

NOTES ON EQUIVARIANT HIGHER CHOW GROUPS

NGUYEN MANH TOAN

ABSTRACT. In this short note, we prove a comparison theorem between Levine-Serpé's equivariant higher Chow groups of an algebraic variety equipped with an action of a finite group and ordinary higher Chow groups of its fixed points. As a consequence, we show that the equivariant motivic spectral sequence degenerates rationally. This yields a Riemann-Roch Theorem for equivariant algebraic K -theory.

1. INTRODUCTION

Let k be a field and G a finite group of order coprime to the exponential characteristic of k . Levine-Serpé have defined in [11] equivariant higher Chow groups $CH_p(G, X, r)$ for any separated noetherian scheme X which is essentially of finite type over k equipped with an action of G (see Definition 2.1 below). They are a generalization of Bloch's higher Chow groups in the equivariant setting. Equivariant higher Chow groups can be considered as a motivic Borel-Moore homology theory on the category of G -schemes over k . Moreover, these groups have a close relation with the higher K -theory of G -equivariant coherent sheaves on X (cf. [12]) by a spectral sequence

$$(1.1) \quad E_1^{p,q} = CH_{-p}(G, X, -p - q) \Rightarrow G_{-p-q}(G, X)$$

([11, Corollary 3.8]). In the case of the trivial group, this reduces to the motivic spectral sequence constructed by Bloch-Lichtenbaum [3], Friedlander-Suslin [5] and Levine [8].

The groups $CH_p(G, X, r)$ are interesting objects which contain information about X as well as the action of G on X . They are, of course, very hard to compute in general.

In this note, we establish the following reconstruction theorem for equivariant higher Chow groups.

Theorem (Theorem 3.3). Let n be the order of G and assume that k contains n -th roots of unity. For any G -scheme X over k , there is a natural isomorphism of $\mathbb{Z}[1/n]$ -modules

$$(1.2) \quad \pi(X) : \prod_{\sigma \in \Gamma} (CH_p(X^\sigma, r) \otimes \tilde{R}\sigma)^{N_G(\sigma)} \xrightarrow{\sim} CH_p(G, X, r) \otimes \mathbb{Z}[1/n]$$

where Γ is a set of representatives for the conjugacy classes of cyclic subgroups of G , X^σ is the *fixed point subscheme* of X under the action of σ , $\tilde{R}\sigma$ is the m -th cyclotomic subring of the representation ring $R\sigma$ tensored with $\mathbb{Z}[1/n]$ where m is the order of σ and $N_G(\sigma)$ is the *normalizer* of σ in G .

The left-hand side of (1.2) can be easily expressed in the form of

$$\left(\prod_{g \in G} CH_p(X^g, r) \otimes \mathbb{Z}[1/n] \right)^G.$$

For X smooth, $r = 0$ with rational coefficients, $(\prod_{g \in G} CH_*(X^g, 0) \otimes \mathbb{Q})^G$ is exactly the (*small orbifold Chow ring* $CH_*^{orb}([X/G])$) of the global quotient $[X/G]$ studied by Abramovich-Graber-Vistoli [1] and Jarvis-Kaufmann-Kimura [7] which is an algebraic version of Chen-Ruan cohomology.

Theorem 3.3 is an analog of Vistoli's theorem for equivariant algebraic K -theory [13, Theorem 1 and 2] and Segal's theorem for equivariant topological K -theory [6]. Using this result we show that the spectral sequence (1.1) degenerates with rational coefficients for X smooth which yields a Riemann-Roch theorem for equivariant algebraic K -theory.

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Theorem (Theorem 4.4). If X is a smooth G -scheme over k , there is a natural isomorphism of \mathbb{Q} -vector spaces

$$(1.3) \quad \bigoplus_{p,q} CH_p(G, X, q) \otimes \mathbb{Q} \cong \bigoplus_q K_q(G, X) \otimes \mathbb{Q}.$$

Moreover, Theorem 3.3 provides two multiplicative structures on rational equivariant higher Chow groups. Hence we obtain complete answers to questions proposed in [11, Introduction].

This paper is organized as follows. In Section 2, we will briefly review Levine-Serpé's construction of equivariant higher Chow groups and discuss various functorial properties. Most of the things can be found in [11] except the construction and functorialities of the *induction map* that is new. In Section 3, we establish the comparison morphism from higher Chow groups of fixed point subschemes to equivariant higher Chow groups. This is modelled by Vistoli's construction for the K groups [13, Section 3]. Using localization property for (equivariant) higher Chow groups, we will show that the comparison morphism is an isomorphism. In Section 4, we will lift Vistoli's construction to the K -theory spectra. This allows us to compare the equivariant motivic spectral sequence with the ordinary one. From there, some applications will be derived, including a Riemann-Roch theorem for equivariant algebraic K -theory and multiplicative structures on equivariant higher Chow groups with rational coefficients. We will also see how to understand Levine-Serpé's equivariant higher Chow groups from orbifold theories.

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Notation and conventions. Throughout this paper, k is a fixed base field and G is a finite group whose order is coprime to the exponential characteristic of k .

We write \mathbf{Sch}_k^G for the category whose objects are separated schemes essentially of finite type and quasi-projective over k equipped with a left G -action and morphisms are G -equivariant morphisms. Set $\Lambda := \mathbb{Z}[1/|G|]$ for our coefficient ring.

For any $X \in \mathbf{Sch}_k^G$, the *abelian category* of G -equivariant coherent sheaves on X is denoted by $\mathbf{Coh}_G(X)$ and the *exact category* of G -equivariant vector bundles on X is denoted by $\mathbf{Vect}_G(X)$. In the case of the trivial group G , we denote $\mathbf{Coh}(X)$ for $\mathbf{Coh}_G(X)$.

The connected K -theory spectra of $\mathbf{Coh}_G(X)$ and $\mathbf{Vect}_G(X)$ are denoted by $G(G, X)$ and $K(G, X)$, respectively. We will use the notation $G_i(G, X)$ to denote the i -th homotopy group of $G(G, X)$ and similarly for $K(G, X)$. When X is smooth, the natural inclusion $\mathbf{Vect}_G(X) \hookrightarrow \mathbf{Coh}_G(X)$ induces an isomorphism $K_i(G, X) \xrightarrow{\sim} G_i(G, X)$. This identification will be used frequently in this paper.

For any group H , the ring of representations of H over k tensored with Λ is denoted by RH , i.e., $RH := K_0(H, \text{Spec}(k)) \otimes \Lambda$.

2. EQUIVARIANT HIGHER CHOW GROUPS

We recall Levine-Serpé's definition of equivariant higher Chow groups [11] and discuss some of their properties which will be used in the following sections.

2.1. Construction. Let Δ^\bullet be the standard cosimplicial scheme with

$$\Delta^r := \text{Spec}(k[t_0, \dots, t_r] / (\sum t_i = 1))$$

equipped with the trivial action of G . A *face* of Δ^r is a closed subscheme defined by $t_{i_1} = \dots = t_{i_j} = 0$.

For any $X \in \mathbf{Sch}_k^G$, we set

$$S_{(p)}^{G,X}(r) := \left\{ W \subset X \times \Delta^r \left| \begin{array}{l} W \text{ is a closed } G\text{-stable subset} \\ \text{and } \dim_{X \times F}(W \cap X \times F) \leq p + \dim F \\ \text{for all faces } F \subset \Delta^r \end{array} \right. \right\}.$$

The group G acts obviously on the set $(X \times \Delta^r)_{(p+r)}$ of dimension $(p+r)$ points on $X \times \Delta^r$. Let

$$X_{(p)}^G(r) := \{[x] \in (X \times \Delta^r)_{(p+r)}/G \mid \overline{G \cdot x} \in S_{(p)}^{G,X}(r)\}$$

where $\overline{G \cdot x}$ stands for the closure of the orbit $G \cdot x$ in $X \times \Delta^r$.

We define

$$z_p(G, X, r) := \bigoplus_{[x] \in X_{(p)}^G(r)} K_0(G_x, \text{Spec}(k(x))),$$

where $k(x)$ is the residue field of x and G_x is the *set-theoretic stabilizer group* of x in G , i.e., $G_x := \{g \in G \mid gx = x\}$. The assignment $r \rightarrow z_p(G, X, r)$ forms a simplicial abelian group ([11, Proposition 3.2]) which is denoted by $z_p(G, X, \bullet)$.

Definition 2.1. [11, Definition 3.4] The *equivariant cycle complex (of Bredon type)* $z_p(G, X, *)$ is the complex associated to $z_p(G, X, \bullet)$. The *equivariant higher Chow groups (of Bredon type)* are defined by

$$CH_p(G, X, r) := H_r(z_p(G, X, *)).$$

In the case of the trivial group, we recover the set $X_{(p)}(r)$ of dimension $(p+r)$ subvarieties of $X \times \Delta^r$ meeting all faces properly, the cycle complex $z_p(X, *)$ and the higher Chow group $CH_p(X, r)$ defined in [2] (or [10]). Note that $CH_p(X, 0)$ is exactly the usual Chow group $CH_p(X)$ of dimension p cycles on X .

When G acts *trivially* on X , there is a natural isomorphism

$$K_0(X) \otimes K_0(G, \text{Spec}(k)) \cong K_0(G, X)$$

(cf. [13, Proposition 1.6]) which yields an isomorphism

$$z_p(X, r) \otimes RG \cong z_p(G, X, r) \otimes \Lambda.$$

Therefore,

$$(2.1) \quad CH_p(X, r) \otimes RG \cong CH_p(G, X, r) \otimes \Lambda.$$

Since X is quasi-projective and G is finite, the quotient X/G exists as a scheme. If G acts *freely* on X , then $X_{(p)}^G(r) = (X/G)_{(p)}(r)$ and $K_0(G_x, \text{Spec } k(x)) = K_0(\text{Spec } k(x)) \cong \mathbb{Z}$. This gives a natural isomorphism

$$z_p(G, X, r) \cong z_p(X/G, r),$$

and hence

$$CH_p(G, X, r) \cong CH_p(X/G, r).$$

2.2. Functoriality with respect to \mathbf{X} . If $f : Y \rightarrow X$ is a *proper* G -equivariant morphism in \mathbf{Sch}_k^G , there is the *push-forward homomorphism*

$$f_*(r) : z_p(G, Y, r) \rightarrow z_p(G, X, r)$$

defined by

$$[\alpha \in K_0(G_z, \text{Spec}(k(Z)))] \mapsto [(f \times \text{id})_*(\alpha) \in K_0(G_{(f \times \text{id})(z)}, \text{Spec}(k((f \times \text{id})(Z))))]$$

if $Z \mapsto (f \times \text{id})(Z)$ is generically finite and sending α to zero if not. These maps form a simplicial map

$$f_*(-) : z_p(G, Y, -) \rightarrow z_p(G, X, -)$$

which yields the *push-forward map* for equivariant higher Chow groups

$$(2.2) \quad f_* : CH_p(G, Y, r) \rightarrow CH_p(G, X, r).$$

If $f : Y \rightarrow X$ is a *flat* G -equivariant morphism of relative dimension d , the *pull-back homomorphism*

$$f^*(r) : z_p(G, X, r) \rightarrow z_{p+d}(G, Y, r)$$

is given by

$$[\alpha \in K_0(G_z, \text{Spec}(k(Z)))] \rightarrow [(f \times \text{id})^*(\alpha) \in \bigoplus_{[y] \in Y_{(p+d)}^G(r), (f \times \text{id})(y) \in G \cdot x} K_0(G_y, \text{Spec}(k(y)))].$$

These maps form a simplicial map

$$f^*(-) : z_p(G, X, -) \rightarrow z_{p+d}(G, Y, -)$$

which yields the *pull-back map* for equivariant higher Chow groups

$$(2.3) \quad f^* : CH_p(G, X, r) \rightarrow CH_p(G, Y, r).$$

For proper composable morphisms f and g , we have $(f \circ g)_* = f_* \circ g_*$. If f and g are flat composable morphisms then $(f \circ g)^* = g^* \circ f^*$. Moreover, if

$$\begin{array}{ccc} W & \xrightarrow{f'} & Z \\ \downarrow g' & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

is a G -equivariant cartesian squares in which f is proper and g is flat then $g^* \circ f_* = f'^* \circ g'_*$. These properties are well-known for (equivariant) algebraic K -theory and higher Chow groups. See [11, 3.2] for more details.

2.3. Functoriality with respect to G . If $\phi : H \rightarrow G$ is a group homomorphism and $X \in \mathbf{Sch}_k^G$ is a G -scheme then ϕ gives an action of H on X . The collection of maps

$$\phi_x^* : K_0(G_x, \text{Spec}(k(x))) \rightarrow K_0(H_x, \text{Spec}(k(x)))$$

induces the map of simplicial abelian groups

$$\phi^* : z_p(G, X, r) \rightarrow z_p(H, X, r)$$

which yields the *restriction map*

$$\phi^* : CH_p(G, X, r) \rightarrow CH_p(H, X, r).$$

It is easy to see that the maps ϕ^* are natural with respect to the proper push-forward and flat pull-back (cf. [11, Section 3.2]).

We now want to construct a group homomorphism going in the other direction when $H \subset G$ be a subgroup.

Recall that the *induction morphism* for equivariant algebraic K -theory

$$\text{Ind}_H^G : K_0(H, X) \rightarrow K_0(G, X)$$

is the push-forward

$$\pi_* : K_0(G, (X \times G)/H) \rightarrow K_0(G, X)$$

along the (G -equivariant) projection $\pi : (X \times G)/H \rightarrow X$ together with a natural identification

$$K_0(G, (X \times G)/H) \cong K_0(H, X)$$

where the (free) action of H on $X \times G$ is given by

$$h(x, g) = (hx, gh^{-1}).$$

For more details, see [13, Section 2].

The analogous operations are valid for equivariant cycle complex. If $[x] \in (X \times \Delta^r)_{(p+r)}/H$ such that $\overline{H.x} \in S_{(p)}^{H, X}(r)$ then $\overline{G.x} \in S_{(p)}^{G, X}(r)$. It is obvious that each orbit $H.x$ gives rise to an orbit $G.x$ and the stabilizer group H_x is a subgroup of G_x . Therefore, we have the induction map for equivariant K -theory:

$$\text{Ind}_{H_x}^{G_x}(\text{Spec}(k(x))) : K_0(H_x, \text{Spec}(k(x))) \rightarrow K_0(G_x, \text{Spec}(k(x))).$$

Take the sum over all orbits on both sides to obtain a map on the elements of the equivariant cycle complexes

$$\text{Ind}_H^G(X, r) : z_p(H, X, r) \rightarrow z_p(G, X, r).$$

The maps $\text{Ind}_H^G(X, r)$ are 'push-forward' maps, so they form a simplicial map

$$\text{Ind}_H^G(X, -) : z_p(H, X, -) \rightarrow z_p(G, X, -)$$

which yields the *induction map* for equivariant higher Chow groups

$$(2.4) \quad \text{Ind}_H^G(X) : CH_p(H, X, r) \rightarrow CH_p(G, X, r).$$

Lemma 2.2. (1) *If $f : Y \rightarrow X$ is a flat G -morphism of G -schemes, then for any subgroup $H \subset G$ the diagram*

$$\begin{array}{ccc} CH_p(H, X, r) & \xrightarrow{\text{Ind}_H^G(X)} & CH_p(G, X, r) \\ \downarrow f_* & & \downarrow f_* \\ CH_p(H, Y, r) & \xrightarrow{\text{Ind}_H^G(Y)} & CH_p(G, Y, r) \end{array}$$

commutes.

(2) *If $g : Y \rightarrow X$ is a proper G -morphism of G -schemes, then for any subgroup $H \subset G$ the diagram*

$$\begin{array}{ccc} CH_p(H, Y, r) & \xrightarrow{\text{Ind}_H^G(Y)} & CH_p(G, Y, r) \\ \downarrow g_* & & \downarrow g_* \\ CH_p(H, X, r) & \xrightarrow{\text{Ind}_H^G(X)} & CH_p(G, X, r) \end{array}$$

commutes.

Proof. The proof is straightforward, using the fact that the induction morphism

$$\text{Ind}_H^G(X) : CH_p(H, X, r) \rightarrow CH_p(G, X, r).$$

is the push-forward

$$\pi_* : CH_p(G, (X \times G)/H, r) \rightarrow CH_p(G, X, r).$$

along the (G -equivariant) projection $\pi : (X \times G)/H \rightarrow X$, together with an identification

$$CH_p(G, (X \times G)/H, r) \cong CH_p(H, X, r).$$

□

2.4. Equivariant motivic spectral sequence. Equivariant higher Chow groups enjoy certain good properties as their ordinary counterparts.

Proposition 2.3 (Localization theorem). [11, Theorem 4.1] *Let $X \in \mathbf{Sch}_k^G$, $i : W \rightarrow X$ a G -invariant closed subscheme with open complement $j : U \rightarrow X$. Then for each p , there is a long exact sequence*

$$\dots \rightarrow CH_p(G, W, r) \xrightarrow{i^*} CH_p(G, X, r) \xrightarrow{j^*} CH_p(G, U, r) \xrightarrow{\delta} CH_p(G, W, r-1) \rightarrow \dots$$

Using this, Levine and Serpé show that

Proposition 2.4 (Equivariant motivic spectral sequence). [11, Corollary 3.8] *Let $X \in \mathbf{Sch}_k^G$. There is a strongly convergent spectral sequence*

$$(2.5) \quad E_1^{p,q} = CH_{-p}(G, X, -p-q) \Rightarrow G_{-p-q}(G, X).$$

3. RECONSTRUCTION THEOREM

From now on, we will assume that k contains all the n -th roots of unity where $n := |G|$. With our hypotheses, the ring RG only depends on the characteristic of k . In this section, we will investigate the relation between equivariant higher Chow groups and the ordinary higher Chow groups of fixed point subschemes under cyclic subgroups.

3.1. Comparison morphism. For any cyclic subgroup $\sigma \subset G$ of order m , let t be the generator of the dual group $\hat{\sigma}$ of homomorphisms $\sigma \rightarrow k^*$. We have

$$R\sigma \cong \Lambda\hat{\sigma} \cong \Lambda[t]/(t^m - 1) \cong \prod_{d|m} \Lambda[t]/(\Phi_d(t))$$

where $\Phi_d(t)$ is the d -th *cyclotomic polynomial*. Denote by $\tilde{R}\sigma$ the factor of $R\sigma$ corresponding to $\Lambda[t]/(\Phi_m(t))$ which is independent of the choice of the generator t . The *normalizer* $N(\sigma)$ of σ acts naturally on $R\sigma$ and $\tilde{R}\sigma$. Moreover, $N(\sigma)$ acts on the *fixed point subscheme* X^σ of X under σ hence acts on $CH_*(X^\sigma, *)$. Note that by our assumption, the functor $- \otimes \tilde{R}\sigma$ is exact on the category of Λ -modules and for any subgroup $H \subset G$, taking H -invariants $(-)^H$ is an exact functor from the category of ΛH -modules to the category of Λ -modules.

Let Γ be a set of representatives for the conjugacy classes of cyclic subgroups of G . We define the morphism

$$(3.1) \quad \pi(X) : \prod_{\sigma \in \Gamma} (CH_p(X^\sigma, r) \otimes \tilde{R}\sigma)^{N(\sigma)} \rightarrow CH_p(G, X, r) \otimes \Lambda$$

as the composition of the inclusion

$$\prod_{\sigma \in \Gamma} (CH_p(X^\sigma, r) \otimes \tilde{R}\sigma)^{N(\sigma)} \hookrightarrow \prod_{\sigma \in \Gamma} (CH_p(X^\sigma, r) \otimes R\sigma)^{N(\sigma)}$$

given by the embedding $\tilde{R}\sigma \hookrightarrow R\sigma$, the isomorphism

$$\prod_{\sigma \in \Gamma} (CH_p(X^\sigma, r) \otimes R\sigma)^{N(\sigma)} \xrightarrow{\sim} \prod_{\sigma \in \Gamma} ((CH_p(\sigma, X^\sigma, r)) \otimes \Lambda)^{N(\sigma)}$$

given by (2.1), the obvious inclusion

$$\prod_{\sigma \in \Gamma} (CH_p(\sigma, X^\sigma, r) \otimes \Lambda)^{N(\sigma)} \hookrightarrow \prod_{\sigma \in \Gamma} CH_p(\sigma, X^\sigma, r) \otimes \Lambda,$$

the push-forward

$$\prod_{\sigma \in \Gamma} CH_p(\sigma, X^\sigma, r) \otimes \Lambda \rightarrow \prod_{\sigma \in \Gamma} CH_p(\sigma, X, r) \otimes \Lambda$$

along closed embeddings $X^\sigma \hookrightarrow X$, and the product of induction maps (2.4)

$$\prod_{\sigma \in \Gamma} CH_p(\sigma, X, r) \otimes \Lambda \rightarrow CH_p(G, X, r) \otimes \Lambda.$$

Replace equivariant higher Chow groups by the K -groups of equivariant cohenrent sheaves everywhere, we obtain the homomorphism

$$(3.2) \quad \alpha_*(X) : \prod_{\sigma \in \Gamma} (G_*(X^\sigma) \otimes \tilde{R}\sigma)^{N(\sigma)} \rightarrow G_*(G, X) \otimes \Lambda$$

defined by Vistoli [13, Section 3]. We will show that $\pi(X)$ is an isomorphism of Λ -modules.

3.2. A simple case.

Proposition 3.1. *Let $X := \coprod_y \text{Spec}(k(y))$ be a 0-dimensional scheme where G acts transitively on a set of points then*

$$(3.3) \quad \alpha_0(X) : \prod_{\sigma \in \Gamma} (K_0(X^\sigma) \otimes \tilde{R}\sigma)^{N(\sigma)} \rightarrow K_0(G, X) \otimes \Lambda$$

is an isomorphism.

Proof. If G acts trivially on X , then $X = \text{Spec}(F)$ for a field extension F/k . In this case, both sides of (3.3) are free Λ -modules whose rank are the number of conjugacy classes of G . The injectivity of $\alpha_0(\text{Spec}(F))$ is proved in [13, Proposition 1.5].

In general, we fix an arbitrary element x . Since G acts transitively on the index set, we have

$$K_0(G, X) = K_0(G_x, \text{Spec } k(x)).$$

Let \mathcal{C} be the set of cyclic subgroups of G and $\mathcal{D}(y)$ the set of cyclic subgroups of G_y . We also have

$$\prod_{\sigma \in \mathcal{C}} K_0(X^\sigma) \cong \prod_y \prod_{\sigma \in \mathcal{D}(y)} K_0(\text{Spec } k(y))^\sigma.$$

Denote Ψ for a set of representatives for the conjugacy classes of cyclic subgroups of G_x and let $\mathcal{D} := \mathcal{D}(x)$. By Lemma 3.2 below, there are isomorphisms

$$\begin{aligned} \prod_{\sigma \in \Gamma} (K_0(X^\sigma) \otimes \tilde{R}\sigma)^{N_G(\sigma)} &\cong \left(\prod_{\sigma \in \mathcal{C}} K_0(X^\sigma) \otimes \tilde{R}\sigma \right)^G \\ &\cong \left(\prod_y \prod_{\sigma \in \mathcal{D}} K_0(\text{Spec } k(x)^\sigma) \otimes \tilde{R}\sigma \right)^G \\ &\cong \left(\prod_{\sigma \in \mathcal{D}} K_0((\text{Spec } k(x))^\sigma) \otimes \tilde{R}\sigma \right)^{G_x} \\ &\cong \prod_{\sigma \in \Psi} (K_0((\text{Spec } k(x))^\sigma) \otimes \tilde{R}\sigma)^{N_{G_x}(\sigma)}. \end{aligned}$$

Remark that if σ acts non-trivially on $\text{Spec } k(x)$ then $(\text{Spec } k(x))^\sigma = \emptyset$. Hence, by replacing G with G_x , X with $\text{Spec } k(x)$ and changing notation, one reduces the statement to the case of a point, i.e., to prove that

$$(3.4) \quad \prod_{\sigma \in \Gamma'} (K_0(\text{Spec } F) \otimes \tilde{R}\sigma)^{N_G(\sigma)} \cong K_0(G, \text{Spec } F) \otimes \Lambda$$

where F/k is a finitely generated field extension and $\Gamma' \subset \Gamma$ is the subset consisting of cyclic subgroups of G which act *trivially* on F .

Denote $G_{(F)}$ for the *inertia group* of $\text{Spec } F$, i.e.,

$$G_{(F)} := \ker(G \rightarrow \text{Aut}(F/k)),$$

and \mathcal{I} for the set of cyclic subgroup of $G_{(F)}$. We have

$$\begin{aligned} \prod_{\sigma \in \Gamma'} (K_0(\text{Spec } F) \otimes \tilde{R}\sigma)^{N_G(\sigma)} &\cong \left(\prod_{\sigma \in \mathcal{I}} K_0(\text{Spec } F) \otimes \tilde{R}\sigma \right)^G \\ &\cong \left(\left(\prod_{\sigma \in \mathcal{I}} K_0(\text{Spec } F) \otimes \tilde{R}\sigma \right)^{G_{(F)}} \right)^{G/G_{(F)}} \\ &\cong (K_0(G_{(F)}, \text{Spec } F) \otimes \Lambda)^{G/G_{(F)}} \\ &\cong K_0(G, \text{Spec } F) \otimes \Lambda. \end{aligned}$$

The first isomorphism holds by Lemma 3.2. The third isomorphism holds because $G_{(F)}$ acts trivially on F . The last isomorphism is a special case of the descent property for equivariant algebraic K -theory with $\Lambda = \mathbb{Z}[1/n]$ -coefficients. Hence (3.4) is an isomorphism as claimed. \square

Lemma 3.2. *Let the group G act on the left on a set \mathcal{S} and let \mathcal{T} be a set of representatives for the orbits. Assume that G acts on the left on a product of abelian groups of the type $\prod_{s \in \mathcal{S}} M_s$ in such a way that for any $g \in G$*

$$gM_s = M_{gs}.$$

For each $t \in \mathcal{T}$ let G_t be the stabilizer of t in G . Then the canonical projection

$$\prod_{s \in \mathcal{S}} M_s \rightarrow \prod_{t \in \mathcal{T}} M_t$$

induces an isomorphism

$$\left(\prod_{s \in \mathcal{S}} M_s \right)^G \rightarrow \prod_{t \in \mathcal{T}} (M_t)^{G_t}.$$

Proof. It is straightforward. \square

In particular, if \mathcal{C} is the set of cyclic subgroups of G and Γ is a set of representatives for the conjugacy classes of cyclic subgroups of G , then

$$\prod_{\sigma \in \Gamma} (K_0(X^\sigma) \otimes \tilde{R}\sigma)^{N_G(\sigma)} \cong \left(\prod_{\sigma \in \mathcal{C}} K_0(X^\sigma) \otimes \tilde{R}\sigma \right)^G \cong \left(\prod_{g \in G} K_0(X^g) \otimes \Lambda \right)^G.$$

3.3. The general case.

Theorem 3.3 (Reconstruction Theorem). *The map $\pi(X)$ in (3.1) is an isomorphism for any $X \in \mathbf{Sch}_k^G$.*

Proof. Both sides of (3.1) satisfy localization and $\pi(-)$ is natural with localization. Indeed, by Lemma 2.2, each component of π is compatible with the push-forward i_* given by closed immersion $i : Z \hookrightarrow X$ and the pull-back j^* given by open immersion $j : U \hookrightarrow X$. Hence, we only need to show that $\pi(X)$ is an isomorphism for $X = \coprod_x \text{Spec}(k(x))$ when G acts transitively on a set of points.

By the same argument as in Proposition 3.1, we reduce our problem to the case of a point, i.e., $\pi(\text{Spec}(F))$ is an isomorphism for any finitely generated field extension F/k . In this case

$$(CH_p((\text{Spec } F)^\sigma, r) \otimes \tilde{R}\sigma)^{N(\sigma)} = (CH_p(\text{Spec } F), r) \otimes \tilde{R}\sigma^{N(\sigma)}$$

if σ acts trivially on $\text{Spec}(F)$ and

$$(CH_p((\text{Spec } F)^\sigma, r) \otimes \tilde{R}\sigma)^{N(\sigma)} = 0$$

otherwise.

Denote by Γ' the subset of Γ consisting of cyclic subgroups of G which act trivially on F . We have

$$\begin{aligned} \prod_{\sigma \in \Gamma'} (z_p(\text{Spec } F, r) \otimes \tilde{R}\sigma)^{N(\sigma)} &= \prod_{\sigma \in \Gamma'} \left(\bigoplus_{x \in (\text{Spec } F)_{(p)}(r)} \mathbb{Z}.x \otimes \tilde{R}\sigma \right)^{N(\sigma)} \\ &= \prod_{\sigma \in \Gamma'} \left(\bigoplus_{x \in (\text{Spec } F)_{(p)}(r)} K_0(\text{Spec } k(x)) \otimes \tilde{R}\sigma \right)^{N(\sigma)} \\ &= \prod_{\sigma \in \Gamma'} \left(\bigoplus_{[x] \in (\text{Spec } F)_p^G(r)} \bigoplus_{y \in G.x} K_0(\text{Spec } k(y)) \otimes \tilde{R}\sigma \right)^{N(\sigma)} \\ &= \prod_{\sigma \in \Gamma'} \left(\bigoplus_{[x]} K_0 \left(\prod_{y \in G.x} \text{Spec } k(y) \right) \otimes \tilde{R}\sigma \right)^{N(\sigma)} \\ &= \bigoplus_{[x]} \prod_{\sigma \in \Gamma'} \left(K_0 \left(\prod_{y \in G.x} \text{Spec } k(y) \right) \otimes \tilde{R}\sigma \right)^{N(\sigma)}. \end{aligned}$$

If $\sigma \notin \Gamma'$, i.e., σ acts non-trivially on F , then the scheme $\prod_{y \in G.x} \text{Spec } k(y)$ with $[x] \in (\text{Spec } F)_p^G(r)$ has no fixed point subscheme under σ . If $\sigma \in \Gamma'$ then σ acts trivially on $\prod_{y \in G.x} \text{Spec } k(y)$. Hence,

$$\begin{aligned} \bigoplus_{[x]} \prod_{\sigma \in \Gamma'} \left(K_0 \left(\prod_{y \in G.x} \text{Spec } k(y) \right) \otimes \tilde{R}\sigma \right)^{N(\sigma)} &= \bigoplus_{[x]} \prod_{\sigma \in \Gamma} \left(K_0 \left(\prod_{y \in G.x} \text{Spec } k(y) \right)^\sigma \otimes \tilde{R}\sigma \right)^{N(\sigma)} \\ &= \bigoplus_{[x]} K_0(G, \prod_{y \in G.x} \text{Spec } k(y)) \otimes \Lambda \text{ (by Proposition 3.1)} \\ &= \bigoplus_{[x]} K_0(G_x, \text{Spec } k(x)) \otimes \Lambda \\ &= z_p(G, \text{Spec } F, r) \otimes \Lambda. \end{aligned}$$

Take homologies of the associated complexes, one obtains isomorphisms

$$\prod_{\sigma \in \Gamma} (CH_p(\text{Spec } F, r) \otimes \tilde{R}\sigma)^{N(\sigma)} \cong CH_p(G, \text{Spec } F, r) \otimes \Lambda.$$

for each p and r . □

4. APPLICATIONS

Levine has constructed in [10] a spectral sequence

$$(4.1) \quad E_1^{p,q} = CH_{-p}(X, -p-q) \Rightarrow G_{-p-q}(X)$$

for any quasi-projective scheme X over k . In this section we will show that the isomorphism (3.1) is compatible with the spectral sequences (4.1) and (2.4). We then show that the equivariant motivic spectral sequence (2.4) degenerates rationally to obtain a Riemann-Roch theorem for equivariant algebraic K -theory.

4.1. **Comparison morphism revisited.** In Section 3 we defined the homomorphism (3.2)

$$\alpha_*(X) : \prod_{\sigma \in \Gamma} (G_*(X^\sigma) \otimes \tilde{R}\sigma)^{N(\sigma)} \rightarrow G_*(G, X) \otimes \Lambda.$$

We show now that this morphism can actually be expressed on the level of spectra.

Assume that H is a finite group which acts *trivially* on a k -scheme Y (in our application, σ acts trivially on X^σ). Let \mathcal{I} be the set of *irreducible representations* of H over k . For any representation ρ in \mathcal{I} with representation space V , we define

$$\rho^* : \mathbf{Coh}(Y) \rightarrow \mathbf{Coh}_H(Y)$$

by mapping any coherent sheaf \mathcal{F} on Y to $\mathcal{F} \otimes_k V$ with the action of H induced by the action of H on V . It is obvious that ρ^* is an exact functor, hence it induces a map between spectra

$$\rho^* : G(Y) \rightarrow G(H, Y).$$

Assume further that there is another group Π acting (possibly *non-trivially*) on Y and H in such a way that the action respects the group structure of H , i.e.,

$$\alpha(h_1, h_2) = \alpha h_1 \alpha h_2$$

for any $\alpha \in \Pi$ and $h_1, h_2 \in H$. This induces actions of Π on the spectra $G(Y)$, $G(H, Y)$ and on the set \mathcal{I} . Let \mathcal{J} be a subset of \mathcal{I} which is closed under the action of Π , then $\prod_{\rho \in \mathcal{J}} \mathbf{Coh}(Y)$ and $\mathbf{Coh}_H(Y)$ have a natural action of Π and the map

$$\prod_{\rho \in \mathcal{J}} \rho^* : \prod_{\rho \in \mathcal{J}} \mathbf{Coh}(X) \rightarrow \mathbf{Coh}_H(X)$$

is an Π -equivariant map. This induces an Π -equivariant map between spectra

$$(4.2) \quad \prod_{\rho \in \mathcal{J}} \rho^* : \prod_{\mathcal{J}} G(Y) \rightarrow G(H, Y).$$

Moreover, the inclusion $\mathcal{J} \hookrightarrow \mathcal{I}$ induces an Π -equivariant map

$$\prod_{\mathcal{J}} G(Y) \rightarrow \prod_{\mathcal{I}} G(Y)$$

and the following diagram

$$\begin{array}{ccc} \prod_{\mathcal{J}} G(Y) & \hookrightarrow & \prod_{\mathcal{I}} G(Y) \\ & \searrow & \downarrow \\ & & G(H, Y) \end{array}$$

commutes. Hence we obtain a commutative diagram between *homotopy fixed point spectra*

$$\begin{array}{ccc} (\prod_{\mathcal{J}} G(Y))^{h\Pi} & \hookrightarrow & (\prod_{\mathcal{I}} G(Y))^{h\Pi} \\ & \searrow & \downarrow \\ & & (G(H, Y))^{h\Pi} \end{array}$$

Replacing H by σ , Y by X^σ and Π by $N(\sigma)$. Since k contains enough roots of unity, it is well-known that the set of irreducible representations of σ over k has m elements where m is the order of σ and every irreducible representation of σ is one dimensional given by multiplication by a m -th root of unity. Let \mathcal{J} be the set of representations given by multiplication by a *primitive* m -th root of unity. It is clear that $|\mathcal{J}| = \deg(\Phi_m(t))$, the degree of the m -th cyclotomic ring $\Phi_m(t)$. Define the map

$$(4.3) \quad \alpha(X) : \prod_{\sigma \in \Gamma} \left(\prod_{\mathcal{J}} G(X^\sigma)[1/n] \right)^{hN(\sigma)} \rightarrow G(G, X)[1/n]$$

($n = |G|$ is inverted) as the composition of the inclusion

$$\prod_{\sigma} \left(\prod_{\mathcal{J}} G(X^\sigma)[1/n] \right)^{hN(\sigma)} \rightarrow \prod_{\sigma} \left(\prod_{\mathcal{I}} G(X^\sigma)[1/n] \right)^{hN(\sigma)}$$

given by the inclusion $\mathcal{J} \hookrightarrow \mathcal{I}$, the morphism

$$\prod_{\sigma} \left(\prod_{\mathcal{I}} G(X^\sigma)[1/n] \right)^{hN(\sigma)} \rightarrow \prod_{\sigma} (G(\sigma, X^\sigma)[1/n])^{hN(\sigma)}$$

given by (4.2), the natural morphism

$$\prod_{\sigma} (G(\sigma, X^\sigma)[1/n])^{hN(\sigma)} \rightarrow \prod_{\sigma} G(\sigma, X^\sigma)[1/n],$$

the push-forward

$$\prod_{\sigma} (G(\sigma, X^\sigma)[1/n]) \rightarrow \prod_{\sigma} G(\sigma, X)[1/n]$$

given by the closed embedding $X^\sigma \hookrightarrow X$, and the wedge sum of induction maps

$$\prod_{\sigma} G(\sigma, X)[1/n] \rightarrow G(G, X)[1/n].$$

Proposition 4.1. *The map $\alpha(X)$ of (4.3) induces $\alpha_*(X)$ of (3.2) on homotopy groups.*

Proof. We only have to show that

$$\pi_d \left(\prod_{\mathcal{J}} G(X^\sigma)[1/n] \right)^{hN(\sigma)} \cong (G_d(X^\sigma) \otimes \tilde{R}\sigma)^{N(\sigma)}$$

and

$$\pi_d(G(G, X)[1/n]) \cong G_d(G, X) \otimes \Lambda.$$

The last identity is clear by definition. For the first identity, note that if S is a spectrum with an action of a finite group G of order n , then there is a natural isomorphism

$$(4.4) \quad \pi_d(S^{hG}[1/n]) \cong (\pi_d S)^G \otimes \mathbb{Z}[1/n].$$

Indeed, the spectral sequence

$$E_2^{p,q} = H^{-p}(G, \pi_q(S) \otimes \mathbb{Z}[1/n]) \Rightarrow \pi_{p+q}(S^{hG}[1/n])$$

has $E_2^{p,q} = (\pi_q(S))^G \otimes \mathbb{Z}[1/n]$ if $p = 0$ and equals 0 otherwise. Moreover, it is easy to see that

$$(4.5) \quad \pi_d \left(\prod_{\mathcal{J}} G(X^\sigma)[1/n] \right) = \prod_{\mathcal{J}} G_d(X^\sigma) \otimes \mathbb{Z}[1/n] \cong G_d(X^\sigma) \otimes \tilde{R}\sigma$$

with a compatible action of $N(\sigma)$ on both sides. Our claim follows from (4.4) and (4.5). \square

4.2. A Riemann-Roch theorem. The way we define $\alpha(X)$ extends naturally to a family of compatible maps $\alpha_{(*)}(X)$ between the two towers

$$\begin{array}{ccc} \prod_{\sigma} \left(\prod_{\mathcal{J}} G_{(p)}(X^\sigma, -)[1/n] \right)^{hN(\sigma)} & \longrightarrow \dots \longrightarrow & \prod_{\sigma} \left(\prod_{\mathcal{J}} (X^\sigma)[1/n] \right)^{hN(\sigma)} \\ \downarrow \alpha_{(p)}(X) & & \downarrow \alpha(X) \\ G_{(p)}(G, X, -)[1/n] & \longrightarrow \dots \longrightarrow & G(G, X)[1/n]. \end{array}$$

The notation $G_{(p)}(G, X, -)$ used here is inherited from [11, Section 2.1]. Namely, $G_{(p)}(G, X, -)$ is the simplicial spectrum whose r -simplices is

$$G_{(p)}(G, X, r) := \text{hocolim}_W G^W(G, X \times \Delta^r)$$

the K -theory of equivariant coherent sheaves on $X \times \Delta^r$ with supports in W where W runs over all the closed G -stable subsets of $X \times \Delta^r$ such that $\dim_k W \cap X \times F \leq p + \dim_k F$ for all faces $F \subset \Delta^r$.

The top tower induces the spectral sequence

$$(4.6) \quad E_1^{p,q} = \prod_{\sigma} (CH_{-p}(X^\sigma, -p-q) \otimes \tilde{R}\sigma)^{N(\sigma)} \Rightarrow \prod_{\sigma} (G_{-p-q}(X^\sigma) \otimes \tilde{R}\sigma)^{N(\sigma)}$$

by applying operations to (4.1).

The bottom tower induces the spectral sequence

$$(4.7) \quad E_1^{p,q} = CH_{-p}(G, X, -p-q) \otimes \Lambda \Rightarrow G_{-p-q}(G, X) \otimes \Lambda$$

which is essentially the spectral sequence (2.4) with $\Lambda = \mathbb{Z}[1/n]$ -coefficients.

This shows that the map $\pi(X)$ (induced by the family $(\alpha_{(p)}(X))_p$) is natural with respect to the spectral sequences (4.6) and (4.7). Hence, one obtains Vistoli's reconstruction theorem for equivariant algebraic G -theory:

Corollary 4.2. [13, Theorem 2] *For any $X \in \mathbf{Sch}_k^G$ the map*

$$\alpha_*(X) : \prod_{\sigma \in \Gamma} (G_*(X^\sigma) \otimes \tilde{R}\sigma)^{N(\sigma)} \rightarrow G_*(G, X) \otimes \Lambda$$

is an isomorphism of graded Λ -modules which is compatible with localization sequence.

The following lemma is a weak version of [11, Corollary 5.6] but the proof is simpler:

Corollary 4.3. *Let $X \in \mathbf{Sch}_k^G$ and let G act on $X \times \mathbb{A}^1$ via the given action on X and the trivial action on \mathbb{A}^1 . Then the pull-back via the projection $p : X \times \mathbb{A}^1 \rightarrow X$ induces an isomorphism*

$$(4.8) \quad p_X^* : CH_p(G, X, r) \otimes \Lambda \rightarrow CH_{p+1}(G, X \times \mathbb{A}^1, r) \otimes \Lambda.$$

Proof. Since G acts trivially on \mathbb{A}^1 , we have $(X \times \mathbb{A}^1)^\sigma = X^\sigma \times \mathbb{A}^1$. The diagram

$$\begin{array}{ccc} \prod_{\sigma \in \Gamma} (CH_p(X^\sigma, r) \otimes \tilde{R}\sigma)^{N(\sigma)} & \xrightarrow{\pi(X)} & CH_p(G, X, r) \otimes \Lambda \\ \downarrow \Pi p_{X^\sigma}^* & & \downarrow p_X^* \\ \prod_{\sigma \in \Gamma} (CH_{p+1}(X^\sigma \times \mathbb{A}^1, r) \otimes \tilde{R}\sigma)^{N(\sigma)} & \xrightarrow{\pi(X \times \mathbb{A}^1)} & CH_{p+1}(G, X \times \mathbb{A}^1, r) \otimes \Lambda \end{array}$$

is commutative. The maps $\pi(X)$ and $\pi(X \times \mathbb{A}^1)$ are isomorphisms by Theorem 3.3. Each map $p_{X^\sigma}^*$ is an isomorphism by homotopy invariance for higher Chow groups [2, Theorem (2.1)]. Therefore, p_X^* is an isomorphism. \square

The reconstruction theorem for equivariant higher Chow groups yields a Riemann-Roch theorem for equivariant algebraic K -theory:

Theorem 4.4 (Riemann-Roch). *If X is a smooth G -scheme over k then the spectral sequence (2.4) degenerates rationally, i.e., there are isomorphisms of \mathbb{Q} -vector spaces*

$$(4.9) \quad \bigoplus_q K_q(G, X) \otimes \mathbb{Q} \cong \bigoplus_q G_q(G, X) \otimes \mathbb{Q} \cong \bigoplus_{p,q} CH_p(G, X, q) \otimes \mathbb{Q}.$$

Proof. Since X is smooth and G is finite, every G -equivariant coherent sheaf on X has a resolution by G -equivariant locally free sheaves ([12, Corollary 5.8]). Quillen's *Dévissage theorem* provides for any $q \in \mathbb{N}$ an isomorphism

$$K_q(G, X) \cong G_q(G, X).$$

For any $\sigma \in \Gamma$, the scheme X^σ is smooth (cf. [4, Proposition 3.4]). Hence, the spectral sequence

$$E_1^{p,q} = CH_{-p}(X^\sigma, -p-q) \Rightarrow G_{-p-q}(X^\sigma)$$

degenerates rationally to get

$$\bigoplus_{p,q} CH_p(X^\sigma, q) \otimes \mathbb{Q} \cong \bigoplus_q G_q(X^\sigma) \otimes \mathbb{Q}$$

(cf. [9, Theorem 14.8]). Taking product over all $\sigma \in \Gamma$ yields

$$\prod_{\sigma \in \Gamma} \left(\bigoplus_{p,q} (CH_p(X^\sigma, q) \otimes \mathbb{Q}) \otimes_\Lambda \tilde{R}\sigma \right)^{N(\sigma)} \cong \prod_{\sigma \in \Gamma} \left(\bigoplus_q (G_q(X^\sigma) \otimes \mathbb{Q}) \otimes_\Lambda \tilde{R}\sigma \right)^{N(\sigma)}.$$

Equivalently,

$$\bigoplus_{p,q} \left(\prod_{\sigma \in \Gamma} CH_p(X^\sigma, q) \right) \otimes \tilde{R}^{N(\sigma)} \otimes_\Lambda \mathbb{Q} \cong \bigoplus_q \left(\prod_{\sigma \in \Gamma} G_q(X^\sigma) \right) \otimes \tilde{R}^{N(\sigma)} \otimes_\Lambda \mathbb{Q},$$

i.e.,

$$\bigoplus_{p,q} CH_p(G, X, q) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_q G_q(G, X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

\square

Remark 4.5. When $X \in \mathbf{Sch}_k^G$ which is not necessarily smooth, there is still an isomorphism

$$\bigoplus_q G_q(G, X) \otimes \mathbb{Q} \cong \bigoplus_{p,q} CH_p(G, X, q) \otimes \mathbb{Q}$$

by Theorem 3.3, Corollary 4.2 and Bloch-Riemann-Roch isomorphism

$$G_q(X^\sigma) \otimes \mathbb{Q} \xrightarrow{\sim} \bigoplus_p CH_p(X^\sigma, q) \otimes \mathbb{Q}$$

[2, Theorem (9.1)]. However, we do not know how to define directly an *equivariant Riemann-Roch map*

$$G_q(G, X) \rightarrow \bigoplus_p CH_p(G, X, q) \otimes \mathbb{Q}.$$

4.3. Multiplicative structure and further remarks. When X is smooth, there are certainly two ways to equip a multiplicative structure on equivariant higher Chow groups with rational coefficients (the integral case is still unknown).

The first structure is obtained through the isomorphism (4.9) where the multiplication on $CH_*(G, X, *) \otimes \mathbb{Q}$ inherits the ordinary multiplication on $K_*(G, X) \otimes \mathbb{Q}$ (given by tensor product over \mathcal{O}_X).

The second structure is obtained by the reconstruction theorem for equivariant higher Chow groups. Recall that Jarvis-Kaufmann-Kimura have considered in [7] the *stringy Chow ring*

$$\prod_{g \in G} CH_*(X^g) \otimes \mathbb{Q}$$

with the multiplicative structure given by the *stringy product* [7, Definition 1.6]. This can be extended to define a multiplicative structure on

$$\prod_{g \in G} CH_*(X^g, *) \otimes \mathbb{Q},$$

and hence on the algebra of invariants

$$\left(\prod_{g \in G} CH_*(X^g, *) \otimes \mathbb{Q} \right)^G.$$

By Theorem 3.3, this gives another multiplicative structure on $CH_*(G, X, *) \otimes \mathbb{Q}$.

These two structures are different in general even in the case of Chow groups $CH_*(G, X, 0) \otimes \mathbb{Q}$. The first one is the usual multiplication on the Grothendieck group of the quotient stack $[X/G]$. The second one should be the usual multiplication on the Grothendieck group of a (and all) hyper-Kähler resolution of the coarse moduli space X/G of $[X/G]$. This is the content of the *K-theoretic hyper-Kähler resolution conjectures* [7, Conjecture 1.2] which has been verified in certain interesting cases.

Remark 4.6. As the reader might guess, the isomorphism $\pi(X)$ in Theorem 3.3 need not to be true in general if we index equivariant higher Chow groups by codimension rather than dimension as in [11]. Some degree shifts are needed to get right indexes. The reason is that $\pi(X)$ is defined by using push-forward for algebraic cycles and K -theory that do not preserve codimension in general. For instance, let $G = \mathbb{Z}/2$ act on $X = \mathbb{A}^1 := \text{Spec}(k[t])$ by sending $t \rightarrow -t$. A simple calculation gives

$$CH^1(\mathbb{Z}/2, \mathbb{A}^1, 0) = \mathbb{Z}.$$

(cf. [11, Example 6.17]). The group $\mathbb{Z}/2$ has only two (cyclic) subgroups 0 and $\mathbb{Z}/2$. We have

$$CH^1((\mathbb{A}^1)^0, 0) = CH^1(\mathbb{A}^1, 0) = CH^1(\text{Speck}, 0) = 0$$

and

$$CH^1((\mathbb{A}^1)^{\mathbb{Z}/2}, 0) = CH^1(\text{Speck}, 0) = 0$$

by dimension reason. Therefore,

$$\prod_{\sigma} (CH^1((\mathbb{A}^1)^{\sigma}, 0) \otimes \tilde{R}\sigma)^{N(\sigma)} = 0.$$

It is also worth mentioning that the two multiplicative structures on rational equivariant higher Chow groups considered above do not respect the grading by codimension. For the first multiplicative structure, the reason is that the ring structure on $K_0(G, X)$ does not in general respect the topological filtration [11, Remark 3.6]. For the second multiplicative structure, this failure is measured by 'age' (or 'degree shifting number') [7, Definition 1.3].

Remark 4.7. In the non-equivariant case for X smooth, the spectral sequence (4.1) admits actions of Adams operations which implies its degeneration with rational coefficients. It would be interesting to see how to equip Adams operations on the equivariant motivic spectral sequence (2.5). It might be possible to follow the construction given in [9, Theorem 9.7], but there are some technical annoyances we have to overcome. Even if we are in a good situation, there is no reason to expect that $\pi(X)$ preserves these operations because it is defined using push-forwards which are not ring homomorphisms in general.

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- Current address:* Institut für Mathematik, Universität Osnabrück, Albrechtstr. 28a, 49076, Osnabrück.
E-mail address: toan.nguyen@uni-osnabrueck.de