}}(X)$ is split surjective, and it remains to show that i_0^* is injective. Consider the commutative diagram

$$\begin{array}{ccc} \mathscr{F}_{\mathrm{Zar}}(X \times \mathbf{A}^1) & \xrightarrow{(\eta\times 1)^*} & \mathscr{F}_{\mathrm{Zar}}(\mathbf{A}^1_{k(X)}) \\ {\scriptstyle i_0^*}\downarrow & & \downarrow{\scriptstyle i_0^*} \\ \mathscr{F}_{\mathrm{Zar}}(X) & \xrightarrow{\eta^*} & \mathscr{F}_{\mathrm{Zar}}(k(X)). \end{array}$$

The map $\eta^*\colon \mathscr{F}_{\mathrm{Zar}}(X) \to \mathscr{F}_{\mathrm{Zar}}(k(X))$ is injective by Corollary 12.2. Moreover, we can write

$$\mathscr{F}_{\mathrm{Zar}}(\mathbf{A}^1_{k(X)}) = \operatorname*{colim}_{\varnothing\neq W\subseteq X} \mathscr{F}_{\mathrm{Zar}}(W \times \mathbf{A}^1).$$

Hence Corollary 12.3 tells us that the upper horizontal morphism $(\eta \times 1)^*$ is also injective. By homotopy invariance of $\mathscr{F}$ and Lemma 12.4 we have

$$\mathscr{F}_{\mathrm{Zar}}(\mathbf{A}^1_{k(X)}) \cong \mathscr{F}(\mathbf{A}^1_{k(X)}) \cong \mathscr{F}(k(X)) = \mathscr{F}_{\mathrm{Zar}}(k(X)).$$

Hence the right hand vertical map is an isomorphism. It follows that $i_0^*\colon \mathscr{F}_{\mathrm{Zar}}(X\times\mathbf{A}^1) \to \mathscr{F}_{\mathrm{Zar}}(X)$ is injective. □

Homotopy invariance of $\mathscr{F}_{\mathrm{Nis}}$. We proceed to prove homotopy invariance of the associated Nisnevich sheaf $\mathscr{F}_{\mathrm{Nis}}$, the proof being similar to the one for Zariski sheafification using Nisnevich excision. If A is a local ring, let A^h denote the henselization of A. We write $\mathscr{F}(\operatorname{Spec}\mathcal{O}^h_{X,x})$ or $\mathscr{F}(\mathcal{O}^h_{X,x})$ for the stalk of $\mathscr{F}$ at x in the Nisnevich topology. Thus $\mathscr{F}(\mathcal{O}^h_{X,x}) = \operatorname{colim}_V \mathscr{F}(V)$, where the colimit runs over the filtered system of Nisnevich neighborhoods of x.

Lemma 12.6. *For $U_x^h := \operatorname{Spec}\mathcal{O}^h_{X,x}$, the natural map $\mathscr{F}(U_x^h) \to \mathscr{F}(k(U_x^h))$ is injective.*

Proof. Suppose that $s \in \mathscr{F}(U_x^h)$ maps to 0 in $\mathscr{F}(k(U_x^h))$. This means that there is some Nisnevich neighborhood $p\colon W \to X$ such that $s|_W = 0$. Replacing W by its open image, we may assume that $W \subseteq X$. Let Z be the closed complement of W in X. If $x \in W$ then $s = 0$ in $\mathscr{F}(U_x^h)$; if not then $x \in Z$, and thus Corollary 8.2 shows that $s|_V = 0$ for some Zariski neighborhood V of x. Since V is also a Nisnevich neighboorhood, it follows that $s = 0$ in $\mathscr{F}(U_x^h)$. □

The next two lemmas follow from Lemma 12.6 similarly to the Zariski case.

Corollary 12.7. *The map $\eta^*\colon \mathscr{F}_{\mathrm{Nis}}(X) \to \mathscr{F}_{\mathrm{Nis}}(k(X))$ is injective.*

Corollary 12.8. *For any nonempty open subscheme $i\colon V \subseteq X$, the map $i^*\colon \mathscr{F}_{\mathrm{Nis}}(X) \to \mathscr{F}_{\mathrm{Nis}}(V)$ is injective.*

Lemma 12.9. *The sheafification map $\psi\colon \mathscr{F}(\mathbf{A}^1_{k(X)}) \to \mathscr{F}_{\mathrm{Nis}}(\mathbf{A}^1_{k(X)})$ is an isomorphism.*

Proof. Let $K := k(X)$. By the same reasoning as in the proof of Lemma 12.4, the map $\mathscr{F}(\mathbf{A}^1_K) \to \mathscr{F}_{\mathrm{Nis}}(\mathbf{A}^1_K)$ is injective, and it remains to show surjectivity. Let $s \in \mathscr{F}_{\mathrm{Nis}}(\mathbf{A}^1_K)$ be a section. Since the stalks $\mathscr{F}(K)$ and $\mathscr{F}_{\mathrm{Nis}}(K)$ coincide, there exists an open $V \subseteq \mathbf{A}^1_K$ and a section $s' \in \mathscr{F}(V)$ such that $s' = s|_V$ in $\mathscr{F}_{\mathrm{Nis}}(V)$. We wish to extend $s' \in \mathscr{F}(V)$ to a global section $s'' \in \mathscr{F}(\mathbf{A}^1_K)$. Considering one point at a time, we may assume that $V = \mathbf{A}^1_K \setminus x$ for some closed point x. Let $U_x := \operatorname{Spec}(\mathcal{O}_{\mathbf{A}^1_K,x})$ and $U_x^h := \operatorname{Spec}(\mathcal{O}^h_{\mathbf{A}^1_K,x})$. A lift of s' to a section in $\mathscr{F}(\mathbf{A}^1_K)$ exists if and only if s' maps to 0 in the quotient $\mathscr{F}(V)/\mathscr{F}(\mathbf{A}^1_K)$. Consider the chain of isomorphisms

$$\frac{\mathscr{F}(V)}{\mathscr{F}(\mathbf{A}^1_K)} \xrightarrow[\text{Zar. exc.}]{\cong} \frac{\mathscr{F}(U_x \setminus x)}{\mathscr{F}(U_x)} \xrightarrow[\text{Nis. exc.}]{\cong} \frac{\mathscr{F}(U_x^h \setminus x)}{\mathscr{F}(U_x^h)} \xrightarrow{\cong} \frac{\mathscr{F}_{\mathrm{Nis}}(U_x^h \setminus x)}{\mathscr{F}_{\mathrm{Nis}}(U_x^h)}.$$

The left and the middle map are isomorphisms by Zariski respectively Nisnevich excision, while the right hand map is an isomorphism since both $\mathscr{F}(U_x^h\setminus x)$ and $\mathscr{F}(U_x^h)$ are stalks in the Nisnevich topology. Thus it is enough to show that $s' \in \mathscr{F}(V)$ maps to 0 in $\mathscr{F}_{\mathrm{Nis}}(U_x^h \setminus x)/\mathscr{F}(U_x^h)$. But this follows from the commutative diagram

$$\begin{array}{ccc} \mathscr{F}_{\mathrm{Nis}}(V) & \longrightarrow & \mathscr{F}_{\mathrm{Nis}}(U_x^h \setminus x) \\ \uparrow & & \uparrow \\ \mathscr{F}_{\mathrm{Nis}}(\mathbf{A}^1_K) & \longrightarrow & \mathscr{F}_{\mathrm{Nis}}(U_x^h). \end{array}$$

Hence we can lift s' to $s'' \in \mathscr{F}(\mathbf{A}^1_K)$. It remains to check that s'' maps to $s \in \mathscr{F}_{\mathrm{Nis}}(\mathbf{A}^1_K)$. Knowing that $s''|_V = s|_V \in \mathscr{F}_{\mathrm{Nis}}(V)$, we must show that $s''_x = s_x \in \mathscr{F}_{\mathrm{Nis}}(U_x^h) = \mathscr{F}(U_x^h)$. As $\mathscr{F}(U_x^h)$ injects into $\mathscr{F}(U_x^h \setminus x) = \mathscr{F}(k(U_x^h))$ by Lemma 12.6, it is sufficient to prove the equality in the latter stalk. But this follows from the commutativity of the above diagram, using that both s and s'' map to $s|_V$ in $\mathscr{F}_{\mathrm{Nis}}(V)$. □

Theorem 12.10. *If $\mathscr{F}$ is a homotopy invariant presheaf on $\widetilde{\mathrm{Cor}}_k$, then $\mathscr{F}_{\mathrm{Nis}}$ is also homotopy invariant.*

Proof. We must show that the map $i_0^* \colon \mathscr{F}_{\mathrm{Nis}}(X \times \mathbf{A}^1) \to \mathscr{F}_{\mathrm{Nis}}(X)$ induced by the zero section is injective. Consider the commutative diagram

$$\begin{array}{ccc} \mathscr{F}_{\mathrm{Nis}}(X \times \mathbf{A}^1) & \xrightarrow{(\eta\times 1)^*} & \mathscr{F}_{\mathrm{Nis}}(\mathbf{A}^1_{k(X)}) \\ {\scriptstyle i_0^*}\downarrow & & \downarrow{\scriptstyle i_0^*} \\ \mathscr{F}_{\mathrm{Nis}}(X) & \xrightarrow{\eta^*} & \mathscr{F}_{\mathrm{Nis}}(k(X)). \end{array}$$

By Corollary 12.7, the lower horizontal map η^* is injective, while Corollary 12.8 assures that the upper horizontal map $(\eta \times 1)^*$ is injective. By homotopy invariance of $\mathscr{F}$, the right hand vertical map i_0^* is an isomorphism. Hence $i_0^* \colon \mathscr{F}_{\mathrm{Nis}}(X \times \mathbf{A}^1) \to \mathscr{F}_{\mathrm{Nis}}(X)$ is injective. □

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