ON THE VISIBILITY OF PLANAR SETS

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ABSTRACT. Assume that $E, K \subset \mathbb{R}^2$ are Borel sets with $\dim_{\mathrm{H}} K > 0$. Is a positive dimensional part of K visible from some point in E? Not necessarily, since E can be zero-dimensional, or E and K can lie on a common line. I prove that these are the only obstructions: if $\dim_{\mathrm{H}} E > 0$, and E does not lie on a line, then there exists a point in E from which a $(\dim_{\mathrm{H}} K)/2$ dimensional part of K is visible. Applying the result with $E = K$ gives the following corollary: if $K \subset \mathbb{R}^2$ is Borel set, which does not lie on a line, then the set of directions spanned by K has Hausdorff dimension at least $(\dim_{\mathrm{H}} K)/2$.

1. INTRODUCTION

This paper studies visibility and radial projections in the plane. Given $p \in \mathbb{R}^2$, define the radial projection $\pi_p \colon \mathbb{R}^2 \setminus \{p\} \to S^1$ by

$$\pi_p(q) = \frac{p-q}{|p-q|}.$$

A Borel set $K \subset \mathbb{R}^2$ will be called

- *invisible from* p, if $\mathcal{H}^1(\pi_p(K \setminus \{p\})) = 0$, and
- *totally invisible from* p, if $\dim_{\mathrm{H}} \pi_p(K \setminus \{p\}) = 0$.

Above, $\dim_{\mathrm{H}}$ and $\mathcal{H}^s$ stand for Hausdorff dimension and s-dimensional Hausdorff measure, respectively. I will only consider Hausdorff dimension in this paper, as many of the results below would be much easier for box dimension. The study of (in-)visibility has a long tradition in geometric measure theory. For many more results and questions than I can introduce here, see Section 6 of Mattila's survey [8]. The basic question is the following: given a Borel set $K \subset \mathbb{R}^2$, how large can the sets

$$\mathrm{Inv}(K) = \{p \in \mathbb{R}^2 : K \text{ is invisible from } p\}$$

and

$$\mathrm{Inv}_T(K) := \{p \in \mathbb{R}^2 : K \text{ is strongly invisible from } p\}$$

be? Clearly $\mathrm{Inv}_S(K) \subset \mathrm{Inv}(K)$, and one generally expects $\mathrm{Inv}_S(K)$ to be significantly smaller than $\mathrm{Inv}(K)$. The existing results fall roughly into the following three categories:

(1) What happens if $\dim_{\mathrm{H}} K > 1$?
(2) What happens if $\dim_{\mathrm{H}} K \leq 1$?
(3) What happens if $0 < \mathcal{H}^1(K) < \infty$?

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Cases (1) and (3) are the most classical, having already been studied in the 1954 paper [6] of Marstrand. Given $s > 1$, Marstrand proved that any Borel set $K \subset \mathbb{R}^2$ with $0 < \mathcal{H}^s(K) < 1$ is visible (that is, not invisible) from Lebesgue almost every point $p \in \mathbb{R}^2$, and also from $\mathcal{H}^s$ almost every point $p \in K$. Unifying Marstrand's results, the following sharp bound was recently established by Mattila and the author in [9] and [10]:

$$\dim_{\mathrm{H}} \mathrm{Inv}(K) \leq 2 - s, \qquad \dim_{\mathrm{H}} K = s > 1. \tag{1.1}$$

The visibility of sets K in Case (3) depends on their rectifiability. It is easy to show that 1-rectifiable sets, which are not $\mathcal{H}^1$ almost surely covered by a single line, are visible from all points in $\mathbb{R}^2$, with possibly one exception, see [11]. On the other hand, if $K \subset \mathbb{R}^2$ is purely 1-unrectifiable, then the sharp bound

$$\dim_{\mathrm{H}}[\mathbb{R}^2 \setminus \mathrm{Inv}(K)] = \dim_{\mathrm{H}}\{p \in \mathbb{R}^2 : K \text{ is visible from } p\} \leq 1.$$

was obtained by Marstrand, building on Besicovitch's projection theorem. For generalisations, improvements and constructions related to the bound above, see [7, Theorem 5.1], and [3, 4]. Marstrand raised the question – which remains open to the best of my knowledge – whether it is possible that $\mathcal{H}^1(\mathbb{R}^2 \setminus \mathrm{Inv}(K)) > 0$: in particular, can a purely 1-unrectifiable set be visible from a positive fraction of its own points? For purely 1-unrectifiable self-similar sets $K \subset \mathbb{R}^2$ one has $\mathrm{Inv}(K) = \mathbb{R}^2$, as shown by Simon and Solomyak [13].

Case (3) has received less attention. To simplify the discussion, assume that $\dim_{\mathrm{H}} K = 1$ and $\mathcal{H}^1(K) = 0$, so that the considerations of Case (3) no longer apply, and $\mathrm{Inv}(K) = \mathbb{R}^2$. Then, the relevant question becomes the size of $\mathrm{Inv}_T(K)$. The radial projections π_p fit the influential *generalised projections* framework of Peres and Schlag [12], so one should start by checking, what bounds follow from [12, Theorem 7.3]. If $K \subset \mathbb{R}^2$ is a Borel set with arbitrary dimension $s \in [0, 2]$, then it follows from [12, Theorem 7.3] that

$$\dim_{\mathrm{H}} \mathrm{Inv}_T(K) = \dim_{\mathrm{H}}\{p \in \mathbb{R}^2 : \dim_{\mathrm{H}} \pi_p(K) = 0\} \leq 2 - s. \tag{1.2}$$

When $s > 1$, the bound (1.2) is a weaker version of (1.1), but the benefit of (1.2) is that it holds without any restrictions on s. In particular, if $s = 1$, one obtains

$$\dim_{\mathrm{H}} \mathrm{Inv}_T(K) \leq 1. \tag{1.3}$$

This bound is sharp, and quite trivially so: consider the case, where K lies on a single line $\ell \subset \mathbb{R}^2$. Then, $\mathrm{Inv}_T(K) = \ell$. The starting point for this paper was the question: are there essentially different examples manifesting the sharpness of (1.3)? The answer turns out to be negative in a very strong sense. Here are the main results of the paper:

Main Theorem 1.4 (Weak version). *Assume that $K \subset \mathbb{R}^2$ is a Borel set with $\dim_{\mathrm{H}} K > 0$. Then, at least one of the following holds:*

- $\dim_{\mathrm{H}} \mathrm{Inv}_T(K) = 0$.
- $\mathrm{Inv}_T(K)$ *is contained on a line.*

In fact, more is true. For $K \subset \mathbb{R}^2$, define

$$\mathrm{Inv}_{1/2}(K) := \left\{p \in \mathbb{R}^2 : \dim_{\mathrm{H}} \pi_p(K \setminus \{p\}) < \tfrac{\dim_{\mathrm{H}} K}{2}\right\}.$$

Then, if $\dim_{\mathrm{H}} K > 0$, one evidently has $\mathrm{Inv}_T(K) \subset \mathrm{Inv}_{1/2}(K) \subset \mathrm{Inv}(K)$.

Main Theorem 1.5 (Strong version). *Theorem 1.4 holds with* $\mathrm{Inv}_T(K)$ *replaced by* $\mathrm{Inv}_{1/2}(K)$. *That is, if* $E \subset \mathbb{R}^2$ *is a Borel set with* $\dim_{\mathrm{H}} E > 0$, *not contained on a line, then there exists* $p \in E$ *such that* $\dim_{\mathrm{H}} \pi_p(K \setminus \{p\}) \geq (\dim_{\mathrm{H}} K)/2$.

Remark 1.6. A closely related result is Theorem 1.6 in the paper [1] of Bond, Łaba and Zahl; with some imagination, Theorem 1.6(a) in [1] can be viewed as a "single scale" variant of Theorem 1.5. As far as I can tell, proving the Hausdorff dimension statement in this context presents a substantial extra challenge, so Theorem 1.5 is not easily implied by the results in [1].

Example 1.7. *Figure 1 depicts the main challenge in the proofs of Theorems 1.4 and 1.5. The*

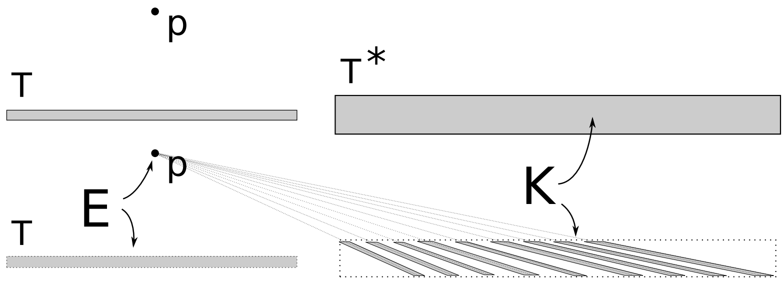

FIGURE 1. What is the next step in the construction of E?

set E *has* $\dim_{\mathrm{H}} E > 0$, *and consists of something inside a narrow tube* T, *plus a point* $p \notin T$. *Then, Theorem 1.4 states that* $E \not\subset \mathrm{Inv}_T(K)$ *for any compact set* $K \subset \mathbb{R}^2$ *with* $\dim_{\mathrm{H}} K > 0$. *So, in order to find a counterexample to Theorem 1.5, all one needs to do is find* K *by a standard "Venetian blind" construction, in such a way that* $\dim_{\mathrm{H}} K > 0$ *and* $\dim_{\mathrm{H}} \pi_q(K) = 0$ *for all* $q \in E$. *The first steps are obvious: to begin with, require that* $K \subset T^*$ *for another narrow tube parallel to* T, *see Figure 1. Then* $\pi_q(K)$ *is small for all* $q \in T$. *To handle the remaining point* $p \in E$, *split the contents of* T^* *into a finite collection of new narrow tubes in such a way that* $\pi_p(K)$ *is small. In this manner,* $\pi_q(K)$ *can be made arbitrarily small for all* $q \in E$ *(in the sense of* ϵ*-dimensional Hausdorff content, for instance, for any prescribed* $\epsilon > 0$*). It is quite instructive to think, why the construction cannot be completed: why cannot the "Venetian blinds" be iterated further (for both* E *and* K*) so that, at the limit,* $\dim_{\mathrm{H}} \pi_q(K) = 0$ *for all* $q \in E$*?*

Theorem 1.5 has the following immediate consequence:

Corollary 1.8 (Corollary to Theorem 1.5). *Assume that* $K \subset \mathbb{R}^2$ *is a Borel set, not contained on a line. Then the set of unit vectors spanned by* K, *namely*

$$S(K) := \left\{ \tfrac{p-q}{|p-q|} \in S^1 : p, q \in K \text{ and } p \neq q \right\},$$

satisfies $\dim_{\mathrm{H}} S(K) \geq \frac{\dim_{\mathrm{H}} K}{2}$.

Proof. If $\dim_{\mathrm{H}} K = 0$, there is nothing to prove. Otherwise, Theorem 1.5 implies that $K \not\subset \mathrm{Inv}_{1/2}(K)$, whence $\dim_{\mathrm{H}} S(K) \geq \dim_{\mathrm{H}} \pi_p(K \setminus \{p\}) \geq (\dim_{\mathrm{H}} K)/2$ for some $p \in K$. $\square$

Corollary 1.8 is probably not sharp, and the following conjecture seems plausible:

Conjecture 1.9. *Assume that* $K \subset \mathbb{R}^2$ *is a Borel set, not contained on a line. Then* $\dim_{\mathrm{H}} S(K) = \min\{\dim_{\mathrm{H}} K, 1\}$.

This follows from Marstrand's result, discussed in Case (1) above, when $\dim_{\mathrm{H}} K > 1$. For $\dim_{\mathrm{H}} K \leq 1$, Conjecture 1.9 is closely connected with continuous sum-product problems, which means that significant improvements over Corollary 1.8 will, most likely, require new technology. An ϵ-improvement may be possible, combining the proof below with ideas from the paper [5] of Katz and Tao, and using the discretised sum-product theorem of Bourgain [2].

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2. PROOFS

If $\ell \subset \mathbb{R}^2$ is a line, I denote by $T(\ell, \delta)$ the open (infinite) tube of width 2δ, with ℓ "running through the middle", that is, $\operatorname{dist}(\ell, \mathbb{R}^2 \setminus T(\ell, \delta)) = \delta$. The notation $B(x, r)$ stands for a closed ball with centre $x \in \mathbb{R}^2$ and radius $r > 0$. The notation $A \lesssim B$ means that there is an absolute constant $C \geq 1$ such that $A \leq CB$.

Lemma 2.1. *Assume that μ is a Borel probability measure on $B(0,1) \subset \mathbb{R}^2$, and $\mu(\ell) = 0$ for all lines $\ell \subset \mathbb{R}^2$. Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(T(\ell, \delta)) \leq \epsilon$ for all lines $\ell \subset \mathbb{R}^2$.*

Proof. Assume not, so there exists $\epsilon > 0$, a sequence of positive numbers $\delta_1 > \delta_2 > \ldots > 0$, and a sequence of lines $\{\ell_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$ with $\mu(T(\ell_i, \delta_i)) \geq \epsilon$. Since $\operatorname{spt} \mu \subset B(0,1)$, one has $\ell_i \cap B(0,1) \neq \emptyset$ for all $i \in \mathbb{N}$. Consequently, there exists a subsequence $(i_j)_{j \in \mathbb{N}}$, and a line $\ell \subset \mathbb{R}^2$ such that $\ell_j \to \ell$ in the Hausdorff metric. Then, for any given $\delta > 0$, there exists $j \in \mathbb{N}$ such that

$$B(0,1) \cap T(\ell_{i_j}, \delta_{i_j}) \subset T(\ell, \delta),$$

so that $\mu(T(\ell, \delta)) \geq \epsilon$. It follows that $\mu(\ell) \geq \epsilon$, a contradiction. $\square$

The following lemma contains most of the proof of Theorem 1.5:

Lemma 2.2. *Assume that μ, ν are Borel probability measures with compact supports $K, E \subset B(0,1)$, respectively. Assume that both measures μ and ν satisfy a Frostman condition with exponents $\kappa_\mu, \kappa_\nu \in (0, 2]$, respectively:*

$$\mu(B(x,r)) \leq C_\mu r^{\kappa_\mu} \quad \text{and} \quad \nu(B(x,r)) \leq C_\nu r^{\kappa_\nu} \tag{2.3}$$

for all balls $B(x,r) \subset \mathbb{R}^2$, and for some constants $C_\mu, C_\nu \geq 1$. Assume further that $\mu(\ell) = 0$ for all lines $\ell \subset \mathbb{R}^2$. Fix also

$$0 < \tau < \tfrac{\kappa_\mu}{2} \quad \text{and} \quad \epsilon > 0,$$

and write $\delta_k := 2^{-(1+\epsilon)^k}$.

Then, there exist numbers $\beta = \beta(\kappa_\mu, \kappa_\nu, \tau) > 0$, $\eta = \eta(\epsilon, \kappa_\mu, \kappa_\nu, \tau) > 0$, and an index $k_0 = k_0(\epsilon, \mu, \kappa_\nu, \tau) \in \mathbb{N}$ with the following properties. For all $k \geq k_0$, there exist

(a) compact sets $K \supset K_{k_0} \supset K_{k_0+1} \ldots$ with

$$\mu(K_k) \geq 1 - \sum_{k_0 \leq j < k} (\tfrac{1}{4})^{j - k_0 + 1} \geq \frac{1}{2}, \tag{2.4}$$

(b) compact sets $E \supset E_{k_0} \supset E_{k_0+1} \ldots$ with $\nu(E_k) \geq \delta_k^{\beta}$
with the following property: if $k > k_0$, $p \in E_k$, and $T(\ell_1, \delta_k), \ldots, T(\ell_N, \delta_k)$ is a family of tubes of cardinality $N \leq \delta_k^{-\tau}$, each containing p, then

$$\mu\left(K_k \cap \bigcup_{j=1}^{N} T(\ell_j, \delta_k)\right) \leq \delta_k^{\eta}. \tag{2.5}$$

Remark 2.6. The index k_0 can be chosen as large as desired; this will be clear from the proof below. It will also be used on many occasions, without separate remark, that δ_k can be assumed very small for all $k \geq k_0$.

Proof. The proof is by induction, starting at the largest scale k_0, which will be presently defined. Fix $\eta = \eta(\epsilon, \kappa_\mu, \kappa_\nu, \tau) > 0$ and

$$\Gamma = \Gamma(\epsilon, \kappa_\mu, \kappa_\nu, \tau). \in \mathbb{N} \tag{2.7}$$

The number Γ will be specified at the very end of the proof, right before (2.32), and there will be several requirements for the number η, see (2.22), (2.28), and (2.31). Applying Lemma 2.1, first pick an index $k_1 = k_1(\epsilon, \mu, \kappa_\nu, \tau) \in \mathbb{N}$ such that $\mu(T(\ell, \delta_{k_1})) \leq (\frac{1}{4})^{\Gamma+1}$ for all tubes $T(\ell, \delta_{k_1}) \subset \mathbb{R}^2$, and

$$\delta_{k-\Gamma}^{\eta} \leq (\tfrac{1}{4})^{k-\Gamma+1}, \qquad k \geq k_1. \tag{2.8}$$

Set $k_0 := k_1 + \Gamma$. Then, the following holds for all $k \in \{k_0, \ldots, k_0 + \Gamma\}$. For any subset $K' \subset K$, and any tube $T(\ell, \delta_{k-\Gamma}) \subset \mathbb{R}^2$, one has

$$\mu(K' \cap T(\ell, \delta_{k-\Gamma})) \leq \mu(T(\ell, \delta_{k_1})) \leq (\tfrac{1}{4})^{\Gamma+1} \leq (\tfrac{1}{4})^{k-k_0+1}. \tag{2.9}$$

Define

$$K_k := K \quad \text{and} \quad E_k := E, \qquad k_1 \leq k \leq k_0.$$

(The definitions of E_k, K_k for $k_1 \leq k < k_0$ are only given for notational convenience.)

I start by giving an outline of how the induction will proceed. Assume that, for a certain $k \geq k_0$, the sets K_k and E_k have been constructed such that

(i) the condition (2.9) is satisfied with $K' = K_k$, and for all tubes $T(\ell, \delta_{k-\Gamma})$ with $T(\ell, \delta_{k-\Gamma}) \cap E_{k-\Gamma} \neq \emptyset$.

(ii) K_k and E_k satisfy the measure lower bounds (a) and (b) from the statement of the lemma.

Under the conditions (i)-(ii), I claim that it is possible to find subsets $K_{k+1} \subset K_k$ and $E_{k+1} \subset E_k$, satisfying (ii) at level $k+1$, and also the non-concentration condition (2.5) at level $k+1$. This is why (2.5) is only claimed to hold for $k > k_0$, and no one is indeed claiming that it holds for the sets K_{k_0} and E_{k_0}. These sets satisfy (i), however, which should be viewed as a weaker substitute for (2.5) at level k, which is just strong enough to guarantee (2.5) at level $k+1$. There is one obvious question at this point: if (i) at level k gives (2.5) at level $k+1$, then where does one get (i) back at level $k+1$?

If $k+1 \in \{k_0, \ldots, k_0+\Gamma\}$, the condition (i) is simply guaranteed by the choice of k_0 (one does not even need to assume that $T(\ell, \delta_{k-\Gamma}) \cap E_{k-\Gamma} \neq \emptyset$). For $k+1 > k_0 + \Gamma$, this is no longer true. However, for $k+1 > \Gamma + k_0$, one has $k+1-\Gamma > k_0$, and thus $K_{k+1-\Gamma}$ and $E_{k+1-\Gamma}$ have already been constructed to satisfy (2.5). In particular, if $E_{k+1-\Gamma} \cap T(\ell, \delta_{k+1-\Gamma}) \neq \emptyset$, then

$$\mu(K_{k+1} \cap T(\ell, \delta_{k+1-\Gamma})) \leq \mu(K_{k+1-\Gamma} \cap T(\ell, \delta_{k+1-\Gamma})) \leq \delta_{k+1-\Gamma}^{\eta} \leq (\tfrac{1}{4})^{(k+1)-k_0+1} \tag{2.10}$$

by (2.5) and (2.8). This means that (i) is satisfied at level $k+1$, and the induction may proceed.

So, it remains to prove that (i)–(ii) at level k imply (ii) and (2.5) at level $k+1$. To avoid clutter, I write

$$\delta := \delta_{k+1}.$$

Assume that the sets K_k, E_k have been constructed for some $k \geq k_0$, satisfying (i)–(ii). The main task is to understand the structure of the set of points $p \in E_k$ for which (2.5) fails, and these points are denoted by $\mathbf{Bad}_k$. More precisely, $p \in \mathbf{Bad}_k$, if and only if $p \in E_k$, and there exist $N \leq \delta^{-\tau}$ tubes $T(\ell_1, \delta), \ldots, T(\ell_N, \delta)$, each containing p, such that

$$\mu\left(K_k \cap \bigcup_{j=1}^{N} T(\ell_j, \delta)\right) > \delta^{\eta}. \tag{2.11}$$

Note that if $\mathbf{Bad}_k = \emptyset$, then one can simply define $E_{k+1} := E_k$ and $K_{k+1} := K_k$, and (ii) and (2.5) (at level $k+1$) are clearly satisfied.

Instead of analysing $\mathbf{Bad}_k$ directly, it is useful to split it up into "directed" pieces, and digest the pieces individually. To make this precise, let S be the "space of directions"; for concreteness, I identify S with the upper half of the unit circle. Then, if $T = T(\ell, \delta) \subset \mathbb{R}^2$ is a tube, I denote by $\mathrm{dir}(T)$ the unique vector $e \in S$ such that $\ell \| e$.

Recall the small parameter $\eta > 0$, and partition S into $D = \delta^{-\eta}$ arcs $J_1, \ldots, J_D$ of length $\sim \delta^{\eta}$.[1] For $d \in \{1, \ldots, D\}$ fixed ("d" for "direction"), consider the set $\mathbf{Bad}_k^d$: it consists of those points $p \in E_k$ such that there exist $N \leq \delta^{-\tau}$ tubes $T(\ell_1, \delta), \ldots, T(\ell_N, \delta)$, each containing p, with $\mathrm{dir}(T(\ell_i, \delta)) \in J_d$, and satisfying

$$\mu\left(K_k \cap \bigcup_{j=1}^{N} T(\ell_j, \delta)\right) > \delta^{2\eta}.$$

Since the direction of every possible tube in $\mathbb{R}^2$ belongs to one of the arcs J_i, and there are only $D = \delta^{-\eta}$ arcs in total, one has

$$\mathbf{Bad}_k \subset \bigcup_{d=1}^{D} \mathbf{Bad}_k^d. \tag{2.12}$$

The next task is to understand the structure of $\mathbf{Bad}_k^d$ for a fixed direction $d \in \{1, \ldots, D\}$. I claim that $\mathbf{Bad}_k^d$ looks like a garden of flowers, with all the petals pointing in direction J_d, see Figure 2 for a rough idea. To make the statement more precise, I introduce an additional piece of notation. Fox $X \subset K_k$, let $B_d(X)$ consist of those points $p \in E_k$ such that X can be covered by $N \leq \delta^{-\tau}$ tubes $T(\ell_1, \delta), \ldots, T(\ell_N, \delta)$, with directions $\mathrm{dir}(T(\ell_i, \delta)) \in J_d$, and each containing p. Then, note that

$$\mathbf{Bad}_k^d = \{p \in E_k : \exists X \subset K_k \text{ with } \mu(X) > \delta^{2\eta} \text{ and } p \in B_d(X)\}. \tag{2.13}$$

The sets $B_d(X)$ also has the trivial but useful property that

$$X \subset X' \subset K_k \quad \Longrightarrow \quad B_d(X') \subset B_d(X).$$

[1]Here, it might be better style to pick another letter, say $\alpha > 0$, in place of η, since the two parameters play slightly different roles in the proof. Eventually, however, one would end up considering $\min\{\eta, \alpha\}$, and it seems a bit cleaner to let $\eta > 0$ be a "jack of all trades" from the start.

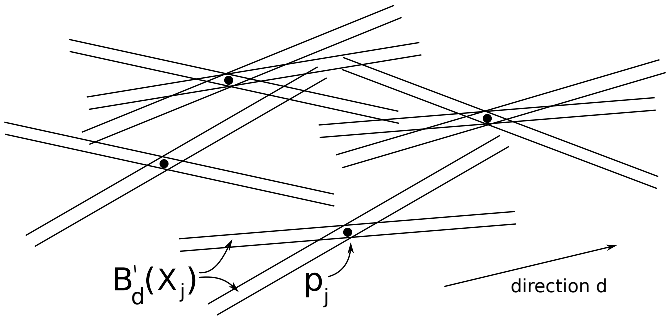

FIGURE 2. The set $\mathbf{Bad}_k^d$.

There are two steps in establishing the "garden" structure of $\mathbf{Bad}_k^d$: first, one needs to find the "flowers", and second, one needs to check that the sets obtained actually look like flowers in a non-trivial sense. I start with the former task. Assuming that $\mathbf{Bad}_k^d \neq \emptyset$, pick any point $p_1 \in \mathbf{Bad}_k^d$, and an associated subset $X_1 \subset K_k$ with

$$\mu(X_1) > \delta^{2\eta} \quad \text{and} \quad p_1 \in B_d(X_1).$$

Then, assume that $p_1, \ldots, p_m \in \mathbf{Bad}_k^d$ and $X_1, \ldots, X_m$ have already been chosen with the properties above, and further satisfying

$$\mu(X_i \cap X_j) \leq \delta^{4\eta}/2, \qquad 1 \leq i < j \leq m. \tag{2.14}$$

Then, see if there still exists a subset $X_{m+1} \subset K_k$ with the following three properties: $\mu(X_{m+1}) > \delta^{2\eta}$, $B_d(X_{m+1}) \neq \emptyset$, and $\mu(X_{m+1} \cap X_i) \leq \delta^{4\eta}/2$ for all $1 \leq i \leq m$. If such a set no longer exists, stop; if it does, pick $p_{m+1} \in B_d(X_{m+1})$, and add X_{m+1} to the list.

It follows from the "competing" conditions $\mu(X_i) > \delta^{2\eta}$, and (2.14), that the algorithm needs to terminate in at most

$$M \leq 2\delta^{-4\eta} \tag{2.15}$$

Indeed, assume that the sets $X_1, \ldots, X_M$ have already been constructed, and consider the following chain of inequalities:

$$\begin{aligned}
\frac{1}{M} + \frac{1}{M(M-1)} \sum_{i_1 \neq i_2} \mu(X_{i_1} \cap X_{i_2}) &\geq \frac{1}{M^2} \sum_{i_1, i_2 = 1}^{M} \mu(X_{i_1} \cap X_{i_2}) \\
&= \frac{1}{M^2} \int \sum_{i_1, i_2 = 1}^{M} \mathbf{1}_{X_{i_1} \cap X_{i_2}}(x) \, d\mu(x) \\
&= \frac{1}{M^2} \int [\operatorname{card}\{1 \leq i \leq M : x \in X_i\}]^2 \, d\mu(x) \\
&\geq \frac{1}{M^2} \left(\int \operatorname{card}\{1 \leq i \leq M : x \in X_i\} \, d\mu(x) \right)^2 \\
&= \frac{1}{M^2} \left(\sum_{i=1}^{M} \mu(X_i) \right)^2 > \delta^{4\eta}.
\end{aligned}$$

Thus, if $M > 2\delta^{-4\eta}$, there exists a pair X_{i_1}, X_{i_2} with $i_1 \neq i_2$ such that $\mu(X_{i_1} \cap X_{i_2}) > \delta^{4\eta}/2$, and the algorithm has already terminated earlier. This proves (2.15).

With the sets $X_1, \ldots, X_M$ now defined, write

$$B'_d(X_j) := \{p \in E_k : \exists X' \subset X_j \text{ with } \mu(X') > \delta^{4\eta}/2 \text{ and } p \in B_d(X')\}.$$

I claim that

$$\mathbf{Bad}_k^d \subset \bigcup_{j=1}^{M} B'_d(X_j). \tag{2.16}$$

Indeed, if $p \in \mathbf{Bad}_k^d$, then $p \in B_d(X)$ for some $X \subset K_k$ with $\mu(X) > \delta^{2\eta}$ by (2.13). It follows that

$$\mu(X \cap X_j) > \delta^{4\eta}/2 \tag{2.17}$$

for one of the sets X_j, $1 \leq j \leq M$, because either $X \in \{X_1, \ldots, X_M\}$, and (2.17) is clear (all the sets X_j even satisfy $\mu(X_j) > \delta^{2\eta}$), or else (2.17) must hold by virtue of X **not** having been added to the list $X_1, \ldots, X_M$ in the algorithm. But (2.17) implies that $p \in B'_d(X_j)$, since $X' = X \cap X_j \subset X_j$ satisfies $\mu(X') > \delta^{4\eta}/2$ and $p \in B_d(X) \subset B_d(X')$.

According to (2.15) and (2.16) the set $\mathbf{Bad}_k^d$ can be covered by $M \leq 2\delta^{-4\eta}$ sets of the form $B'_d(X_j)$, see Figure 2. These sets are the "flowers", and their structure is explored in the next lemma:

Lemma 2.18. *The following holds, if $\delta = \delta_{k+1}$ is small enough. For $1 \leq d \leq D$ and $1 \leq j \leq M$ fixed, the set $B'_d(X_j)$ can be covered by $\leq 4\delta^{-8\eta}$ tubes of the form $T = T(\ell, \delta^\rho)$, where* $\operatorname{dir}(T) \in J_d$*, and $\rho = \rho(\kappa_\mu, \tau) > 0$. The tubes can be chosen to contain the point $p_j \in B_d(X_j)$.*

Proof. Fix $1 \leq j \leq M$ and $p \in B'_d(X_j)$. Recall the point $p_j \in B_d(X_j)$ from the definition of X_j. By definition of $p \in B'_d(X_j)$, there exists a set $X' \subset X_j$ with $\mu(X') > \delta^{4\eta}/2$ and $p \in B_d(X')$. Unwrapping the definitions further, there exist $N \leq \delta^{-\tau}$ tubes $T(\ell_1, \delta), \ldots, T(\ell_N, \delta)$, the union of which covers X', and each satisfies $\operatorname{dir}(T(\ell_i, \delta)) \in J_d$ and $p \in T(\ell_i, \delta)$. In particular, one of these tubes, say $T_p = T(\ell_i, \delta)$, has

$$\mu(X_j \cap T_p) \geq \mu(X' \cap T_p) \geq \mu(X') \cdot \delta^\tau \geq \delta^{4\eta+\tau}/2 \geq \delta^{8\eta+\tau}/4. \tag{2.19}$$

(The final inequality is for just a triviality at this point, but is useful for later technical purposes later.) Here comes perhaps the most basic geometric observation in the proof: if the measure lower bound (2.19) holds for some δ-tube T – this time T_p – and a sufficiently small $\eta > 0$ (crucially so small that $8\eta + \tau < \kappa/2$), then the whole set $B_d(X_j)$ is actually contained in a neighbourhood of T, called T^*, because $X_j \cap T$ is so difficult to cover by δ-tubes centred at points outside T^*, see Figure 3. In particular, in the present case,

$$p_j \in B_d(X_j) \subset T(\ell_i, \delta^{4\rho}) =: T_p^* \tag{2.20}$$

for a suitable constant $\rho = \rho(\kappa_\mu, \tau) > 0$, specified in (2.22). To see this formally, pick $q \in B(0,1) \setminus T_p^*$, and argue as follows to show that $q \notin B_d(X_j)$. First, any δ-tube T containing q, and intersecting $T_p \cap B(0,1)$, makes an angle of at least $\gtrsim \delta^{4\rho}$ with T_p. It follows that

$$\operatorname{diam}(T \cap T_p \cap B(0,1)) \lesssim \delta^{1-4\rho},$$

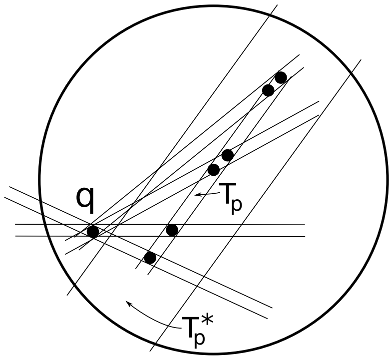

FIGURE 3. Covering $X_j \cap T_p$ by tubes centred at points outside T_p^*.

and consequently $\mu(T \cap T_p \cap B(0,1)) \lesssim C_\mu \delta^{\kappa_\mu(1-4\rho)}$. So, in order to cover $X_j \cap T_p$ (let alone the whole set X_j) it takes by (2.19) at least

$$\gtrsim \frac{\mu(X_j \cap T_p)}{C_\mu \delta^{\kappa_\mu(1-4\rho)}} \geq \frac{\delta^{8\eta+\tau-\kappa_\mu(1-4\rho)}}{C_\mu} \geq \frac{\delta^{8\eta-\kappa_\mu/2+8\rho}}{C_\mu} \tag{2.21}$$

tubes T containing q. But if

$$0 < 8\eta < \frac{\frac{\kappa_\mu}{2} - \tau}{2} \quad \text{and} \quad 8\rho = \frac{\frac{\kappa_\mu}{2} - \tau}{2}, \tag{2.22}$$

then the number on the right hand side of (2.21) is far larger than $\delta^{-\tau}$, which means that $q \notin B_d(X_j)$, and proves (2.20).

Recall the statement of the Lemma 2.18, and compare it with the previous accomplishment: (2.20) states that whenever $p \in B_d'(X_j)$, then p lies in a certain tube of width $\delta^{4\rho}$ (namely T_p), which has direction in J_d, and also contains p_j. This sounds a bit like the statement of the lemma, but there is a problem: in principle, every point $p \in B'(X_j)$ could give rise to a different tube T_p. So, it essentially remains to show that all these $\delta^{4\rho}$-tubes T_p can be covered by a small number of tubes of width δ^ρ. To begin with, note that the ball $B_j := B(p_j, \delta^{2\rho})$ can be covered by a single tube of width δ^ρ, in any direction desired. So, to prove the lemma, it remains to cover $B_d'(X_j) \setminus B_j$.

Note that if p, q satisfy $|p - q| \geq \delta^{2\rho}$, then the direction of any $\delta^{4\rho}$-tube containing both p, q lies in a fixed arc $J(p,q) \subset S$ of length $|J(p,q)| \lesssim \delta^{4\rho}/\delta^{2\rho} = \delta^{2\rho}$. As a corollary, the union of all $\delta^{4\rho}$-tubes containing p, q, intersected with $B(0,1)$, is contained in a single tube of width $\sim \delta^{2\rho}$. In particular, this union (still intersected with $B(0,1)$) is contained in a single δ^ρ-tube, assuming that $\delta > 0$ is small; this tube can be chosen to be a δ^ρ-tube around an arbitrary $\delta^{4\rho}$-tube containing both p and q.

The tube-cover of $B_d'(X_j) \setminus B_j$ can now be constructed by adding one tube at a time. First, assume that there is a point $q_1 \in B_d'(X_j) \setminus B_j$, and find a tube $T(\ell_1, \delta^{4\rho})$ containing both q and p_j, with direction in J_d; existence follows from (2.20). Add the tube $T(\ell_1, \delta^\rho)$ to the the tube-cover of $B_d'(X_j) \setminus B_j$, and recall from the previous paragraph that $T(\ell_1, \delta^\rho)$ now contains $T \cap B(0,1)$ for **any** $\delta^{4\rho}$-tube $T \supset \{q_1, p_j\}$ (of which $T = T(\ell_1, \delta^{4\rho})$ is just one example). Finally, by definition of $q_1 \in B_d'(X_j)$, associate to q_1 a subset $X_1' \subset X_j$ with

$$\mu(X_1') > \delta^{4\eta}/2 \quad \text{and} \quad q_1 \in B_d(X_1'). \tag{2.23}$$

Assume that the points $q_1, \ldots, q_H \in B'_d(X_j) \setminus B_j$, along with the associated tubes $\{q_i, p_j\} \subset T(\ell_i, \delta^{4\rho}) \subset T(\ell_i, \delta^{\rho})$, and subsets $X'_i \subset X_j$, as in (2.23), have already been constructed. Assume inductively that

$$\mu(X'_{i_1} \cap X'_{i_2}) \leq \delta^{8\eta}/4, \qquad 1 \leq i_1 < i_2 \leq H. \tag{2.24}$$

To proceed, pick any point $q_{H+1} \in B'_d(X_j) \setminus B_j$, and associate to q_{H+1} a subset $X'_{H+1} \subset X_j$ with $\mu(X'_{H+1}) > \delta^{4\rho}/2$ and $q_{H+1} \in B_d(X'_{H+1})$. Then, test whether (2.24) still holds, that is, whether $\mu(X'_{H+1} \cap X'_i) \leq \delta^{8\eta}_{k+1}/4$ for all $1 \leq i \leq H$. If such a point q_{H+1} can be chosen, run the argument from the previous paragraph, first locating a tube $T(\ell_{H+1}, \delta^{4\rho})$ containing both q_{H+1} and p_j, with direction in J_d, and finally adding $T(\ell_{H+1}, \delta^{\rho})$ to the tube-cover under construction.

The "competing" conditions $\mu(X'_i) > \delta^{4\eta}/2$, and (2.24), guarantee that the the algorithm terminates in

$$H \leq 4\delta^{-8\eta}$$

steps. The argument is precisely the same as used to prove (2.15), so I omit it. Once the algorithm has terminated, I claim that all points of $B'_d(X_j) \setminus B_j$ are covered by the tubes $T(\ell_i, \delta^{\rho})$, with $1 \leq i \leq H$. To see this, pick $q \in B'_d(X_j) \setminus B_j$, and a subset $X' \subset X_j$ with $\mu(X') > \delta^{4\eta}/2$, and $q \in B_d(X')$. Since the algorithm had already terminated, it must be the case that

$$\mu(X' \cap X'_i) > \delta^{8\eta}/4$$

for some index $1 \leq i \leq H$. Since $X'' := X' \cap X'_i \subset X'$ and consequently $q \in B_d(X'')$, one can find a tube $T_q = T(\ell_q, \delta) \ni q$ with $\mathrm{dir}(T_q) \in J_d$, and satisfying

$$\mu(X'_i \cap T_q) \geq \mu(X'' \cap T_q) \geq \mu(X'') \cdot \delta^{\tau} > \delta^{8\eta+\tau}/4.$$

This lower bound is precisely the same as in (2.19). Hence, it follows from the same argument, which gave (2.20), that

$$q_i \in B_d(X'_i) \subset T(\ell_q, \delta^{4\rho}).$$

Since $X'_i \subset X_j$, also $p_j \in B_d(X_j) \subset B_d(X'_i) \subset T(\ell_q, \delta^{4\rho})$. So,

$$\{q, q_i, p_j\} \subset B(0,1) \cap T(\ell_q, \delta^{4\rho}). \tag{2.25}$$

In particular, $T(\ell_q, \delta^{4\rho})$ is a $\delta^{4\rho}$-tube containing both q_i, p_j, and hence

$$B(0,1) \cap T(\ell_q, \delta^{4\rho}) \subset T(\ell_i, \delta^{\rho}).$$

Combined with (2.25), this yields $q \in T(\ell_i, \delta^{\rho})$, as claimed. This concludes the proof of Lemma 2.18. □

Combining (2.15)-(2.16) with Lemma 2.18, the structural description of $\mathbf{Bad}^d_k$ is now complete: $\mathbf{Bad}^k_d$ is covered by

$$\leq M \cdot 4\delta^{-8\eta} \leq 8\delta^{-12\eta} \tag{2.26}$$

tubes of width δ^{ρ}, with directions in J_d. For non-adjacent $d_1, d_2 \in \{1, \ldots, D\}$ (the ordering of indices corresponds to the ordering of the arcs $J_d \subset S$), the covering tubes are then fairly transversal. This is can be used to infer that most point in E_k do not lie in many different sets $\mathbf{Bad}^d_k$. Indeed, consider the set $\mathbf{BadBad}_k$ of those points in $\mathbb{R}^2$, which lie in (at least) two sets $\mathbf{Bad}^{d_1}_k$ and $\mathbf{Bad}^{d_2}_k$ with $|d_2 - d_1| > 1$. By Lemma 2.18, such points lie in the intersection of some pair of tubes $T_1 = T(\ell_1, \delta^\rho)$ and $T_2 = T(\ell_2, \delta^\rho)$ with $\mathrm{dir}(T_i) \in J_{d_i}$. The angle between these tubes is $\gtrsim \delta^\eta$, whence

$$\mathrm{diam}(T_1 \cap T_2) \lesssim \delta^{\rho-\eta},$$

and consequently

$$\nu(T_1 \cap T_2) \lesssim C_\nu \delta^{\kappa_\nu(\rho-\eta)} \le C_\nu \delta^{\kappa_\nu \rho - 2\eta}. \tag{2.27}$$

For $d \in \{1, \ldots, D\}$ fixed, there correspond $\lesssim \delta^{-12\eta}$ tubes in total, as pointed out in (2.26). So, the number of pairs T_1, T_2, as above, is bounded by

$$\lesssim D^2 \cdot \delta^{-24\eta} \le \delta^{-26\eta}.$$

Consequently, by (2.27),

$$\nu(\mathbf{BadBad}_k) \lesssim C_\nu \delta^{-28\eta + \kappa_\nu \rho}.$$

This upper bound is far smaller than $\nu(E_k)/2 \ge \delta_k^\beta/2$, assuming that

$$0 < \beta < \kappa_\nu \rho - 28\eta. \tag{2.28}$$

Given that $28\eta < \kappa_\nu \rho/2$, one is free to make such an assumption on β (it holds for $k = k_0$, since $\nu(E_{k_0}) = 1$), but the smaller β is, the more difficult it becomes be to ensure that $\nu(E_{k+1}) \ge \delta_{k+1}^\beta$. To see that this can be done, start by writing $G_k := E_k \setminus \mathbf{BadBad}_k$, so that

$$\nu(G_k) \ge \nu(E_k)/2 \ge \delta_k^\beta/2$$

by the choice of β. Now, either

$$\nu\left(G_k \cap \mathbf{Bad}_k\right) \ge \frac{\nu(G_k)}{2} \quad \text{or} \quad \nu\left(G_k \cap \mathbf{Bad}_k\right) < \frac{\nu(G_k)}{2}. \tag{2.29}$$

The latter case is quick and easy: set $E_{k+1} := G_k \setminus \mathbf{Bad}_k$ and $K_{k+1} := K_k$. Then $\nu(E_{k+1}) \ge \nu(E_k)/4 \ge \delta_{k+1}^\beta$ (assuming that $k \ge k_0$ is large enough). Moreover, the set E_{k+1} no longer contains any points in $\mathbf{Bad}_k$, so (2.5) is satisfied at level $k+1$, by the very definition of $\mathbf{Bad}_k$, see (2.11).

So, it remains to treat the first case in (2.29). Start by recalling from (2.12) that $\mathbf{Bad}_k$ is covered by the sets $\mathbf{Bad}_k^d$, $1 \le d \le D$, so

$$\nu(G_k \cap \mathbf{Bad}_k^d) \ge \frac{\nu(G_k)}{2D} \ge \frac{\delta^\eta \delta_k^\beta}{4} = \frac{\delta^{\eta + \beta/(1+\epsilon)}}{4}$$

for some fixed $d \in \{1, \ldots, D\}$. Then, recall from (2.26) that $\mathbf{Bad}_k^d$ can be covered by $\le 8\delta^{-12\eta}$ tubes of the form $T(\ell, \delta^\rho)$, with directions in J_d. It follows that there exists a fixed tube $T_0 = T(\ell_0, \delta^\rho)$ such that

$$\mathrm{dir}(T_0) \in J_d \quad \text{and} \quad \nu(G_k \cap T_0 \cap \mathbf{Bad}_k^d) \ge \frac{\delta^{13\eta + \beta/(1+\epsilon)}}{32}. \tag{2.30}$$

So, to ensure $\nu(G_k \cap T_0 \cap \mathbf{Bad}_k^d) \ge \delta^\beta$, choose $\eta > 0$ so small that

$$13\eta + \beta/(1+\epsilon) < \beta. \tag{2.31}$$

To convince the reader that there is no circular reasoning at play, I gather here all the requirements for β and η (harvested from (2.22), (2.28), and (2.31)):

$$0 < \beta < \frac{\kappa_\nu \rho}{2} \quad \text{and} \quad 0 < \eta < \min\left\{\frac{\kappa_\mu/2 - \tau}{2}, \frac{\kappa_\nu \rho}{56}, \frac{\epsilon\beta}{1+\epsilon}\right\}$$

With such choices of β, η, recalling (2.30), and assuming that δ is small enough, the set

$$E_{k+1} := G_k \cap T_0 \cap \mathbf{Bad}_k^d.$$

satisfies $\nu(E_{k+1}) \geq \delta^{\beta}$, which is statement (b) from the lemma. It remains to define K_{k+1}. To this end, recall that T_0 is a tube around the line $\ell_0 \subset \mathbb{R}^2$. Define

$$K_{k+1} := K_k \setminus T(\ell_0, \delta^{\eta/2}).$$

Then, assuming that $\eta/2$ has the form $\eta/2 = (1+\epsilon)^{-\Gamma-1}$ for an integer $\Gamma = \Gamma(\epsilon, \kappa_\mu, \kappa_\nu, \tau) \in \mathbb{N}$ (this is finally the integer from (2.7)), one has

$$\delta^{\eta/2} = \delta_{k-\Gamma}. \tag{2.32}$$

Since $T(\ell_0, \delta_{k-\Gamma}) \cap E_{k-\Gamma} \neq \emptyset$, it follows from the induction hypothesis (i) that

$$\mu(K_k \cap T(\ell, \delta_{k-\Gamma})) \leq (\tfrac{1}{4})^{k-k_0+1}.$$

Consequently,

$$\mu(K_{k+1}) \geq \mu(K_k) - (\tfrac{1}{4})^{k-k_0+1} \geq 1 - \sum_{k_0 \leq j < k+1} (\tfrac{1}{4})^{j-k_0+1},$$

which is the desired lower bound from (a) of the statement of the lemma. So, it remains to verify the non-concentration condition (2.5) for E_{k+1} and K_{k+1}. To this end, pick $p \in E_{k+1}$. First, observe that every tube $T = T(\ell, \delta)$, which contains p and has non-empty intersection with $K_{k+1} \subset B(0,1) \setminus T(\ell, \delta^{\eta/2})$, forms an angle $\gtrsim \delta^{\eta/2}$ with T_0. In particular, this angle is far larger than δ^{η}. Since $\mathrm{dir}(T_0) \in J_d$ by (2.30), this implies that $\mathrm{dir}(T) \in J_{d'}$ for some $|d' - d| > 1$.

Now, if the non-concentration condition (2.5) still failed for $p \in E_{k+1}$, there would exist $N \leq \delta^{-\tau}$ tubes $T(\ell_1, \delta), \ldots, T(\ell_N, \delta)$, each containing p, and with

$$\mu\left(K_{k+1} \cap \bigcup_{i=1}^{N} T(\ell_i, \delta)\right) > \delta^{\eta}.$$

By the pigeonhole principle, it follows that the tubes $T(\ell_i, \delta)$ with $\mathrm{dir}(T_i) \in J_{d'}$, for some fixed arc $J_{d'}$, cover a set $X \subset K_{k+1} \subset K_k$ of measure $\mu(X) > \delta^{2\eta}$. This means precisely that $p \in \mathbf{Bad}_k^{d'}$, and by the observation in the previous paragraph, $|d - d'| > 1$. But $p \in E_{k+1} \subset \mathbf{Bad}_k^d$ by definition, so this would imply that $p \in \mathbf{BadBad}_k$, contradicting the fact that $p \in E_{k+1} \subset G_k$. This completes the proof of (2.5), and the lemma. $\square$

The proof of Theorem 1.5 is now quite standard:

Proof of Theorem 1.5. Write $s := \dim_{\mathrm{H}} K$, and assume that $s > 0$ and $\dim_{\mathrm{H}} E > 0$. Make a counter assumption: E is not contained on a line, but $\dim_{\mathrm{H}} \pi_p(K) < s/2$ for all $p \in E$. Then, find $t < s/2$, and a positive-dimensional subset $\tilde{E} \subset E$, not contained on any single line, with $\dim_{\mathrm{H}} \pi_p(K) \leq t$ for all $p \in \tilde{E}$ (if your first attempt at $\tilde{E}$ lies on some line ℓ, simply add a point $p_0 \in E \setminus \ell$ to $\tilde{E}$, and replace t by $\max\{t, \dim_{\mathrm{H}} \pi_{p_0}(K)\} < s/2$). So, now $\tilde{E}$ satisfies the same hypotheses as E, but with "$< s/2$" replaced by "$\leq t < s/2$". Thus, without loss of generality, one may assume that

$$\dim_{\mathrm{H}} \pi_p(K) \leq t < s/2, \qquad p \in E. \tag{2.33}$$

Using Frostman's lemma, pick probability measures μ, ν with $\operatorname{spt}\mu \subset K$ and $\operatorname{spt}\nu \subset E$, and satisfying the growth bounds (2.3) with exponents $0 < \kappa_\mu < s$ and $\kappa_\nu > 0$. Pick, moreover, κ_μ so close to s that

$$\kappa_\mu/2 > t. \tag{2.34}$$

Observe that $\mu(\ell) = 0$ for all lines $\ell \subset \mathbb{R}^2$. Indeed, if $\mu(\ell) > 0$ for some line $\ell \subset \mathbb{R}^2$, then there exists $p \in E \setminus \ell$ by assumption, and

$$\dim_{\mathrm{H}} \pi_p(K) \geq \dim_{\mathrm{H}} \pi_p(\operatorname{spt}\mu \cap \ell) \geq \kappa_\mu > t,$$

violating (2.33) at once. Finally, by restricting the measures μ and ν slightly, one may assume that they have disjoint supports.

In preparation for using Lemma 2.2, fix $\epsilon > 0$, $0 < \tau < \kappa_\mu/2$ in such a way that

$$\frac{\tau}{(1+\epsilon)^2} > t. \tag{2.35}$$

This is possible by (2.34). Then, apply Lemma 2.2 to find the parameters $\beta, \eta > 0$, $k_0 \in \mathbb{N}$, and the sets $\operatorname{spt}\mu \supset K_{k_0} \supset K_{k_0+1} \supset \ldots$ and $\operatorname{spt}\nu \supset E_{k_0} \supset E_{k_0+1} \supset \ldots$. Write

$$K' := \bigcap_{k \geq k_0} K_k \subset \operatorname{spt}\mu \quad \text{and} \quad E' := \bigcap_{k \geq k_0} E_k \subset \operatorname{spt}\nu.$$

Both sets are non-empty and compact (being intersections of nested sequences of non-empty compact sets), $\mu(K') \geq \frac{1}{2}$, and $K' \cap E' = \emptyset$. Pick $p \in E'$. I claim that

$$\dim_{\mathrm{H}} \pi_p(K') \geq \frac{\tau}{(1+\epsilon)^2}, \tag{2.36}$$

which violates (2.33) by (2.35). If not, cover $\pi_p(K)$ by efficiently by arcs $J_1, J_2, \ldots$ of lengths restricted to the values $\delta_k = 2^{-(1+\epsilon)^k}$, with $k \geq k_0$. More precisely: assuming that (2.36) fails, start with an arbitrary efficient cover $\tilde{J}_1, \tilde{J}_2, \ldots$ by arcs of length $|\tilde{J}_i| \leq \delta_{k_0}$, satisfying

$$\sum_{j \geq 1} |\tilde{J}_j|^{\tau/(1+\epsilon)^2} \leq 1.$$

Then, replace each $\tilde{J}_j$ by the shortest concentric arc $J_j \supset \tilde{J}_j$, whose length is of the form δ_k. Note that $\ell(J_j) \leq \ell(\tilde{J}_j)^{1/(1+\epsilon)}$, so that

$$\sum_{j \geq 1} |J_j|^{\tau/(1+\epsilon)} \leq \sum_{j \geq 1} |\tilde{J}_j|^{\tau/(1+\epsilon)^2} \leq 1.$$

The arcs $J_1, J_2, \ldots$ now cover $\pi_p(K')$, and there are $\leq \delta_k^{-\tau/(1+\epsilon)}$ arcs of any fixed length δ_k. Since $p \notin K'$, for every $k \geq k_0$ there exists a collection of tubes $\mathcal{T}_k$ of the form $T(\ell, \delta_k) \ni p$, such that $|\mathcal{T}_k| \lesssim \delta_k^{-\tau/(1+\epsilon)}$ (the implicit constant depends on $\operatorname{dist}(p, K')$), and

$$K' \subset \bigcup_{k \geq k_0} \bigcup_{T \in \mathcal{T}_k} T.$$

In particular $|\mathcal{T}_k| \leq \delta_k^{-\tau}$, assuming that δ_k is small enough for all $k \geq k_0$. Recall that $\mu(K') \geq \frac{1}{2}$. Hence, by the pigeonhole principle, one can find $k \in \mathbb{N}$ such that the following holds: there is a subset $K'_k \subset K'$ with $\mu(K'_k) \geq \frac{1}{100k^2}$ such that K'_k is covered by the tubes in $\mathcal{T}_k$. But $1/(100k^2)$ is far larger than δ_k^η, so this is explicitly ruled out by non-concentration estimate in Lemma 2.2, namely (2.5). This contradiction completes the proof. □

REFERENCES

[1] M. BOND, I. ŁABA, AND J. ZAHL: *Quantitative visibility estimates for unrectifiable sets in the plane*, Trans. Amer. Math. Soc. **368**(8) (2016), 5475-5513

[2] J. BOURGAIN: *On the Erdős-Volkmann and Katz-Tao ring conjectures*, Geom. Funct. Anal. **13**(2) (2003), 334-365

[3] M. CSÖRNYEI: *On the visibility of invisible sets*, Ann. Acad. Sci. Fenn. Math. **25**(2) (2000), 417-421

[4] M. CSÖRNYEI: *How to mame Davies' theorem visible*, Bull. London Math. Soc. **33**(1) (2001), 59-66

[5] N. H. KATZ AND T. TAO: *Some connections between Falconer's distance set conjecture and sets of Furstenburg type*, New York J. Math. **7** (2001), 149-187

[6] J. M. MARSTRAND: *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proc. London Math. Soc. **4** (3) (1954), 257-302

[7] P. MATTILA: *Integral geometric properties of capacities*, Trans. Amer. Math. Soc. **226**(2) (1981), 539-554

[8] P. MATTILA: *Hausdorff dimension, projections, and the Fourier transform*, Publ. Mat. **48**(1) (2004), 3-48

[9] P. MATTILA AND T. ORPONEN: *Hausdorff dimension, intersections of projections and exceptional plane sections*, Proc. Amer. Math. Soc. **144**(8) (2016), 3419-3430

[10] T. ORPONEN: *A sharp exceptional set estimate for visibility*, Bull. London Math. Soc. (to appear), arXiv:1602.07629

[11] T. ORPONEN AND T. SAHLSTEN: *Radial projections of rectifiable sets*, Ann. Acad. Sci. Fenn. Math. **36**(2) (2011), 677-681

[12] Y. PERES AND W. SCHLAG: *Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions*, Duke Math. J. **102**(2) (2000), 193-251

[13] K. SIMON AND B. SOLOMYAK: *Visibility for self-similar sets of dimension one in the plane*, Real Anal. Exchange **32**(1) (2006/2007), 67-78

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