

# INFINITELY SUPPORTED LIOUVILLE MEASURES OF SCHREIER GRAPHS

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ABSTRACT. We provide equivalent conditions for Liouville property of actions of groups. As an application, we show that there is a Liouville measure for the action of the Thompson group  $F$  on dyadic rationals. This result should be compared with a recent result of Kaimanovich, where he shows that the action of the Thompson group  $F$  on dyadic rationals is not Liouville for all finitely supported measures. As another application we show that there is a Liouville measure for lamplighter actions. This gives more examples of non-amenable Liouville actions.

## 1. INTRODUCTION

Let a discrete group  $G$  act on a countable set  $X$ , denoted by  $G \curvearrowright X$ , and let  $\mu$  be a probability measure on  $G$ . A measure  $\mu$  on  $G$  is non-degenerate if  $\text{supp} \mu$  generate the group  $G$ . We denote by  $P_\mu$  the transition matrix on  $X$  induced by  $\mu$ , that is

$$P_\mu(x, y) = \sum_{g \in G} \mathbf{1}_{\{g \cdot x = y\}} \mu(g).$$

For the simplicity of the notations we write  $\mu \cdot x = P_\mu(x, \cdot)$ . A function  $f : X \rightarrow \mathbb{R}$  is  $P_\mu$ -harmonic if  $f(x) = \sum_{y \in X} f(y) P_\mu(x, y)$ , and  $(X, P_\mu)$  is Liouville if all bounded  $P_\mu$ -harmonic functions are constant. The action  $G \curvearrowright X$  is  $\mu$ -Liouville if  $(X, P_\mu)$  is Liouville, and if this is the case we say  $\mu$  is a Liouville measure for the action. We call an action Liouville if there is a measure  $\mu$  on  $G$  which makes it  $\mu$ -Liouville.

Note that, one can make several definitions for Liouville actions by adapting the definition of Liouville measures on Cayley graphs. While many of the definitions are equivalent for Cayley graphs, they are not equivalent when we pass to actions. The main reason of our current definition is a recent approach to amenability developed by Kaimanovich in [5]. In order to show that a group is not amenable it is sufficient to find an action which does not admit any non-degenerate Liouville measure. Indeed, this will insure that there is no non-degenerate Liouville measure on the group itself, thus, by renowned result of Kaimanovich and Vershik, [6], the group is not amenable. The problem of amenability of Thompson's group  $F$  can be approached with this technique. In particular, Kaimanovich showed that every finitely supported non-degenerate measure on Thompson group  $F$  is not Liouville for it's action on dyadic rationals. In this paper we show that this action admits infinitely supported Liouville measure.

In fact, our methods are more general, we give a criteria for a measure to be Liouville for the action. As another application of this method we show that certain Schreier graphs of lamplighter group are non-amenable but  $\mu$ -Liouville for some

infinitely supported measure  $\mu$ . We note that examples of non-amenable graphs with Liouville measures of finite support were previously have been known, see for example [2] and references therein.

**Acknowledgements:** We are grateful to Kaimanovich for several interesting and motivating discussions. We also thank Nico Matte Bon and Omer Tamuz for remarks on earlier versions of the paper.

## 2. LIOUVILLE MEASURES ON SCHREIER GRAPHS

We fix a finite generating set  $S$  on  $G$ . Consider an action of  $G$  on a countable set  $X$ . Let  $\|\cdot\|_1$  be the  $\ell^1$  norm with respect to counting measure on  $X$ , that is  $\|f\|_1 = \sum_{x \in X} |f(x)|$ .

**Lemma 1.** Suppose  $P_\mu$  is irreducible and lazy such that  $P_\mu(x, x) \geq \frac{1}{2}$  for all  $x \in X$ . If  $(X, P_\mu)$  is Liouville, then for any two points  $x, y \in X$ ,

$$\lim_{n \rightarrow \infty} \left\| \mu^{(n)} \cdot x - \mu^{(n)} \cdot y \right\|_1 = 0.$$

*Proof.* Since  $P_\mu$  is irreducible, it is sufficient to prove the claim for  $y = s \cdot x$  with  $s \in \text{supp}(\mu)$ . Let  $(W_n)$  be a left random walk on  $G$  with step distribution  $\mu$ .

By Corollary 14.13 of [7],  $(X, P_\mu)$  is Liouville if and only if the invariant  $\sigma$ -field  $\mathcal{I}$  is trivial. Since  $P_\mu$  is lazy, by Theorem 14.18 [7], the completion of the tail  $\sigma$ -field coincides with the completion of the invariant  $\sigma$ -field. Thus Liouville property of  $(X, P_\mu)$  implies that the tail  $\sigma$ -field  $\mathcal{T}$  is trivial. Therefore,

$$\mathbb{P}(W_1 \cdot x = y_1 | W_n \cdot x) \rightarrow \mathbb{P}(W_1 \cdot x = y_1)$$

almost surely when  $n \rightarrow \infty$ . Since

$$\mathbb{P}(W_1 \cdot x = y_1 | W_n \cdot x = y_n) = \frac{P_\mu(x, y_1) P_\mu^{n-1}(y_1, y_n)}{P_\mu^n(x, y_n)},$$

we have for  $y_1 = s \cdot x$ ,

$$\lim_{n \rightarrow \infty} \frac{P_\mu^{n-1}(s \cdot x, W_n \cdot x)}{P_\mu^n(x, W_n \cdot x)} = 1 \text{ a.s.}$$

Then it implies for any  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mu^{(n)} \left( \left\{ g : \left| \frac{P_\mu^{n-1}(s \cdot x, g \cdot x)}{P_\mu^n(x, g \cdot x)} - 1 \right| \geq \epsilon \right\} \right) = 0.$$

It follows that

$$\lim_{n \rightarrow \infty} \left\| P_\mu^{n-1}(s \cdot x, \cdot) - P_\mu^n(x, \cdot) \right\|_1 = 0.$$

Finally, by Theorem 14.16 in [7], laziness of  $P_\mu$  guarentees that

$$\left\| P_\mu^{n-1}(s \cdot x, \cdot) - P_\mu^n(s \cdot x, \cdot) \right\|_1 \rightarrow 0$$

when  $n \rightarrow \infty$ . □

In the other direction, we can build a measure  $\mu$  on  $G$  such that  $P_\mu$  is Liouville from pieces that have good coupling properties.

**Lemma 2.** Suppose there exists an increasing sequence of finite subsets  $(K_n)$  exhausting  $X$  and a sequence  $(\epsilon_n)$  decreasing to 0 such that for each  $n$ , there exists a probability measure  $\nu_n$  of finite support on  $G$  such that for any  $x, y \in K_n$  such that  $y = s \cdot x$  for some  $s \in S$ , we have

$$\|\nu_n \cdot x - \nu_n \cdot y\|_1 < \epsilon_n.$$

Then there exists a non-degenerate probability measure  $\mu$  on  $G$  such that  $(X, P_\mu)$  is Liouville.

*Proof.* Our proof is reminiscent to the proof of Theorem 4.3 in [6]. The measure  $\mu$  is obtained as a convex combination of a subsequence of  $(\nu_n)$ ,

$$\mu = \sum_{j=0}^{\infty} c_j \zeta_j, \quad \zeta_j = \nu_{n_j}$$

To make  $\mu$  non-degenerate, we take  $\nu_0$  to be uniform on  $S \cup S^{-1}$ .

First note that to show  $P_\mu$  is Liouville, it suffices to show for any  $x, y \in X$  connected by an edge,  $y = s \cdot x$  for some  $s \in S$ , we have

$$\liminf_{m \rightarrow \infty} \left\| \mu^{(m)} \cdot x - \mu^{(m)} \cdot y \right\|_1 = 0.$$

(This implies that for any bounded  $P_\mu$ -harmonic function  $h$ ,  $h(x) = h(y)$  for neighboring points, thus the function must be constant.)

Since each  $\nu_n$  is assumed to be of finite support on  $G$ , let  $B(e, r_n)$  be a ball large enough on  $G$  such that  $\text{support} \nu_n \subset B(e, R_n)$ . Let  $r_n$  be the largest ball such that  $B(o, r_n) \subseteq K_n$ , we have  $r_n \rightarrow \infty$  since  $(K_n)$  exhausts  $X$ . Fix a consequence of weights  $(c_j)$ , select  $(n_j)$  inductively as follows: let  $m_j$  be the smallest integer such that  $(c_0 + \dots + c_{j-1})^{m_j} \leq 1/j$ , take  $n_j$  to be the least integer such that

$$m_j R_{n_{j-1}} \leq r_{n_j} \text{ and } m_j R_{n_{j-1}} \epsilon_{n_j} \leq 1/j.$$

For  $m$ -th convolution power of  $\mu$ ,

$$\mu^{(m)} = \sum_k c_{k_1} \dots c_{k_m} \zeta_{k_m} * \dots * \zeta_{k_1}.$$

Consider two parts,  $\mu_1^{(m)}$  consists these terms with  $\max_{1 \leq i \leq m} k_i \geq j$ , and  $\mu_2^{(m)} = \mu^{(m)} - \mu_1^{(m)}$ . For  $m = m_j$ , the total mass of the first part is

$$\left\| \mu_1^{(m_j)} \right\|_1 = (c_1 + \dots + c_{j-1})^{m_j}.$$

For each term in the second part, let  $i = i(k)$  be the lowest index such that  $k_i \geq j$ . Starting at two neighboring points  $x, y$ , consider the distribution induced by  $\zeta_{k_i} * \dots * \zeta_{k_1}$  on these two points. Since for  $i < k$ ,  $k_i < j$ , it follows that the support of  $\zeta_{k_{i-1}} * \dots * \zeta_{k_1} \cdot x$  and  $\zeta_{k_{i-1}} * \dots * \zeta_{k_1} \cdot y$  are contained in the ball  $B_X(o, d(o, x) + 1 + (m_j - 1)R_{n_{j-1}})$ . By the choice of the  $(\zeta_n)$ , we have for  $k_i \geq j$ ,

$$\left\| \zeta_{k_i} * \zeta_{k_{i-1}} * \dots * \zeta_{k_1} \cdot x - \zeta_{k_i} * \zeta_{k_{i-1}} * \dots * \zeta_{k_1} \cdot y \right\|_1 \leq 2(m_j - 1)R_{n_{j-1}} \epsilon_{n_j}.$$

Combine the two parts, we have

$$\left\| \mu^{(m_j)} \cdot x - \mu^{(m_j)} \cdot y \right\|_1 \leq (c_1 + \dots + c_{j-1})^{m_j} + 2(m - 1)R_{\ell_{m-1}} \epsilon_{\ell_m} \leq 3/j$$

□

### 3. APPLICATIONS TO THE THOMPSON GROUP $F$

We denote by  $F$  the Thompson group  $F$ . In [5] and [8], authors show that the Schreier graph of the action of  $F$  on the orbit of  $1/2$  is not Liouville with respect to any measure of finite support. Here we show that there are measures with infinite support that make this action Liouville. In fact, one can even choose symmetric ones.

**Theorem 3.** There is a non-degenerate symmetric measure  $\mu$  on Thompson group  $F$  such that the action on  $Orb(1/2)$  is  $\mu$ -Liouville.

*Proof.* By Lemma 2, we have to find a measure that approximates any finite subset in  $Orb(1/2)$ . The Schreier graph of  $Orb(1/2)$  was described by Savchuk, see [9]. There are two parts of the graph: the one that corresponds to the binary tree and another is rays attached to every node of the tree. These rays imitate positive part of the Cayley graph of  $\mathbb{Z}$ , and we will call them hairs.

Let  $K \subset Orb(1/2)$ . Since  $F$  is strongly transitive (see [1]), we can find an element  $g$  that maps this set to the hair. We can assume that this set is mapped deep enough into the hair. One of the generators, say  $g_0$  (in the notations of Savchuk), act as  $\mathbb{Z}$ . The set  $g(K)$  might not be connected, but it is clear that the uniform measure on  $\{g_0^k : k \in [-n, n]\}$  will satisfy Lemma 2 for sufficiently large  $n$ , therefore the uniform measure on  $\{g_0^k g : k \in [-n, n]\}$  is Liouville. We can make it symmetric by taking  $\{g^{-1} g_0^k g : k \in [-n, n]\}$  □

### 4. LIOUVILLE NON-AMENABLE SCHREIER GRAPHS AND LAMPLIGHTERS

As another application of Lemma 2 we show that certain Schreier graphs of lamplighter group are non-amenable but  $\mu$ -Liouville for some infinitely supported measure  $\mu$ . We note that examples of non-amenable graphs with Liouville measures were previously discovered in [2]. There are many examples when a non-amenable group  $G$  admits an action on a set  $X$  such that the induced action of the lamplighter group  $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$  on  $\bigoplus_X \mathbb{Z}/2$  is not amenable. In fact, if the action of  $G$  on  $X$  is not amenable then the action of  $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$  on  $\bigoplus_X \mathbb{Z}/2$ , see for example [3], [4]. However, this action always admits Liouville measure.

**Lemma 4.** Consider the semi-direct product  $G \rtimes A$  with  $G$  discrete and  $A$  amenable. Then there exists a non-degenerate probability measure  $\mu$  on  $G \rtimes A$  such that the action of  $G \rtimes A$  on  $A$  is  $\mu$ -Liouville.

*Proof.* Fix a sequence of finite subsets  $(K_n)$  that exhaust  $A$ . Since  $A$  is amenable, we can find a sequence of measures  $(\nu_n)$  on  $A$  such that

$$\sup_{x, y \in K_n} \|\nu_n \cdot x - \nu_n \cdot y\|_{\ell^1(A)} \leq \frac{1}{n}.$$

Regard  $\nu_n$  as a measure supported on  $\{(e_G, a) : a \in A\}$ , then by Lemma 2 we can find a non-degenerate measure  $\mu$  on  $G \rtimes A$  such that  $P_\mu$  is Liouville. □

Let  $G$  be an amenable group. The every action of  $G$  admits a Liouville measure (possibly infinitely supported), moreover, the stabilizers of the action are amenable.

While the following questions should have a negative answer, we currently don't have any examples to support it.

**Question 5.** Let  $G$  act transitively on a set  $X$  and assume that this action is  $\mu$ -Liouville action of  $G$  on  $X$  for some measure  $\mu$  on  $G$  such that  $Stab_G(x)$  is abelian for some (equivalently for all)  $x$  in  $X$ . Is  $G$  amenable?

**Question 6.** Let  $G$  act transitively on a set  $X$  and assume that this action is  $\mu$ -Liouville action of  $G$  on  $X$  for some measure  $\mu$  on  $G$ . Denote by  $X_n$  the set of all finite subsets of size  $n$ . Then  $G$  acts on  $X_n$ , however, this action may not be transitive. Is it true that the action of  $G$  on orbit of  $x \in X_n$  is Liouville for some measure?

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