MARKOV-DYCK SHIFTS, NEUTRAL PERIODIC POINTS AND TOPOLOGICAL CONJUGACY

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Abstract. We study the neutral periodic points of the Markov-Dyck shifts of finite strongly connected directed graphs. Under certain hypothesis on the structure of the graphs G we show, that the topological conjugacy of their Markov-Dyck shifts $M_D(G)$ implies the isomorphism of the graphs.

1. Introduction

Let Σ be a finite alphabet, and let S be the left shift on $\Sigma^{\mathbb{Z}}$,

$$S((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}, \qquad (x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}.$$

The closed shift-invariant subsystems of the shifts S are called subshifts. For an introduction to the theory of subshifts see [Ki] and [LM]. A finite word in the symbols of Σ is called admissible for the subshift $X \subset \Sigma^{\mathbb{Z}}$ if it appears somewhere in a point of X. A subshift $X \subset \Sigma^{\mathbb{Z}}$ is uniquely determined by its language $\mathcal{L}(X)$ of admissible words.

In this paper we study the topological conjugacy of subshifts that are constructed from finite directed graphs. We denote a finite directed graph G with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ by $G(\mathcal{V},\mathcal{E})$ (in the case of a tree we write $T(\mathcal{V},\mathcal{E})$). The source vertex of an edge $e \in \mathcal{E}$ or of a directed path in the graph we denote by s and its target vertex by t (or by s_G and t_G, in the case that we have to distinguish between graphs.) We consider strongly connected finite directed graphs $G = G(\mathcal{V},\mathcal{E})$. It is assumed tat G is not a cycle. We recall the construction of the Markov-Dyck shift $M_D(G)$ of G (see [M1]). Let $\mathcal{E}^- = \{e^- : e \in \mathcal{E}\}$ be a copy of $\mathcal{E}$. Reverse the directions of the edges in $\mathcal{E}^-$ to obtain the edge set $\mathcal{E}^+ = \{e^+ : e \in \mathcal{E}\}$ of the reversed graph of $G(\mathcal{V},\mathcal{E}^-)$. In this way one has defined a graph $\widetilde{G}(\mathcal{V},\mathcal{E}^- \cup \mathcal{E}^+)$, where

$$\begin{aligned} s_{\widetilde{G}}(e^-) &= s_G(e), \quad t_{\widetilde{G}}(e^-) = t_G(e), \\ s_{\widetilde{G}}(e^+) &= t_G(e), \quad t_{\widetilde{G}}(e^+) = s_G(e), \qquad e \in \mathcal{E}. \end{aligned}$$

With idempotents $\mathbf{1}_V, V \in \mathcal{V}$, the set $\mathcal{E}^- \cup \{\mathbf{1}_V : V \in \mathcal{V}\} \cup \mathcal{E}^+$ is the generating set of the graph inverse semigroup $\mathcal{S}(G)$ of G (see [L, Section 10.7]), where, besides $\mathbf{1}_V^2 = \mathbf{1}_V, V \in \mathcal{V}$, the relations are

$$\mathbf{1}_U\mathbf{1}_W = 0, \qquad U, W \in \mathcal{V}, U \neq W,$$

$$f^-g^+ = \begin{cases} \mathbf{1}_{s_G(f)}, & \text{if } f = g, \\ 0, & \text{if } f \neq g, \quad f, g \in \mathcal{E}, \end{cases}$$

$$\mathbf{1}_{s_G(f)}f^- = f^-\mathbf{1}_{t_G(f)}, \qquad \mathbf{1}_{t_G(f)}f^+ = f^+\mathbf{1}_{s_G(f)}, \qquad f \in \mathcal{E}.$$

The subsemigroup of the semigroup $\mathcal{S}(G)$, that is generated by $\mathcal{E}^-$ ($\mathcal{E}^+$) we denote by $\mathcal{S}^-(G)(\mathcal{S}^+(G))$, and we refer to the elements of $\mathcal{E}^-(\mathcal{E}^+)$ as the generators of $\mathcal{S}^-(G)(\mathcal{S}^+(G))$.

The alphabet of $M_D(G)$ is $\mathcal{E}^- \cup \mathcal{E}^+$, and a word $(e_k)_{1\leq k\leq K}$ is admissible for $M_D(G)$ precisely if

$$\prod_{1\leq k\leq K} e_k \neq 0.$$

The directed graphs with a single vertex and $N > 1$ loops yield the Dyck inverse monoids (the "polycycliques" of [NP]) $\mathcal{D}_N$ and the Dyck shifts D_N [Kr1].

Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we set

$$x_{[i,j]} = (x_k)_{i\leq k\leq j},$$

and

$$X_{[i,j]} = \{x_{[i,j]} : x \in X\}, \quad i, j \in \mathbb{Z}, i \leq j, \qquad x \in X,$$

and we use similar notation in the case that indices range in semi-infinite intervals. Set

$$\Gamma^+_X(a) = \{x^+ \in X_{(j,\infty)} : ax^+ \in X_{[i,\infty)}\}, \quad a \in X_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j.$$

The notation Γ^- has the symmetric meaning. Also set

$$\omega^+_X(a) = \bigcap_{x^-\in\Gamma^-(a)} \{x^+ \in \Gamma^+(a) : x^-ax^+ \in X\}, \quad a \in X_{[i,j]}, \quad i, j \in \mathbb{Z}, i \leq j.$$

The notation ω^- has the symmetric meaning. Also set

$$A_n(X) = \bigcap_{i\in\mathbb{Z}} (\{x \in X : x_{[i,\infty)} \in \omega^+_X(x_{[i-n,i)})\} \cap \{x \in X : x_{(-\infty,i]} \in \omega^-_X(x_{(i,i+n]})\}),$$

and

$$A(X) = \bigcup_{n\in\mathbb{N}} A_n(X).$$

The periodic points in $A(X)$ are called the neutral periodic points of X. In Section 2 we clarify the structure of the set of neutral periodic points of a Markov-Dyck shift. This includes a characterization of the neutral periodic points of the shift.

For a finite directed graph $G(\mathcal{V}, \mathcal{E})$ we denote by $\mathcal{F}_G$ the set of edges that are the single incoming edges of their target vertices and we denote by $\mathcal{R}_G$ the set of $V \in \mathcal{V}_G$ that have more than one incoming edge. The set $\mathcal{R}_G$ is the set of roots of a set of rooted trees. We denote the vertex set of the tree with root $R \in \mathcal{R}_G$ by $\mathcal{V}_R$, and its edge set by $\mathcal{F}_R$. One has

$$\mathcal{V} = \bigcup_{R\in\mathcal{R}_G} \mathcal{V}_R, \qquad \mathcal{F}_G = \bigcup_{R\in\mathcal{R}_G} \mathcal{F}_R.$$

If $\mathcal{V}_R$ contains more than one vertex, then we call the tree with root R a contracting subtree of G.

We denote the graph that is obtained by letting the contracting subtrees of G contract to their roots by $\widehat{G}(\mathcal{R}_G, \mathcal{E} \setminus \mathcal{F}_G)$. In the graph $\widehat{G}$ the source vertex of an edge $e \in \mathcal{E} \setminus \mathcal{F}_G$ is the root of the tree that has $s_G(e)$ as a leave, and its target vertex is $t_G(e)$. In [Kr2] a Property (A) of subshifts, an invariant of topological conjugacy, was introduced, and to a subshift X with Property (A) a semigroup $\mathcal{S}(X)$ was invariantly associated. In [HK] it was shown that Markov-Dyck shifts have Property (A), and that

$$\mathcal{S}(M_D(G(\mathcal{V}, \mathcal{E}))) = \mathcal{S}(\widehat{G}(\mathcal{R}_G, \mathcal{E} \setminus \mathcal{F}_G)).$$

In Section 2 we show that a topological conjugacy of Markov-Dyck shifts of graphs $G(\mathcal{V}, \mathcal{E})$ induces an isomorphism of the graphs $\widehat{G}(\mathcal{R}_G, \mathcal{E} \setminus \mathcal{F}_G)$, that also preserves certain data pertaining to the configuration of the neutral periodic points of the Markov-Dyck shift. For Markov-Motzkin shifts (see [KM2, Section 4.1]) analogous results hold.

In Section 3 we consider three families, $\mathbf{F}_I, \mathbf{F}_{II}$ and $\mathbf{F}_{III}$ of finite directed graphs with a single vertex or a single contracting subtree. We characterize the Markov-Dyck shifts of the graphs in each of these families within the Markov-Dyck shifts. We also show for the graphs in each of these families, that the topological conjugacy class of their Markov-Dyck shifts determines the isomorphism class of the graphs.

The family $\mathbf{F}_I$ contains the finite strongly connected directed non-cyclic graphs with a single vertex or a single contracting subtree, such that all of its vertices, except the root of the contracting subtree, have out-degree one.

The family $\mathbf{F}_{II}$ contains the finite strongly connected directed non-cyclic graphs with a single vertex or with a single contracting subtree, such that all leaves of the contracting subtree have level one.

The family $\mathbf{F}_{III}$ contains the finite strongly connected directed non-cyclic graphs with a single contracting subtree, that has the shape of a "V", and that are such that the two leaves of the contracting subtree have the same out-degree, and all interior vertices of the contracting subtree have out-degree one.

It had been known, that for finite directed graphs, in which every vertex has at least two incoming edges, topological conjugacy of their Markov-Dyck shifts implies the isomorphism of the graphs (see [HIK, pp. 617 - 624] and [Kr3, Corollary 3.2]). Actually, for finite directed graphs, in which every vertex has at least two incoming edges, the flow equivalence of their Markov-Dyck shifts implies the isomorphism of the graphs. For Dyck shifts this follows from [M2], and for the general case see [CS] and [Kr4].

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2. Neutral periodic points of Markov-Dyck shifts

We denote the period of a periodic point of a subshift by π. Given a rooted tree $\mathcal{T}(\mathcal{V},\mathcal{F})$, we denote for $V \in \mathcal{V}$ by $b(V)$ the path from the root to V.

We continue to consider a strongly connected finite directed graph $G = G(\mathcal{V},\mathcal{E})$, card$(\mathcal{E} \setminus \mathcal{F}_G)$. An edge $e \in \mathcal{E} \setminus \mathcal{F}_G$, determines a generator of $\mathcal{S}^-\widehat{G})(S^+\widehat{G}))$, that we denote by $\widehat{e}^-(\widehat{e}^+)$, and we set

$$\lambda(e^-) = \widehat{e}^-, \qquad \lambda(e^+) = \widehat{e}^+, \qquad e \in \mathcal{E} \setminus \mathcal{F}_G.$$

A root $R \in \mathcal{R}_G$ determines an idempotent of $\mathcal{S}(\widehat{G})$, that we denote by $\widehat{\mathbf{1}}_R$, and we set

$$\lambda(f^-) = \lambda(f^+) = \widehat{\mathbf{1}}_R, \qquad f \in \mathcal{F}_R, R \in \mathcal{R}_G.$$

We set

$$(e^-)^{-1} = e^+, \quad (e^+)^{-1} = e^-, \qquad e \in \mathcal{E},$$

and, more generally, for a path $b = (b_i)_{1 \le i \le I} \in \mathcal{L}(M_D(G))$ we use the notation

$$b^{-1} = ((b_i)^{-1})_{I \ge i \ge I}$$

for the reversed path of b.

The set of neutral periodic points of $M_D(G)$ we denote by $P^{(0)}(G)$. For a vertex $V \in \mathcal{V}$, we denote by $P^{(0)}(V)$ the set of periodic points of $M_D(G)$, such that there is an $m \in \mathbb{Z}$, such that

$$V = s_{\widetilde{G}}(p_{[m,m+\pi(p))}),$$

and such that

$$\mathbf{1}_V = \prod_{m \le j < m+\pi(p)} p_j.$$

Theorem 2.1.
$$P^{(0)}(G) = \bigcup_{V\in\mathcal{V}} P^{(0)}(V).$$

Proof. For the proof let there be given a neutral periodic point p of $M_D(G)$. With $\widehat{K}_+ \in \mathbf{Z}_+, \widehat{K}_- \in \mathbf{Z}_+$, and
$$\widehat{e}^+_{\widehat{k}(+)} \in \widehat{\mathcal{E}}^+, \qquad \widehat{K}_+ \geq \widehat{k}(+) > 0,$$
$$\widehat{e}^-_{\widehat{k}(-)} \in \widehat{\mathcal{E}}^-, \qquad 0 < \widehat{k}(-) \leq \widehat{K}_-,$$
and with an $R \in \mathcal{R}_G$, write
$$\prod_{0\leq j<\pi(p)} \lambda(p_j) = (\prod_{\widehat{K}_+\geq\widehat{k}(+)>0} \widehat{e}^+_{\widehat{k}(+)})\ \widehat{\mathbf{1}}_R\ (\prod_{0<\widehat{k}(-)\leq\widehat{K}_-} \widehat{e}^-_{\widehat{k}(-)}).$$
Note that
$$s_{\widehat{G}}(\widehat{e}_{\widehat{K}_+}) = t_{\widehat{G}}(\widehat{e}_{\widehat{K}_-}).$$
Assume that
$$(\prod_{0<\widehat{k}(-)\leq\widehat{K}_-} \widehat{e}^-_{\widehat{k}(-)})(\prod_{\widehat{K}_+\geq\widehat{k}(+)>0} \widehat{e}^+_{\widehat{k}(+)}) \in \mathcal{S}^-(\widehat{G}) \setminus \{\widehat{\mathbf{1}}_R\}. \tag{2.1}$$
Choose an $\widehat{m} \in [0, \pi(p))$, such that
$$\prod_{0\leq j<\widehat{m}} \lambda(p_j) = \prod_{\widehat{K}_+\geq\widehat{k}(+)>0} \widehat{e}^+_{\widehat{k}(+)}.$$
Then also
$$\prod_{\widehat{m}\leq j<\pi(p)} \lambda(p_j) = \prod_{0<\widehat{k}(-)\leq\widehat{K}_-} \widehat{e}^-_{\widehat{k}(-)},$$
and therefore by (2.1)
$$\prod_{\widehat{m}\leq j<\widehat{m}+\pi(p)} \lambda(p_j) = (\prod_{0<\widehat{k}(-)\leq\widehat{K}_-} \widehat{e}^-_{\widehat{k}(-)})(\prod_{\widehat{K}_+\geq\widehat{k}(+)>0} \widehat{e}^+_{\widehat{k}(+)}) \in \mathcal{S}^-(\widehat{G}).$$
It follows that the word $p_{[\widehat{m},\widehat{m}+\pi(p))}$ has a suffix e^-b, that is uniquely determined by the condition, that $e^- \in \mathcal{E}^- \setminus \mathcal{F}^-$ and that b is the empty word, or $b = (b_i)_{1\leq i\leq I}$ is a word in $\mathcal{L}(M_D(G))$ such that
$$\prod_{1\leq i\leq I} \lambda(b_i) = \mathbf{1}_{t(e)}.$$
Let $g \neq e$ be an incoming edge of $t_{\widetilde{G}}(e^-)$. For $K \in \mathbb{N}$ the words
$$e^-bp_{[\widehat{m},\widehat{m}+K\pi(p))}$$
and
$$p_{[\widehat{m},\widehat{m}+K\pi(p))}p^{-1}_{[\widehat{m},\widehat{m}+K\pi(p))}b^{-1}g^+$$
are admissible for $M_D(G)$. However the word
$$e^-bp_{[\widehat{m},\widehat{m}+K\pi(p))}p^{-1}_{[\widehat{m},\widehat{m}+K\pi(p))}b^{-1}g^+$$
is not admissible for $M_D(G)$. This contradicts the neutrality of p. Under the assumption that
$$(\prod_{0<\widehat{k}(-)\leq\widehat{K}_-} \widehat{e}^-_{\widehat{k}(-)})(\prod_{\widehat{K}_+\geq\widehat{k}(+)>0} \widehat{e}^+_{\widehat{k}(+)}) \in \mathcal{S}^+(\widehat{G}) \setminus \{\widehat{\mathbf{1}}_R\},$$
one has the symmetric argument. We have shown that
$$\lambda(p_{[\widehat{m},\widehat{m}+\pi(p))}) = \mathbf{1}_{\widehat{R}}. \tag{2.2}$$

To repeat this reasoning, with $K_+ \in \mathbf{Z}_+, K_- \in \mathbf{Z}_+$, and

$$e^+_{k(+)} \in \mathcal{E}^+, \qquad K_+ \geq k(+) > 0,$$

$$e^-_{k(-)} \in \mathcal{E}^-, \qquad 0 < k(-) \leq K_-,$$

and with a $V \in \mathcal{V}$, write

$$\prod_{\widehat{m} \leq j < \widehat{m}+\pi(p)} p_j = \Big(\prod_{K_+ \geq k(+) > 0} e^+_{k(+)}\Big)\, \mathbf{1}_V \,\Big(\prod_{0<k(-)\leq K_-} e^-_{k(-)}\Big).$$

Note that

$$s_{\widetilde{G}}(e^+_{K(+)}) = t_{\widetilde{G}}(e^-_{K(-)}).$$

Assume that

$$\Big(\prod_{0<k(-)\leq K_-} e^-_{k(-)}\Big)\Big(\prod_{K_+ \geq k(+) > 0} e^+_{k(+)}\Big) \in \mathcal{S}^-(G) \setminus \{\mathbf{1}_V\}. \tag{2.3}$$

Choose an $m \in [0, \pi(p))$, such that

$$\prod_{\widehat{m} \leq j < \widehat{m}+m} p_j = \prod_{K_+ \geq k(+) > 0} e^+_{k(+)}.$$

Then also

$$\prod_{\widehat{m}+m \leq j < \widehat{m}+\pi(p)} p_j = \prod_{0<k(-)\leq K_-} e^-_{k(-)},$$

and therefore by (2.3)

$$\prod_{m \leq j < m+\pi(p)} p_j = \Big(\prod_{0<k(-)\leq K_-} e^-_{k(-)}\Big)\Big(\prod_{K_+ \geq k(+) > 0} e^+_{k(+)}\Big) \in \mathcal{S}^-(G).$$

It follows from this and from (2.2), that there is a directed path $(f_l)_{1\leq l\leq L}, L \in \mathbb{N}$, in one of the contracting subtrees of G, such that

$$\prod_{m \leq j < m+\pi(p)} p_j = \prod_{1\leq l\leq L} f^-_l.$$

This contradicts the periodicity of p. Under the assumption that

$$\Big(\prod_{0<k(-)\leq K_-} e^-_{k(-)}\Big)\Big(\prod_{K_+ \geq k(+) > 0} e^+_{k(+)}\Big) \in \mathcal{S}^+(G) \setminus \{\mathbf{1}_V\}.$$

one has the symmetric argument. This confirms that

$$\mathbf{1}_V = \prod_{m \leq j < m+\pi(p)} p_j,$$

and completes the proof. □

The set of neutral periodic points of a subshift $X \subset \Sigma^{\mathbb{Z}}$ carries a pre-order relation $\lessapprox (X)$ (see [Kr2]). For neutral periodic points q and r of X one has $q \lessapprox (X) r$, if there exists a point in $A(X)$, that is left asymptotic to the orbit of q and right asymptotic to the orbit of r. The equivalence relation that is derived from $\lessapprox (X)$ we denote by $\approx(X)$.

We set

$$P^{(0)}_R = \bigcup_{V \in \mathcal{V}_R:} P^{(0)}(V), \qquad R \in \mathcal{R}_G.$$

The proof of the following lemma is similar to the proof of Theorem 3.2 of [HK].

Lemma 2.2. *The $\approx(X)$-equivalence class of $p \in P^{(0)}_R, R \in \mathcal{R}_G$, coincides with $P^{(0)}_R$.*

Proof. The proof comes in two parts. For the first part let $R \in \mathcal{R}_G$, let $U, W \in \mathcal{V}_R$, and $q, r \in P_R^{(0)}$, and let $j, k \in \mathbb{Z}$, be such that

$$U = s_{\widetilde{G}}(p_{[j,j+\pi(q))}), \qquad W = s_{\widetilde{G}}(r_{[k,k+\pi(r))}),$$

and such that

$$\mathbf{1}_U = \prod_{j \le i < j+\pi(p)} q_i, \qquad \mathbf{1}_W = \prod_{k \le i < k+\pi(r)} r_i.$$

Let a point $x \in M_D(G)$ be given by

$$x_{(-\infty,0]} = q_{(-\infty,j]} b^{-1}(U), \qquad x_{(0,\infty)} = b(W) r_{(k,\infty)}.$$

One has

$$x \in A_{\max\{\pi(q),\pi(r)\}}(M_D(G)).$$

This follows since the edges in the paths $b(U)$ and $b(W)$ are by construction the only incoming edges of their target vertices. We have proved that $q \approx(X) r$.

For the second part let $R, R' \in \mathcal{R}_G$,

$$R \neq R', \tag{2.4}$$

let

$$V \in \mathcal{V}_R, p \in P_R^{(0)}, \quad V' \in \mathcal{V}_{R'}, p' \in P_{R'}^{(0)},$$

and let $j, j' \in \mathbb{Z}$ be such that

$$V = s_{\widetilde{G}}(p_{[j,j+\pi(p))}), \quad V' = s_{\widetilde{G}}(p'_{[j',j'+\pi(p'))}),$$

and

$$\mathbf{1}_V = \prod_{j \le i < j+\pi(p)} p_i, \quad \mathbf{1}_{V'} = \prod_{j' \le i' < j'+\pi(p')} p'_{i'}.$$

We prove that p and p' are $\lessapprox(M_D(G))$-incomparable. Assume that

$$p \lessapprox (M_D(G))\ p',$$

and let $J \in \mathbb{N}$, and

$$x \in A_J(M_D(G)), \tag{2.5}$$

and $m, m' \in \mathbb{Z}, m < m'$, be such that

$$x_{(\infty,m)} = p_{(\infty,j)}, \qquad x_{[j',\infty)} = p'_{[j',\infty)}.$$

With $K, K' \in \mathbb{Z}_+$, and

$$e_k \in \mathcal{E} \setminus \mathcal{F}_G, \qquad K \ge k > 0,$$
$$e'_{k'} \in \mathcal{E} \setminus \mathcal{F}_G, \qquad 0 < k' \le K',$$

and with $Q \in \mathcal{R}_G$,

$$t_G(e_K) = Q = t_G(e_{K'}),$$

write

$$\prod_{m \le i < m'} \lambda(x_i) = (\prod_{K \ge k > 0} \widehat{e}_k^{\,+})\ \widehat{\mathbf{1}}_Q\ (\prod_{0 < k' \le K'} \widehat{e'}_{k'}^{\,-}).$$

Assumption (2.4) implies that $(K, K') \neq (0,0)$. Assume $K > 0$. Then we have an $M \in [m, m')$, such that $x_M = e_K^+$, and

$$\prod_{m \le i < M} \lambda(x_i) = \widehat{\mathbf{1}}_Q.$$

Let $g \neq e_K$, be an incoming edge of Q. The word

$$g^- b(V)^- p_{[m-J\pi(p),m)} x_{[m,M)}$$

is admissible for $M_D(G)$. However the word

$$g^- b(V)^- p_{[m-J\pi(p),m)} x_{[m,M)} e^+_K$$

is not. This contradicts (2.5).Under the assumption that $K' > 0$, one has the symmetric argument. We have shown that $p \not\approx (M_D(G))\, p'$. □

Denoting by $\Pi_n(Y)$ the number of points of period n of a shift-invariant set $Y \subset \Sigma^{\mathbb{Z}}$, the zeta function of Y is given by

$$\zeta_Y(z) = e^{\sum_{n\in\mathbb{N}} \frac{\Pi_n(X) z^n}{n}}.$$

We denote by $I^{(0)}_{2k}(M_D(G))$ the number of neutral periodic points of period $2k$ of $M_D(G), k \in \mathbb{N}$.

For the finite strongly connected directed graph $G(\mathcal{V}, \mathcal{E})$, we vertex weigh the graph $\widehat{G}(\mathcal{R}_G, \mathcal{E} \setminus \mathcal{F}_G)$ by assigning to its vertices $R \in \mathcal{R}_G$ the zeta function of $P^{(0)}_R$.

Corollary 2.3. *For a finite strongly connected directed graphs $G(\mathcal{V}, \mathcal{E})$ the topological conjugacy of the Markov-Dyck shifts $M_D(G)$ implies the isomorphism of the vertex weighted graphs $\widehat{G}(\mathcal{G}, \mathcal{E} \setminus \mathcal{F}_G)$ with weights $(\zeta_{P^{(0)}_R})_{R\in\mathcal{R}_G}$.*

Proof. By Lemma 2.2 there is a one-to-one correspondence between $\mathcal{R}_G$ and the set of $\approx (M_D(G))$-equivalence classes of the neutral periodic points of $M_D(G)$. A topological conjugacy carries $\approx (M_D(G))$-equivalence classes into $\approx (M_D(G))$-equivalence classes. □

To a vertex $V \in \mathcal{V}$ we associate the circular code $\mathcal{C}_V$ that contains the words $(c_i)_{1\le i\le I} \in \mathcal{L}(M_D(G))$ such that

$$s_{\widetilde{G}}(c) = t_{\widetilde{G}}(c) = V,$$

and

$$\prod_{1\le i\le I} c_i = \mathbf{1}_V,$$

$$\prod_{1\le i\le J} c_i \neq \mathbf{1}_V, \qquad 1 < J < I.$$

The generating function of $\mathcal{C}_V$ we denote by φ_V.

Corollary 2.4. *For $G(\mathcal{V}, \mathcal{E})$,*

$$\zeta_{P^{(0)}(M_D(G))} = \prod_{V\in\mathcal{V}} \frac{1}{1-\varphi_V}.$$

Proof. The corollary follows from Theorem 2.1 (e.g. see [P, Section 5] or [KM1, Section 2]). □

3. Families of finite directed graphs

In this section we consider the case of strongly connected finite directed graphs $G = G(\mathcal{V}, \mathcal{E}), \mathrm{card}(\mathcal{E} \setminus \mathcal{F}_G) > 1$, with a single vertex or with a single contracting subtree. These graphs are precisely the graphs, that have a Dyck inverse monoid as the associated semigroup of their Markov-Dyck shifts. Complete invariants for the isomorphism for these directed graphs are known for the case that all leafs of the contracting subtree have the same out-degree (see [GM]). We denote the root of $\mathcal{F}_G$ by V_0, and the out-degree of V_0 by $D(V_0)$.

Following the terminology, that was introduced in [HI], we say that a periodic point p of $M_D(G)$ and its orbit have negative multiplier $e \in \mathcal{E} \setminus \mathcal{F}_G$, if there exists an $i \in \mathbb{Z}$ and an $M \in \mathbb{N}$, such that

$$\lambda(p_{[i,i+\pi(p))}) = (\widehat{e^-})^M.$$

The mapping that assigns to a multiplier $e \in \mathcal{E} \setminus \mathcal{F}_G$ the set of periodic points of $M_D(G)$ with negative multiplier e is an invariant of topological conjugacy [HIK, Proposition 4.2]. We set

$$\nu(M_D(G)) = \operatorname{card}(\mathcal{E} \setminus \mathcal{F}_G), \qquad \mathcal{M}(M_D(G)) = \mathcal{E} \setminus \mathcal{F}_G,$$

We denote for a multiplier $e \in \mathcal{E} \setminus \mathcal{F}_G$, by $I_k^{(e)}(M_D(G))$ the number of periodic points of $M_D(G)$ with negative multiplier e and period k, and we set

$$\Lambda^{(e)}(M_D(G)) = \min\{k \in \mathbb{N} : I_k^{(e)}(M_D(G)) > 0\},$$
$$\mathcal{M}_\ell(M_D(G)) = \{e \in \mathcal{M}(M_D(G)) : \Lambda^{(e)} = \ell\}, \qquad \ell \in \mathbb{N}.$$

The set $\{\Lambda^{(e)}(M_D(G)) : e \in \mathcal{E} \setminus \mathcal{F}_G\}$ is an invariant of topological conjugacy. By the use of the notation $\Lambda(M_D(G))$ we indicate that all lengths $\Lambda^{(e)}(M_D(G)), e \in \mathcal{E} \setminus \mathcal{F}_G$, are equal and that $\Lambda(M_D(G))$ is their common value. We also denote by $\Xi_{2k}^{(e)}(M_D(G))$ the number of orbits of length $2k, k \in \mathbb{N}$, with negative multiplier $e \in \mathcal{E} \setminus \mathcal{F}_G$.

3.1. **A family of finite directed graphs I.** Set

$$\Pi_I = \{(S_\ell)_{\ell \in \mathbb{N}} \in \mathbb{Z}_+^{\mathbb{N}} : 1 < \sum_{\ell \in \mathbb{N}} S_\ell < \infty\}.$$

The data $(S_\ell)_{\ell \in \mathbb{N}} \in \Pi_I$ determine canonical models $G((S_\ell)_{\ell \in \mathbb{N}})$ of the graphs in $\mathbf{F}_I$. We define $G((S_\ell)_{\ell \in \mathbb{N}})$ as the graph with vertices V_0 and

$$V_{\ell,s,t}, \qquad 1 \le t < \ell, 1 \le s \le S_\ell,\ \ell > 1,$$

and edges

$$f_{\ell,s,t}, \qquad 1 \le t < \ell, 1 \le s \le S_\ell,\ \ell > 1,$$

and

$$e_{\ell,s}, \qquad 1 \le s \le S_\ell, \quad \ell \in \mathbb{N}.$$

The source and target mappings are given by

$$s(f_{\ell,s,1}) = V_0,$$
$$s(f_{\ell,s,t}) = V_{\ell,s,t-1}, \qquad 1 < t < \ell,$$
$$t(f_{\ell,s,t}) = V_{\ell,s,t}, \qquad 1 \le t < \ell, \quad 0 < s \le S_\ell,\ \ell > 1,$$

and

$$s(e_{1,s}) = V_0 = t(e_{1,s}), \qquad 0 < s \le S_1,$$

$$s(e_{\ell,s}) = V_{\ell,s,\ell-1}, t(e_{\ell,s}) = V_0, \qquad 0 < s \le S_\ell,\ \ell > 1.$$

The edge set of the contracting subtree of $G((S_\ell)_{\ell \in \mathbb{N}})$ is

$$\{f_{\ell,s,t} : 1 \le t < \ell, 1 \le s \le S_\ell,\ \ell > 1\}.$$

The Dyck shifts $D_N, N > 1$, belong here with the data $S_1 = N, S_\ell = 0, \ell > 1$. Also the Fibonacci-Dyck shift belongs here with the data $S_1 = 1, S_2 = 1, S_\ell = 0, \ell > 2$.

Theorem 3.1. *For a finite directed graph $G = G(\mathcal{V}, \mathcal{E})$ there exist data*

$$(S_\ell)_{\ell \in \mathbb{N}} \in \Pi_I,$$

such that there is a topological conjugacy

$$M_D(G) \simeq M_D(G((S_\ell)_{\ell \in \mathbb{N}})), \tag{3.I.1}$$

if and only if the associated semigroup of $M_D(G)$ is a Dyck inverse monoid, and

$$\tfrac{1}{2} I^{(0)}(M_D(G)) = \nu(M_D(G)) + \sum_{\ell > 1} (\ell - 1)\operatorname{card}(\mathcal{M}_\ell(M_D(G))), \tag{3.I.2}$$

and in this case (3.I.1) holds for

$$S_\ell = \mathrm{card}(\mathcal{M}_\ell(M_D(G))), \qquad \ell \in \mathbb{N}.$$

Proof. The statement holds if G is a single vertex graph. Consider a graph $\widetilde{G} = G(\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}})$ with a single contracting subtree $\mathcal{F}_{\widetilde{G}}$. Denote by $J_{\widetilde{G}}(\ell)$ the number of leafs of $\mathcal{F}_{\widetilde{G}}$ that have level $\ell - 1$. One has that

$$\mathrm{card}(\mathcal{F}_{\widetilde{G}}) \leq \sum_{\ell > 1} (\ell - 1) J_{\widetilde{G}}(\ell), \tag{3.I.3}$$

and

$$J_{\widetilde{G}}(\ell) \leq \mathrm{card}(\mathcal{M}_\ell(M_D(\widetilde{G}))). \qquad \ell > 1, \tag{3.I.4}$$

Therefore

$$\mathrm{card}(\mathcal{F}_{\widetilde{G}}) \leq \sum_{\ell > 1} (\ell - 1) \mathrm{card}(\mathcal{M}_\ell(M_D(\widetilde{G}))). \tag{3.I.5}$$

Equality holds simultaneously in (3.I.3) and (3.I.4) if and only if it holds in (3.I.5) if and only if G belongs to $\mathbf{F}_I$. Equation (3.I.2) implies equality in (3.I.5) and the theorem follows. ☐

Corollary 3.2. *For finite directed graphs $G = G(\mathcal{V}, \mathcal{E})$ such that the associated semigroup of $M_D(G)$ is a Dyck inverse monoid, and such that*

$$\tfrac{1}{2} I^{(0)}(M_D(G)) = \quad \nu(M_D(G)) + \sum_{\ell > 1} (\ell - 1) \mathrm{card}(\{e \in \mathcal{M}(M_D(G)) : \Lambda^{(e)} = \ell\})$$

the topological conjugacy of their Markov-Dyck shifts implies the isomorphim of the graphs.

Proof. In (3.I.2) the data $(M_\ell)_{\ell \in \mathbb{N}}$ are expressed in terms of invariants of topological conjugacy. ☐

3.2. **A family of finite directed graphs II.** We set

$$\Pi = \{(R, (Q_M)_{M \in \mathbb{N}}) \in \mathbb{Z}_+ \times \mathbb{N}^{\mathbb{Z}_+} : 1 < R + \sum_{M \in \mathbb{N}} Q_M < \infty\}.$$

The data $(R, (Q_M)_{M \in \mathbb{N}}) \in \Pi$ determine canonical models $G((R, (Q_M)_{M \in \mathbb{N}}))$ of the graphs in $\mathbf{F}_{II}$. We define $G((R, (Q_M)_{M \in \mathbb{N}}))$ as the directed graph with a vertex $V(0)$ and with vertices

$$V_{M,q}(1), \qquad 1 \leq q \leq Q_M, \quad M \in \mathbb{N},$$

and with edges

$$f_{M,q}, \qquad 1 \leq q \leq Q_M, \quad M \in \mathbb{N},$$

and

$$e_r, \qquad 1 \leq r \leq R,$$

and

$$e_{M,q,m}, \qquad 1 \leq m \leq M, 1 \leq q \leq Q_M, \quad M \in \mathbb{N}.$$

The source and target mappings are given by

$$s(f_{M,q}) = V(0), \quad t(f_{M,q}) = V_{M,q}(1), \qquad 1 \leq q \leq Q_M, \quad M \in \mathbb{N},$$

and

$$s(e_r) = t(e_r) = V(0), \quad 1 \leq r \leq R,$$

and

$$s(e_{M,q,m}) = V_{M,q}(1), \quad t(e_{M,q,m}) = V(0), \qquad 1 \leq m \leq M, 1 \leq q \leq Q_M, \quad M \in \mathbb{N}.$$

The contracting subtree of $G(R, (Q_M)_{M \in \mathbb{N}})$ has the edges $f_{M,q}, 1 \leq q \leq Q_M, M \in \mathbb{N}$. Note the non-empty intersection of $\mathbf{F}_{II}$ with $\mathbf{F}_I$.

Theorem 3.3. *For a finite directed graph $G = G(\mathcal{V}, \mathcal{E})$ there exist data*

$$(R, (Q_M)_{M\in\mathbb{N}}) \in \Pi,$$

such that there is a topological conjugacy

$$M_D(G) \simeq M_D(G((R, (Q_M)_{M\in\mathbb{N}})), \tag{3.II.1}$$

if and only if the associated semigroup of $M_D(G)$ is a Dyck inverse monoid, and

$$\Lambda^{(e)} \leq 2, \qquad e \in \mathcal{E} \setminus \mathcal{F}_G, \tag{3.II.2}$$

and in this case (3.1) holds for

$$\begin{aligned} R =& \text{card}(\{e \in \mathcal{E} \setminus \mathcal{F}_G : \Lambda^{(e)} = 1\}), \\ Q_M =& \tfrac{1}{M}\text{card}(\{e \in \mathcal{E} \setminus \mathcal{F}_G : \Xi_4^{(e)} = \\ & M + I_2^{(0)}(M_D(G)) - \nu(M_D(G)) + \text{card}(\mathcal{M}_1(G))\}), \quad M \in \mathbb{N}. \end{aligned} \tag{3.II.3}$$

Proof. The assumption, that the associated semigroup of $M_D(G)$ is a Dyck inverse monoid, implies, that either G is a one-vertex graph, or G has the single contracting subtree $T(\mathcal{V}, \mathcal{F}_G)$. In case of a contracting subtree, one has from (3.II.2) that all leaves of the subtree are at level 1. Also, for an $e \in \mathcal{E} \setminus \mathcal{F}_G$, one has that

$$\text{card}(\Xi_4^{(e)}) = \text{card}(\{e' \in \mathcal{E} \setminus \mathcal{F}_G : s(e') = s(e)\}) + D(V_0).$$

This follows from the observation, that in this case every orbit of length 4 with negative multiplier $e \in \mathcal{E} \setminus \mathcal{F}_G$ is obtained by inserting into an orbit of length 2 with negative multiplier $e \in \mathcal{E} \setminus \mathcal{F}_G$ either the word $f^-f^+, f \in \mathcal{F}_G, t(f) = s(e)$, or a word $\widetilde{e}^-\widetilde{e}^+, \widetilde{e} \in \mathcal{M}_1(G)$, or a word $\widetilde{e}^-\widetilde{e}^+, \widetilde{e} \in \mathcal{M}_2(G), s(\widetilde{e}) = s(e)$. It is

$$D(V_0) = I_2^{(0)}(M_D(G)) - \nu(M_D(G)) + \text{card}(\mathcal{M}_1(G)),$$

and the equations (3.II.3) follow. □

Corollary 3.4. *For finite directed graphs $G = G(\mathcal{V}, \mathcal{E})$ such that the associated semigroup of $M_D(G)$ is a Dyck inverse monoid, and such that*

$$\Lambda^{(e)} \leq 2, \qquad e \in \mathcal{E} \setminus \mathcal{F}_G,$$

the topological conjugacy of their Markov-Dyck shifts implies the isomorphim of the graphs.

Proof. In (3.II.2) the data $(R, (Q_M)_{M\in\mathbb{N}})$ are expressed in terms of invariants of topological conjugacy. □

3.3. **A family of finite directed graphs III.** We consider the family $\mathbf{F}_{III}$ of finite directed graphs $G = G(\mathcal{V}, \mathcal{G})$ with a single contracting subtree $T(\mathcal{V}, \mathcal{F}_G)$, that contains the graphs $G[\ell, M], \ell, M \in \mathbb{N}$, that we describe as follows: The graph $\ell, M \in \mathbb{N}$ we let $G[\ell, M]$ has a vertex $V(0)$, vertices

$$V_0(l), V_1(l), \qquad 1 \leq l < \ell,$$

and edges

$$f_0(l), f_1(l), \qquad 1 \leq l < \ell,$$

and

$$e_0(m), e_1(m), \qquad 1 \leq m \leq M.$$

The source and target mappings are given by

$$s(f_0(1)) = s(f_1(1)) = V(0),$$

$$s(f_0(l)) = V_0(l-1), \quad s(f_1(l)) = V_1(l-1),$$

$$t(f_0(l)) = V_0(l), \quad t(f_1(l)) = V_1(l), \qquad 1 < l < \ell,$$

and

$$s(e_0(m)) = V_0(\ell-1), s(e_1(m)) = V_1(\ell-1), \quad t(e_0(m)) = t(e_1(m)) = V(0), \\ 1 \le m \le M.$$

The contracting subtree of the graph $G[\ell, M]$ has the root V_0 and the edges $f_0(l)$, $f_1(l), 1 \le l < \ell$. Note the non-empty intersection of $\mathbf{F}_{III}$ with $\mathbf{F}_I$and $\mathbf{F}_{II}$.

Lemma 3.5. *For $G = G[\ell, M]$ one has that*

$$\tfrac{1}{2} I_2^{(0)}(M_D(G)) = \nu(M_D(G)) + 2\Lambda(M_D(G)) - 2.$$

Proof. One observes that one has for $G(\mathcal{V}, \mathcal{E}) = G[\ell, M]$, that

$$2\Lambda(M_D(G)) - 2 = \operatorname{card}(\mathcal{E} \setminus \mathcal{F}_G). \quad \square$$

For $\ell, L, M \in \mathbb{N}$, $\ell \ge 4, L < \ell - 2$, we introduce two auxiliary families of graphs with a unique contracting subtree, that we denote by $G_{2,M}[\ell, L]$ and $G_{M,2}[\ell, L]$. Both graphs $G_{2,M}[\ell, L]$ and $G_{M,2}[\ell, L]$ have vertices

$$V(l), \qquad 0 \le l \le L,$$

and

$$V_0(l, m), V_1(l, m), \qquad 1 \le m \le M, L + 2 \le l < \ell,$$

and edges

$$f(l), \qquad 0 \le l \le L,$$

and

$$f_0(l, m), f_1(l, m), \qquad 1 \le m \le M, L + 2 \le l < \ell,$$

and

$$e_0(m), e_0(m), \qquad 1 \le m \le M.$$

with source and target vertices partially given by

$$s(f(l)) = V(l-1), \qquad 1 \le l \le L,$$

$$s(f_0(l, m)) = V_0(l-1, m), \quad s(f_1(l, m)) = V_1(l-1, m), \qquad 1 \le m \le M, L + 1 < \ell,$$

$$t(f(l)) = V(l), \qquad 1 \le l \le L,$$

$$t(f_0(l, m)) = V_0(l, m), \quad t(f_1(l, m)) = V_1(l, m), \qquad 1 \le m \le M, L + 1 < \ell,$$

and

$$s(e_0(m)) = V_0(\ell-1, m), s(e_1(m)) = V_0(\ell-1, m), t(e_0(m)) = t(e_1(m)) = V(0), \\ 1 \le m \le M.$$

The graph $G_{2,M}[\ell, L]$ has vertices

$$V_0(L+1), V_1(L+1),$$

and edges

$$f_0(L+1), f_1(L+1),$$

and he definition of its source and target mappings is completed by setting

$$s(f_0(L+1)) = s(f_1(L+1)) = V(L),$$

$$t(f_0(L+1)) = V_0(L+1), \qquad t(f_1(L+1)) = V_1(L+1),$$

and

$$s(f_0(L+2, m) = V_0(L+1), \quad s(f_1(L+2, m) = V_1(L+1), \qquad 1 \le m \le M.$$

The graph $G_{M,2}[\ell, L]$ has vertices

$$V(L+1, m), \qquad 1 \le m \le M,$$

and edges

$$f(L+1,m), \qquad 1 \le m \le M,$$

and the definition of its source and target mappings is completed by setting

$$s(f(L+1,m)) = V(L), \quad t(f(L+1,m)) = V(L+1,m), \qquad 1 \le m \le M,$$

and

$$s(f_0(L+2,m) = s(f_1(L+2,m) = V(L+1,m), \qquad 1 \le m \le M.$$

The invariants of topological conjugacy ν and $I_2^{(0)}$ together do not separate the graphs $G[\ell, M]$ from the graphs $G_{2,M}[\ell, L]$ or from the graphs $G_{M,2}[\ell, L]$, but together with the invariant $I_4^{(0)}$ they do, as the next lemma shows.

Lemma 3.6. *(a) For $\ell, M \in \mathbb{N}$, and $G = G[\ell, M]$ one has*

$$I_4^{(0)}(M_D(G)) = 3I_2^{(0)}(M_D(G)) + \nu^2(M_D(G)) - 2\nu(M_D(G)) - 4,$$

(b1) For $\ell, M \in \mathbb{N}, \ell > 3, 2 \le L < \ell - 4$, and $G = G_{2,M}[\ell, L]$, one has

$$I_4^{(0)}(M_D(G)) = 3I_2^{(0)}(M_D(G)) + 4\nu(M_D(G))(\nu(M_D(G)) + 1),$$

(b2) For $\ell, M \in \mathbb{N}, \ell > 3, 2 \le L < \ell - 4$, and $G = G_{M,2}[\ell, L]$ one has

$$I_4^{(0)}(M_D(G)) = 3I_2^{(0)}(M_D(G)) + \tfrac{1}{2}\nu(M_D(G))^2 + 5\nu(M_D(G)) - 4.$$

Proof. The number of neutral periodic orbits of length 4 of

$$M_D(G[\ell, M])$$

$$(M_D(G_{2,M}[\ell, L], M_D(G_{M,2}[\ell, L]),$$

that contain the points that carry the infinite concatenation of words of the form $e^- e^+ \widetilde{e}^- \widetilde{e}^+, e \ne \widetilde{e}$, is equal to

$$1 + M(M-1)$$

$$(1 + 2M(M+1), \tfrac{1}{2}I_2^{(0)} + 2M - 1),$$

and the number of neutral periodic orbits of length 4 of

$$M_D(G[\ell, M])$$

$$(M_D(G_{2,M}[\ell, L]), M_D(G_{M,2}[\ell, L])),$$

that contain the points that carry the infinite concatenation of words of the form $e^- \widetilde{e}^- \widetilde{e}^+ \widetilde{e}^-, t(e) = s(\widetilde{e})$, is equal to

$$\tfrac{1}{2}I_2^{(0)} + M^2 + 3M - 1$$

$$(\tfrac{1}{2}I_2^{(0)} + 2(M-1), \tfrac{1}{2}I_2^{(0)} + 2M - 1). \quad \square$$

Lemma 3.7. *For $\ell > 4$, and $M \in \mathbb{N}$ and for $G(\mathcal{V}, \mathcal{E}) = G[\ell, M]$ one has that*

$$\begin{aligned} \Xi_{\ell+2}^{(e)}(M_D(G)) =& \Lambda(M_D(G)) + \tfrac{1}{2}\nu(M_D(G)), \\ \Xi_{\ell+4}^{(e)}(M_D(G)) =& (\Lambda(M_D(G)) + \tfrac{1}{2}\nu(M_D(G)))^2 + \\ & \Lambda(M_D(G)) + 2\nu(M_D(G)) - 2, \quad e \in \mathcal{E} \setminus \mathcal{F}_G. \end{aligned}$$

Proof. Let $e \in \mathcal{E} \setminus \mathcal{F}_G$, and let $O^{(e)}$ be the shortest periodic orbit of $M_D(G)$ with negative multiplier e.

All periodic orbits of $M_D(G)$ of length $\Lambda(M_D(G)) + 2$ with multiplier e^- are obtained by inserting a word of the form g^-g^+, where the source vertex of the edge g^- is transversed by $O^{(e)}$, into $O^{(e)}$. The number of these words is $\ell + M$.

All periodic orbits of $M_D(G)$ of length $\Lambda(M_D(G)) + 4$ with multiplier e^- are obtained by either inserting two words of the form g^-g^+, where the source vertex of the edge g^- is transversed by $O^{(e)}$, into $O^{(e)}$, or by inserting a word of the form $g^-\widetilde{g}^-\widetilde{g}^+g^+, t(g) = s(\widetilde{g})$, into $O^{(e)}$, where the source vertex of the edge g^- is transversed by $O^{(e)}$, into $O^{(e)}$, and the number of these words is $\ell + 4M - 2$. ☐

Theorem 3.8. *For a finite directed graph $G(\mathcal{V}, \mathcal{E})$ there exist $\ell > 4, M \in \mathbb{N}$, such that there is a topological conjugacy*

$$M_D(G) \simeq M_D(G[\ell, M]), \tag{3.IV.1}$$

if and only if there is a Dyck inverse monoid associated to $M_D(G)$, all $\Lambda(e), e \in \mathcal{E} \setminus \mathcal{F}_G$, have the same value, and

$$\tfrac{1}{2}I_2^{(0)}(M_D(G)) = 2\Lambda(M_D(G)) + \nu(M_D(G)) - 2, \tag{A}$$

$$I_4^{(0)}(M_D(G)) = 3I_2^{(0)}(M_D(G)) + \nu^2(M_D(G)) - 2\nu(M_D(G)) - 4, \tag{B}$$

$$\Xi_{\ell+2}^{(e)}(M_D(G)) = \Lambda(M_D(G)) + \tfrac{1}{2}\nu(M_D(G)), \qquad e \in \mathcal{E} \setminus \mathcal{F}_G, \tag{C}$$

$$\begin{aligned}\Xi_{\ell+4}^{(e)}(M_D(G)) =&(\Lambda(M_D(G)) + \tfrac{1}{2}\nu(M_D(G)))^2 + \\ &\Lambda(M_D(G)) + 2\nu(M_D(G)) - 2, \qquad e \in \mathcal{E} \setminus \mathcal{F}_G.\end{aligned} \tag{D}$$

If conditions (A)(B)(C)(D) are satisfied, then (3.IV.1) holds for

$$\ell = \Lambda(M_D(G)), \qquad M = \tfrac{1}{2}\nu(M_D(G)). \tag{3.IV.2}$$

Proof. Necessity follows from Lemma 3.4, Lemma 3.6. and Lemma 3.7. To prove sufficiency, let $G(\mathcal{V}, \mathcal{E})$ be a graph that satisfies the conditions of the theorem. One has that

$$\tfrac{1}{2}I_2^{(0)}(M_D(G)) =\geq D(V_0)\Lambda(M_D(G)) + \nu(M_D(G)) - D(V_0),$$

and it follows from (A) that

$$D(V_0) \leq 2.$$

In the case $D(V_0) = 2$, one has by (C) that the two leaves of the contracting subtree have the same out-degree and the theorem follows. The task is to exclude the case $D(V_0) = 1$.

Assume, that $D(V_0) = 1$. Let L be maximal such that such that the contracting subtree $\mathcal{F}_G$ has a single vertex at levels up to level L . By (A) $L < \ell - 3$. Denote the vertex at level L by V_L, and the out-degree of V_L by $D(V_L)$. We will exclude each of the following cases by deriving a contradiction to (A, B, C, D):

$$D(V_L) > \tfrac{1}{2}\nu(M_D(G)) + 1, \tag{c1}$$

$$D(V_L) = \tfrac{1}{2}\nu(M_D(G)) + 1, \tag{c2}$$

$$D(V_L) = \tfrac{1}{2}\nu(M_D(G)), \tag{c3}$$

$$\tfrac{1}{2}\nu(M_D(G)) > D(V_L) > 2, \tag{c4}$$

$$D(V_L) = 2. \tag{c5}$$

One has that

$$\Xi^{(e)}_{\ell+2} \geq \Lambda(M_D(G)) + D(V_L) - 1, \quad e \in \mathcal{E} \setminus \mathcal{F}_G,$$

and this means that (c1) contradicts (C).

From (c2) and (C) it follows, that the out-degree of all vertices of $\mathcal{F}_G$ above level L have out-degree one. This implies that

$$\mathrm{card}(\mathcal{E} \setminus \mathcal{F}_G) = \tfrac{1}{2}\nu(M_D(G)) + 1,$$

which is impossible unless $\nu(M_D(G)) = 2$, and in this case the graph G is isomorphic to the graph $G_{2,1}[\Lambda(M_D(G)), 1]$. By statement (b1) of Lemma 3.6 one has a contradiction to (B).

From (c3) and (C) it follows for the target vertex V of an edge with source vertex V_L, and for a path b from V to a leave W of $\mathcal{F}_G$, that the out-degree of exactly one of the vertices V or W or of the vertices that the path b crosses, is equal to two. This vertex with out-degree two must necessarily be V, since otherwise, $\Xi^{(e)}_4(M_D(G)) = \Lambda(M_D(G))+4$ would for all $e \in \mathcal{E}\setminus\mathcal{F}_G$ be less than what is required by (D). The graph G being isomorphic to the graph $G_{\frac{1}{2}\nu(M_D(G)),2}[\Lambda(M_D(G)), 2]$ one has by statement (b1) of Lemma 3.6 a contradiction to (B).

From (c4) and (C) it follows for the target vertex V of an edge with source vertex V_L, and for a path b from V to a leave W of $\mathcal{F}_G$, that the sum of the out-degrees of V and of W and of the vertices that the path b crosses, is equal to

$$\tfrac{1}{2}\nu(M_D(G)) - D(V_L).$$

It then follows that

$$\mathrm{card}(\mathcal{E} \setminus \mathcal{F}_G) \geq D(V_L)(\tfrac{1}{2}\nu(M_D(G)) - D(V_L)).$$

By (c3)

$$D(V_L)(\tfrac{1}{2}\nu(M_D(G)) - D(V_L)) > \nu(M_D(G)),$$

and we have a contradiction.

The argument for case (c5) is the same as for case (c3), with statement (b2) of Lemma 3.6 taking the place of statement (b1). □

Corollary 3.9. *For directed graphs $G = G(\mathcal{V}, \mathcal{E})$ such that to $M_D(G)$ there is associated a Dyck inverse monoid, and that satisfy conditions (A, B, C, D), the topological conjugacy of the Markov-Dyck shifts $M_D(G(\mathcal{V}, \mathcal{E}))$ implies the isomorphism of the graphs $G(\mathcal{V}, \mathcal{E})$.*

Proof. In (3.IV.2) the data $[\ell, M]$ are expressed in terms of invariants of topological conjugacy. □

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