RENEWAL THEOREMS FOR PROCESSES WITH DEPENDENT INTERARRIVAL TIMES

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ABSTRACT. Renewal theorems are developed for point processes with interarrival times $\xi(X_{n+1}X_n\cdots)$, where $(X_n)_{n\in\mathbb{Z}}$ is a stochastic process with finite state space Σ and $\xi\colon\Sigma_A\to\mathbb{R}$ is a Hölder continuous function on a subset $\Sigma_A\subset\Sigma^{\mathbb{N}}$. The theorems developed here unify and generalise the key renewal theorem for discrete measures and Lalley's renewal theorem for counting measures in symbolic dynamics. Moreover, they capture aspects of Markov renewal theory. The new renewal theorems allow for direct applications to problems in fractal and hyperbolic geometry; for instance, to the problem of Minkowski measurability of self-conformal sets.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

We develop renewal theorems for renewal functions $N\colon\mathbb{R}\times\Sigma_A\to\mathbb{R}$ of the form

$$N(t,x)=\sum_{n=0}^{\infty}\sum_{y\in\Sigma_A:\sigma^n y=x}\chi(y)f_y(t-S_n\xi(y))\mathrm{e}^{S_n\eta(y)}, \tag{1.1}$$

where we use the following notation and conditions. $\Sigma_A:=\{x\in\Sigma^{\mathbb{N}}\mid A(x_k,x_{k+1})=1\ \forall\, k\in\mathbb{N}\}$ denotes the set of *one-sided infinite admissible words* over an *alphabet* $\Sigma:=\{1,\ldots,M\}$, $M\geq 2$ consistent with a primitive $(M\times M)$- incidence matrix $A=(A(i,j))_{i,j\in\Sigma}$ of zeros and ones. By *primitive* we mean that there exists $n\in\mathbb{N}$ such that all entries of A^n are positive. Moreover, $\sigma\colon\Sigma_A\to\Sigma_A$ denotes the (left) shift-map on Σ_A defined by $\sigma(\omega_1\omega_2\ldots):=\omega_2\omega_3\ldots$ for $\omega_1\omega_2\cdots\in\Sigma_A$. The functions $\chi,\xi,\eta\colon\Sigma_A\to\mathbb{R}$ are assumed to be α-Hölder continuous for some $\alpha\in(0,1)$ in the sense of Defn. 2.1 and χ shall be non-negative. We write $S_0\xi:=0$ and $S_n\xi:=\sum_{k=0}^{n-1}\xi\circ\sigma^k$ for the *n-th Birkhoff sum* of ξ with $n\in\mathbb{N}$ and suppose that there exists $n\in\mathbb{N}$ for which $S_n\xi$ is strictly positive. Note that this condition is equivalent to ξ being co-homologous to a strictly positive function, see Rem. 2.3. Some mild regularity conditions on the family $\{f_y\colon\mathbb{R}\to\mathbb{R}\mid y\in\Sigma_A\}$ are assumed (see Sec. 3) and $y\mapsto f_y$ is supposed to be α-Hölder continuous.

If, for instance, $\chi=\mathbb{1}_{\Sigma_A}$ and $f_y=\mathbb{1}_{[0,\infty)}$ for all $y\in\Sigma_A$, then $N(t,x)$ gives the expected number of renewals in the time-interval $(0,t]$ provided that the history of the underlying stochastic process $(X_n)_{n\in\mathbb{Z}}$ coincides with x. In this model, $\xi(X_{n+1}X_n\cdots)$ are the interarrival times and $\eta(ix)=\log\mathbb{P}(X_1=i\mid X_0X_{-1}\cdots=x)$ is the logaricmic transition probability. Notice, the current setting extends and unifies the setting of established renewal theorems: In the context of classical renewal theory for finitely supported measures (in particular of the key renewal theorem), η and ξ only depend on the first coordinate. When η and ξ only depend on the first two coordinates, we are in the setting of Markov renewal theory. If η is the constant zero-function and $f_y=\mathbb{1}_{[0,\infty)}$, then we are precisely in the setting of [Lal89], where renewal theorems for counting measures in symbolic dynamics were developed. For more details on these connections, see the survey [Kom].

Renewal theorems deal with the asymptotic behaviour of renewal functions. It is well-known that the limiting behaviour of a renewal function depends on the values which the interarrival times assume. Loosely speaking if the possible values of the interarrival times form a spaced pattern we are in the lattice situation and otherwise in the non-lattice situation, see Defn. 2.2. This lattice – non-lattice dichotomy also occurs in our

2010 *Mathematics Subject Classification.* Primary: 60K05, 60K15. Secondary: 28A80, 28A75.

Key words and phrases. Renewal theorem, dependent interarrival times, symbolic dynamics, Key renewal theorem, Ruelle Perron-Frobenius theory.

This article was completed while the author was staying at the Institut Mittag-Leffler. The author would like to thank the staff at the institute as well as the organisers and participants of the research program *Fractal Geometry and Dynamics* for the stimulating atmosphere, the excellent working environment, warm hospitality and financial support.

main theorems, Thms. 3.2 and 3.3. They are summarised in the following, where we call a function $f\colon \mathbb{R}\to\mathbb{R}$ *asymptotic* to a function $g\colon \mathbb{R}\to\mathbb{R}_{>0}$ as $t\to\infty$, written $f(t)\sim g(t)$ as $t\to\infty$, if $\lim_{t\to\infty} f(t)/g(t)=1$.

Renewal Theorem with dependent interarrival times. Assume that $\{f_x(t)\mid x\in\Sigma_A\}$ satisfies some regularity conditions as described in Sec. 3. There exists $\delta>0$ given in Sec. 2.3 such that the following hold.

(i) If ξ is non-lattice, then there exists a constant $G(x)$ such that

$$N(t,x)\sim \mathrm{e}^{t\delta}G(x)$$

as $t\to\infty$, uniformly for $x\in\Sigma_A$.

(ii) If ξ is lattice, then there exists a periodic function $\widetilde{G}_x$ such that as $t\to\infty$

$$N(t,x)\sim \mathrm{e}^{t\delta}\widetilde{G}_x(t).$$

(iii) We always have

$$\lim_{t\to\infty} t^{-1}\int_0^t \mathrm{e}^{-T\delta}N(T,x)\mathrm{d}T = G(x).$$

We believe that the extension, that we provide with this theorem, is useful in a number of contexts. In Sec. 4 e. g. we will show one application in geometry where the extension is essential, see Rem. 4.2.

This article is organised as follows. In Sec. 2 we introduce the relevant notions from Perron-Frobenius theory, which appear in the statements of the main theorems. Sec. 3 is devoted to the presentation of the main results (Thms. 3.2 and 3.3). In Sec. 4 we present an application in fractal geometry which motivated the development of the present renewal theorem. Finally, in Sec. 5 we provide the proofs of Thms. 3.2 and 3.3.

2. Ruelle-Perron-Frobenius Theory

Here, we assemble preliminaries and fix notation. References for the exposition below are [Bow08, Wal82].

2.1. Subshifts of finite type.

We call (Σ_A,σ) a *subshift of finite type*. If all entries of A are ones, then $\Sigma_A=\Sigma^{\mathbb{N}}$ and (Σ_A,σ) is called the *full shift* on M symbols. The set of *admissible words of length* $n\in\mathbb{N}$ is defined by

$$\Sigma_A^n := \{\omega\in\Sigma^n \mid A(\omega_k,\omega_{k+1})=1 \text{ for } k\le n-1\}. \tag{2.1}$$

If ω has infinite length or length $m\ge n$ we define $\omega|_n:=\omega_1\cdots\omega_n$ to be the subword of length n. Further, $[\omega]:=\{u_1u_2\cdots\in\Sigma_A\mid u_i=\omega_i \text{ for } i\le n\}$ is the ω-*cylinder set* for $\omega\in\Sigma_A^n$.

2.2. Continuous and Hölder-continuous functions.

Equip $\Sigma^{\mathbb{N}}$ with the product topology of the discrete topologies on Σ and equip $\Sigma_A\subset\Sigma^{\mathbb{N}}$ with the subspace topology, i. e. the weakest topology with respect to which the canonical projections onto the coordinates are continuous. The spaces of continuous complex- and real-valued functions on Σ_A are respectively denoted by $\mathcal{C}(\Sigma_A)$ and $\mathcal{C}(\Sigma_A,\mathbb{R})$. Elements of $\mathcal{C}(\Sigma_A,\mathbb{R})$ are called *potential functions*.

Definition 2.1. For $\xi\in\mathcal{C}(\Sigma_A)$, $\alpha\in(0,1)$ and $n\in\mathbb{N}_0$ define

$$\begin{aligned}\mathrm{var}_n(\xi) &:= \sup\{|\xi(\omega)-\xi(u)| \mid \omega,u\in\Sigma_A \text{ and } \omega_i=u_i \text{ for all } i\in\{1,\dots,n\}\},\\ |\xi|_\alpha &:= \sup_{n\ge0}\frac{\mathrm{var}_n(\xi)}{\alpha^n},\\ \mathcal{F}_\alpha(\Sigma_A) &:= \{\xi\in\mathcal{C}(\Sigma_A)\mid |\xi|_\alpha<\infty\} \text{ and } \mathcal{F}_\alpha(\Sigma_A,\mathbb{R}):=\mathcal{F}_\alpha(\Sigma_A)\cap\mathcal{C}(\Sigma_A,\mathbb{R}).\end{aligned}$$

Elements of $\mathcal{F}_\alpha(\Sigma_A)$ are called α-*Hölder continuous* functions on Σ_A.

Definition 2.2. Functions $\xi_1,\xi_2\in\mathcal{C}(\Sigma_A)$ are called *co-homologous*, if there exists $\psi\in\mathcal{C}(\Sigma_A)$ such that $\xi_1-\xi_2=\psi-\psi\circ\sigma$. A function $\xi\in\mathcal{C}(\Sigma_A,\mathbb{R})$ is said to be *lattice*, if it is co-homologous to a function whose range is contained in a discrete subgroup of $\mathbb{R}$. Otherwise, we say that ξ is *non-lattice*.

Remark 2.3. $S_n\xi$ being strictly positive for some $n \in \mathbb{N}$ is equivalent to ξ being co-homologous to a strictly positive function ζ. To see this, note that $\zeta, \xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ are co-homologous if and only if $S_m\zeta(x) = S_m\xi(x)$ for all $m \in \mathbb{N}$ and $x \in \Sigma_A$ with $\sigma^m x = x$. First, suppose that $S_n\xi$ is strictly positive. Let $\zeta := S_n\xi/n$. Then $S_m\xi(x) = S_m\zeta(x)$ for all $x \in \Sigma_A$ with $\sigma^m x = x$. Second, suppose that $\xi = \zeta + \psi - \psi \circ \sigma$ for some $\psi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ and ζ satisfying $\zeta \geq \varepsilon > 0$ for all $x \in \Sigma_A$. Then $S_m\xi(x) = S_m\zeta(x) + \psi(x) - \psi \circ \sigma^{m+1}(x) \geq m\varepsilon + \mathrm{var}_0(\psi) > 0$ for sufficiently large $m \in \mathbb{N}$.

2.3. **Topological pressure function and Gibbs measures.** The *topological pressure function* $P\colon \mathcal{C}(\Sigma_A, \mathbb{R}) \to \mathbb{R}$ is given by the well-defined limit

$$P(\xi) := \lim_{n\to\infty} n^{-1} \log \sum_{\omega \in \Sigma_A^n} \exp \sup_{u \in [\omega]} S_n\xi(u). \tag{2.2}$$

If $\xi, \eta \in \mathcal{C}(\Sigma_A, \mathbb{R})$ are so that $S_n\xi$ is strictly positive on Σ_A, for some $n \in \mathbb{N}$, then $s \mapsto P(\eta + s\xi)$ is continuous, strictly monotonically increasing and convex with $\lim_{s\to-\infty} P(\eta + s\xi) = -\infty$ and $\lim_{s\to\infty} P(\eta + s\xi) = \infty$. Hence, there is a unique $\delta \in \mathbb{R}$ for which $P(\eta - \delta\xi) = 0$.

A finite Borel measure μ on Σ_A is said to be a *Gibbs measure* for $\xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ if there exists a constant $c > 0$ such that

$$c^{-1} \leq \frac{\mu([\omega|_n])}{\exp(S_n\xi(\omega) - n \cdot P(\xi))} \leq c \tag{2.3}$$

for every $\omega \in \Sigma_A$ and $n \in \mathbb{N}$.

2.4. **Ruelle's Perron-Frobenius theorem.** For $s \in \mathbb{C}$ define the *Ruelle-Perron-Frobenius operator* $\mathcal{L}_{\eta+s\xi}\colon \mathcal{C}(\Sigma_A) \to \mathcal{C}(\Sigma_A)$ to the potential function $\eta + s\xi$ by

$$\mathcal{L}_{\eta+s\xi}\chi(x) := \sum_{y \in \Sigma_A : \sigma y = x} \chi(y) \mathrm{e}^{\eta(y)+s\xi(y)}. \tag{2.4}$$

By [Wal01, Thm. 2.16, Cor. 2.17] and [Bow08, Theorem 1.7], for each $\xi \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$, some $\alpha \in (0,1)$, there exists a unique Borel probability measure ν_ξ on Σ_A satisfying $\mathcal{L}_\xi^*\nu_\xi = \gamma_\xi\nu_\xi$ for some $\gamma_\xi > 0$, where $\mathcal{L}_\xi^*$ denotes the dual operator acting on the space of Borel-probability measures supported on Σ_A. This equation uniquely determines γ_ξ, which satisfies $\gamma_\xi = \exp(P(\xi))$ and which coincides with the spectral radius of $\mathcal{L}_\xi$. Further, there exists a unique strictly positive eigenfunction $h_\xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ satisfying $\mathcal{L}_\xi h_\xi = \gamma_\xi h_\xi$ and $\int h_\xi \mathrm{d}\nu_\xi = 1$. Define μ_ξ by $\mathrm{d}\mu_\xi/\mathrm{d}\nu_\xi = h_\xi$. This is the unique σ-invariant Gibbs measure for the potential function ξ. Additionally, under some normalisation assumptions we have convergence of the iterates of the Ruelle-Perron-Frobenius operator to the projection onto the one-dimensional subspace generated by its eigenfunction h_ξ, namely

$$\lim_{m\to\infty} \|\gamma_\xi^{-m}\mathcal{L}_\xi^m\psi - \textstyle\int \psi \mathrm{d}\nu_\xi \cdot h_\xi\|_\infty = 0 \quad \forall\, \psi \in \mathcal{C}(\Sigma_A, \mathbb{R}). \tag{2.5}$$

Sec. 2.3 and the relation $\gamma_\xi = \exp(P(\xi))$ imply the following.

Proposition 2.4. *Let $\xi, \eta \in \mathcal{C}(\Sigma_A, \mathbb{R})$ be such that for some $n \in \mathbb{N}$ the n-th Birkhoff sum $S_n\xi$ of ξ is strictly positive on Σ_A. Then $s \mapsto \gamma_{\eta+s\xi}$ is continuous, strictly monotonically increasing, log-convex in $s \in \mathbb{R}$ with $\lim_{s\to-\infty} \gamma_{\eta+s\xi} = 0$ and satisfies $\lim_{s\to\infty} \gamma_{\eta+s\xi} = \infty$. The unique $\delta \in \mathbb{R}$ from Sec. 2.3 is the unique $\delta \in \mathbb{R}$ for which $\gamma_{\eta-\delta\xi} = 1$.*

Remark 2.5. In the proof of our renewal theorems, we will work with an analytic continuation of the pressure function to the complex domain and with complex Perron Frobenius theory. These complex quantities and their properties will be presented in Sec. 5.1.

3. Renewal theorems with dependent interarrival times

Here, we make our regularity conditions on the functions f_y from (1.1) as well as the statement of the results more precise. We fix $x \in \Sigma_A$ and take $\alpha \in (0,1)$. Recall that, $\xi, \eta, \chi \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$ are so that χ is non-negative but not identically zero and so that $S_n\xi$ is strictly positive on Σ_A for some $n \in \mathbb{N}$, see also Rem. 2.3. Henceforth, we let $\delta > 0$ denote the unique real for which $\gamma_{\eta-\delta\xi} = 1$ (see Prop. 2.4). One of the regularity conditions on $f_y\colon \mathbb{R} \to \mathbb{R}$, for $y \in \Sigma_A$ requires equi directly Riemann integrability, a condition which is motivated by an assumption of the classical key renewal theorem.

Definition 3.1. For a function $f\colon \mathbb{R}\to\mathbb{R}$, $h>0$ and $k\in\mathbb{Z}$ set

$$\underline{m}_k(f,h) := \inf\{f(t) \mid (k-1)h \le t < kh\} \qquad \text{and} \qquad \overline{m}_k(f,h) := \sup\{f(t) \mid (k-1)h \le t < kh\}.$$

The function f is called *directly Riemann integrable (d. R. i.)* if for some sufficiently small $h>0$

$$\underline{R}(f,h) := \sum_{k\in\mathbb{Z}} h\cdot \underline{m}_k(f,h) \qquad \text{and} \qquad \overline{R}(f,h) := \sum_{k\in\mathbb{Z}} h\cdot \overline{m}_k(f,h)$$

are finite and tend to the same limit as $h\to 0$. We call a family of functions $\{f_x\colon \mathbb{R}\to\mathbb{R} \mid x\in I\}$ with some index set I *equi directly Riemann integrable (equi d. R. i.)* if f_x is d. R. i. for all $x\in I$ and if

$$\sum_{k\in\mathbb{Z}} h\cdot \sup_{x\in I}\Big(\underline{m}_k(f_x,h) - \overline{m}_k(f_x,h)\Big)$$

tends to zero as $h\to 0$.

D. R. i. excludes wild oscillations of the function at infinity and is stronger than Riemann integrability. For further insights into this notion we refer the reader to [Fel71, Ch. XI] and [Asm03, Ch. B.V].

The *regularity conditions* are as follows.

(A) *Lebesgue integrability.* For any $x\in\Sigma_A$ the Lebesgue-integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-t\delta}|f_x(t)|\mathrm{d}t$$

exists.

(B) *Boundedness of N.* There exists $\mathfrak{C}>0$ such that $\mathrm{e}^{-t\delta}N^{\mathrm{abs}}(t,x)\le\mathfrak{C}$ for all $x\in\Sigma_A$ and $t\in\mathbb{R}$, where

$$N^{\mathrm{abs}}(t,x) := \sum_{n=0}^{\infty}\sum_{y:\sigma^n y = x}\chi(y)|f_y(t-S_n\xi(y))|\mathrm{e}^{S_n\eta(y)}.$$

(C) *Exponential decay of N on the negative half-axis.* There exist $\widetilde{\mathfrak{C}}>0, s>0$ and $t^*\in\mathbb{R}$ such that $\mathrm{e}^{-t\delta}N^{\mathrm{abs}}(t,x)\le\widetilde{\mathfrak{C}}\mathrm{e}^{st}$ for all $t\le t^*$.

In the non-lattice situation we additionally need the following.

(D) *Non-oscillatory.* The renewal function N does not oscillate wildly at infinity, in particular at least one of the following is satisfied.
  (a) The function $t\mapsto f_x(t)$ is monotonic for any $x\in\Sigma_A$.
  (b) The family $\{t\mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x\in\Sigma_A\}$ is equi d. R. i.
  (c) The oscillation of N is small in the sense that

$$\liminf_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\widetilde{\varepsilon}\in[0,\varepsilon]}\mathrm{e}^{-(r-\widetilde{\varepsilon})\delta}N(r-\widetilde{\varepsilon},x) = \limsup_{\varepsilon\to 0}\liminf_{r\to\infty}\inf_{\widetilde{\varepsilon}\in[0,\varepsilon]}\mathrm{e}^{-(r-\widetilde{\varepsilon})\delta}N(r-\widetilde{\varepsilon},x).$$

Note that Conditions (A) to (D) are a weakening of imposed assumptions of known renewal theorems. In Thm. 3.3 it is for instance shown that equi d. R. i. of $\{t\mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x\in\Sigma_A\}$ and exponential decay of $t\mapsto \mathrm{e}^{-t\delta}f_x(t)$ on the negative half-axis imply (A) to (D). The conditions (A) to (C) and (Da) are motivated by [Lal89].

For $t\in\mathbb{R}$ we write $\lfloor t\rfloor$ for the largest integer $k\in\mathbb{Z}$ satisfying $k\le t$. Moreover, we set $\{t\} := t-\lfloor t\rfloor\in[0,1)$. Notice, for $t\in\mathbb{R}$ positive, $\lfloor t\rfloor$ is the integer part and $\{t\}$ is the fractional part of t.

Theorem 3.2 (Renewal theorem)**.** *Assume that $x\mapsto f_x(t)$ is α-Hölder continuous for any $t\in\mathbb{R}$ and that Conditions (A) to (C) hold.*

(i) If ξ is non-lattice and (D) is satisfied, then

$$N(t,x)\sim \mathrm{e}^{t\delta}\underbrace{\frac{h_{\eta-\delta\xi}(x)}{\int\xi\mathrm{d}\mu_{\eta-\delta\xi}}\int_{\Sigma_A}\chi(y)\int_{-\infty}^{\infty}\mathrm{e}^{-T\delta}f_y(T)\mathrm{d}T\mathrm{d}\nu_{\eta-\delta\xi}(y)}_{=:G(x)}$$

as $t\to\infty$, uniformly for $x\in\Sigma_A$.

(ii) *Assume that ξ is lattice and let $\zeta, \psi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ satisfy the relation*

$$\xi - \zeta = \psi - \psi \circ \sigma,$$

where $\zeta(\Sigma_A) \subseteq a\mathbb{Z}$ for some $a > 0$. Suppose that ξ is not co-homologous to any function with values in a proper subgroup of $a\mathbb{Z}$. Then

$$N(t, x) \sim \mathrm{e}^{t\delta} \widetilde{G}_x(t)$$

as $t \to \infty$, uniformly for $x \in \Sigma_A$. Here $\widetilde{G}_x$ is the following periodic function with period a

$$\widetilde{G}_x(t) = \frac{a\mathrm{e}^{\delta\psi(x) - a\left\{\frac{t+\psi(x)}{a}\right\}\delta}}{\int \zeta \mathrm{d}\mu_{\eta-\delta\zeta}} \cdot h_{\eta-\delta\zeta}(x) \int_{\Sigma_A} \chi(y) \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al\delta} f_y\left(al + a\left\{\tfrac{t+\psi(x)}{a}\right\} - \psi(y)\right) \mathrm{d}\nu_{\eta-\delta\zeta}(y).$$

(iii) *We always have*

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T\delta} N(T, x) \mathrm{d}T = G(x).$$

Theorem 3.3. *Assume that $x \mapsto f_x(t)$ is α-Hölder continuous for $t \in \mathbb{R}$ and that $\{t \mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x \in \Sigma_A\}$ is equi d. R. i. If there exist $\mathfrak{C}', s > 0$ such that $\mathrm{e}^{-t\delta}|f_x(t)| \leq \mathfrak{C}'\mathrm{e}^{st}$ for $t < 0$ and $x \in \Sigma_A$, then (A) to (D) are satisfied and thus the statements of Thm. 3.2 all hold.*

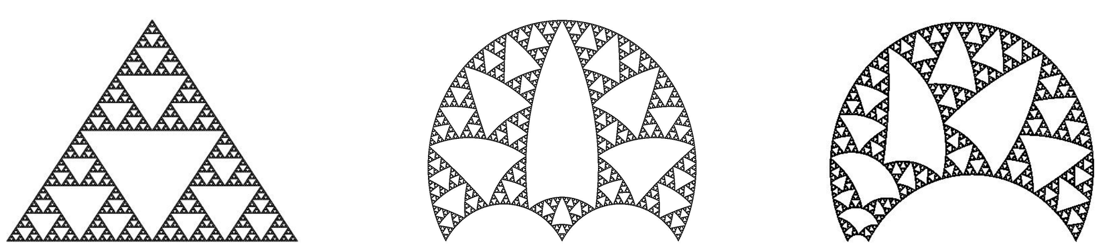
4. An application – Minkowski content of self-conformal sets

A direct application of the renewal theorems with dependent interarrival times in geometry, namely existence of the Minkowski content of self-conformal sets, is the focus of the current section. It is this application which led to developing the renewal theorems.

Let $B \subset \mathbb{R}^d$ be a bounded subset of d-dimensional Euclidean space $(\mathbb{R}^d, \|\cdot\|_2)$. We wish to understand the asymptotic behaviour of the *(e^{-t})-parallel volume* $\lambda_d(B_{\mathrm{e}^{-t}})$ of B as $t \to \infty$. Here, λ_d denotes the d-dimensional Lebesgue measure and $B_{\mathrm{e}^{-t}} := \{x \in \mathbb{R}^d \mid \inf_{b\in B} \|b - x\|_2 \leq \mathrm{e}^{-t}\}$ for $t \in \mathbb{R}$. More precisely, we study existence of the *Minkowski content*

$$\mathcal{M}(B) := \lim_{t\to\infty} \mathrm{e}^{t(d-D)} \lambda_d(B_{\mathrm{e}^{-t}}) \tag{4.1}$$

of B and determine its value when the limit exists. Eq. (4.1) implicitly assumes existence of the *Minkowski dimension* $D := d + \lim_{t\to\infty} t^{-1} \log \lambda_d(B_{\mathrm{e}^{-t}})$, which coincides with the box-counting dimension, see [Fal03, Prop. 3.2]. If $D \in \mathbb{N}$ and $\lambda_D(B) > 0$ then $\mathcal{M}(B) = \lambda_D(B)$. Otherwise $\lambda_d(B) = 0$ and the Minkowski content, when it exists, can be interpreted as D-dimensional volume of B, giving a substitute of the notion of volume for non-integer dimensions. Besides this geometric relevance, the Minkowski content plays a major role in the Weyl-Berry conjecture, which is concerned with spectral asymptotics of Laplace operators on domains with irregular boundaries, see [Ber79, Ber80, LP93]. These are two of the reasons why the Minkowski content has attracted much attention in recent years (see e. g. [Kom13] for a more in depth introduction).



(A) (B) (C)

Figure 1. The self-conformal sets from Ex. 4.1: (a) The Sierpinski gasket (self-similar). (b) and (c) Conformal images of the Sierpinski gasket (self-conformal but not self-similar).

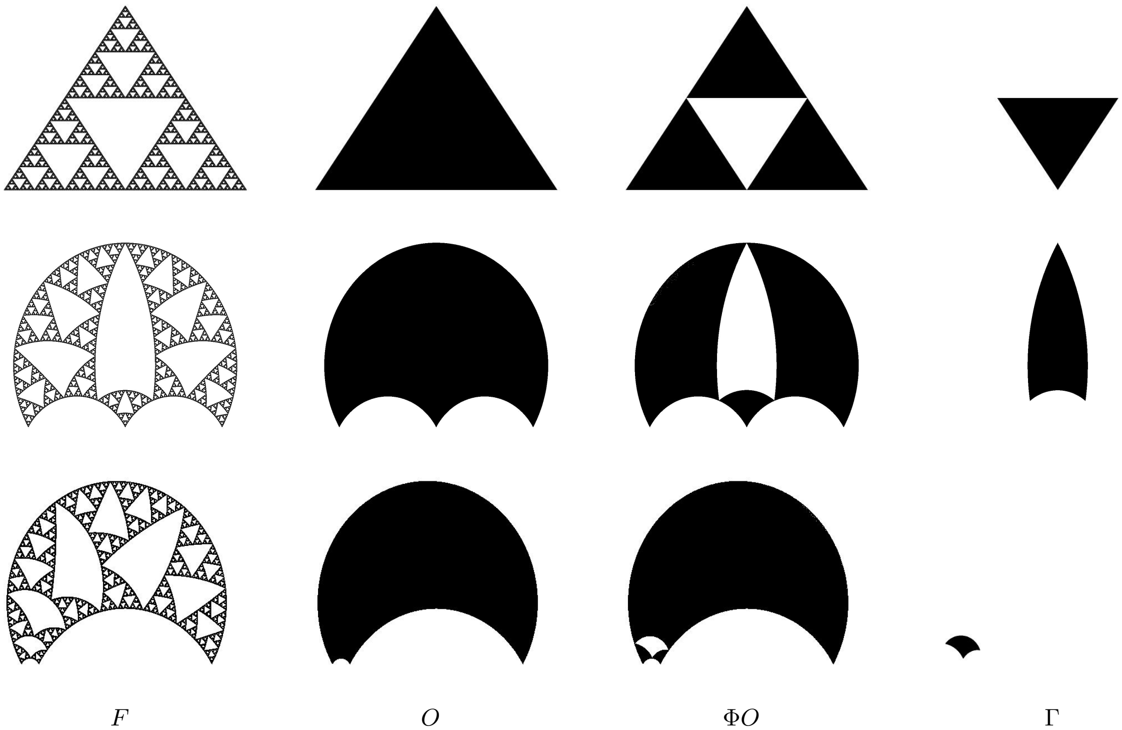


FIGURE 2. The self-conformal sets F from Ex. 4.1 (see Fig. 1), together with examples of associated feasible open sets O, ΦO and Γ.

For defining self-conformal sets let $X \subset \mathbb{R}^d$ be a non-empty compact set with $X = \overline{\text{int}X}$. Define $\Phi := \{\phi_1, \dots, \phi_M : X \to X\}$ to be an iterated function system (IFS) consisting of contractive conformal $\mathcal{C}^{1+\alpha}$-diffeomorphisms ϕ_i with $M \geq 2$, $\alpha \in (0,1)$. Let F be the unique compact non-empty set satisfying $F = \bigcup_{i=1}^M \phi_i F$. It exists due to [Hut81] and is called the *self-conformal set* associated with Φ. F is called *self-similar* if ϕ_i are *similitudes*, i. e. if $\|\phi_i(x) - \phi_i(y)\|_2 = c_i\|x - y\|_2$ for all $x, y \in X$ and $i \in \{1, \dots, M\} =: \Sigma$ with $c_i \in (0,1)$. Self-conformal sets provide prominent examples of fractal sets, see Ex. 4.1 and Fig. 1.

Next, we explain how to obtain the asymptotic behaviour of $\lambda_d(F_{\mathrm{e}^{-t}})$ as $t \to \infty$. Assume that Φ satisfies the *open set condition* (OSC), i.e. there exists a *feasible open set* $O \subset X$, which is non-empty and which satisfies $\phi_i O \subset O$ and $\phi_i O \cap \phi_j O = \varnothing$ for $i \neq j \in \Sigma$. Assume w. l. o. g. that O is bounded. For $\omega = \omega_1 \cdots \omega_n \in \Sigma^n$ write $\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}$. Set $\Gamma := O \setminus \bigcup_{i=1}^M \phi_i O$, $\Phi\Gamma := \bigcup_{i=1}^M \phi_i \Gamma$ and note that

$$O = \dot{\bigcup_{n=0}^{\infty}} \bigcup_{u \in \Sigma^n} \phi_u \Gamma \cup \bigcap_{n=0}^{\infty} \Phi^n O, \tag{4.2}$$

where the unions are disjoint. For the sets from Fig. 1 examples of sets O, with associated sets ΦO and Γ are depicted in Fig. 2. We have $\Phi\left(\overline{\bigcap_{n=0}^\infty \Phi^n O}\right) = \overline{\bigcap_{n=0}^\infty \Phi^n O}$. Thus, $\overline{\bigcap_{n=0}^\infty \Phi^n O}$ is either empty or coincides with F by uniqueness of the self-conformal set. Therefore, $\lambda_d\left(\bigcap_{n=0}^\infty \Phi^n O\right) \leq \lambda_d(F)$. Let D denote the Minkowski dimension of F. If $D < d$ then $\lambda_d(F) = 0$ and whence $\lambda_d\left(\bigcap_{n=0}^\infty \Phi^n O\right) = 0$. In the following we assume that O can be chosen so that $\lambda_d(F_{\mathrm{e}^{-t}} \cap \Gamma) \in \mathfrak{o}(\mathrm{e}^{t(D-d)})$ with the little Landau symbol $\mathfrak{o}$. This is a mild condition, which is always satisfied for self-similar systems with any feasible open set O (see [Win15]). Eq. (4.2) thus implies for $D < d$ that

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap O) = \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_u \Gamma) = \sum_{\omega \in \Sigma^m} \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_u \phi_\omega \Gamma) + \mathfrak{o}(\mathrm{e}^{t(D-d)})$$

for any $m \in \mathbb{N}$. Further, suppose $\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_u\phi_\omega\Gamma) = \lambda_d((\phi_u F)_{\mathrm{e}^{-t}} \cap \phi_u\phi_\omega\Gamma)$. By [Win15], for self-similar systems this assumption always holds true for some feasible open set O. (The open sets O of Fig. 2 satisfy the above conditions, see Ex. 4.1.) If m is large then $\phi_\omega\Gamma$ is small, as each ϕ_i is strictly contracting. Since conformal maps locally behave like similarities and λ_d is homogeneous of degree d, we can approximate $\lambda_d((\phi_u F)_{\mathrm{e}^{-t}} \cap \phi_u\phi_\omega\Gamma)$ by

$$|\phi_u'(\pi\sigma\omega x)|^d \lambda_d(F_{\mathrm{e}^{-t}/|\phi_u'(\pi\sigma\omega x)|} \cap \phi_\omega\Gamma) \tag{4.3}$$

with an arbitrary $x \in \Sigma^{\mathbb{N}}$. Here, $\pi\colon \Sigma^{\mathbb{N}} \to F$ is the *code map* defined by $\{\pi(\omega)\} := \bigcap_{n=0}^{\infty} \phi_{\omega|_n}(X)$. In order to bring (4.3) into a form to apply the renewal theorem we define $\xi\colon \Sigma^{\mathbb{N}} \to \mathbb{R}$ by

$$\xi(\omega) := -\log|\phi_{\omega_1}'(\pi\sigma\omega)|.$$

The function ξ is called the *geometric potential function* associated with the IFS Φ. It carries important geometric information of Φ and F. By definition $\exp(-S_n\xi(u\omega x)) = |\phi_u'(\pi\sigma\omega x)|$. Thus, $\lambda_d(F_{\mathrm{e}^{-t}} \cap O)$ can be approximated by

$$\sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathrm{e}^{-dS_n\xi(u\omega x)} \lambda_d(F_{\mathrm{e}^{-t+S_n\xi(u\omega x)}} \cap \phi_\omega\Gamma) + \mathfrak{o}(\mathrm{e}^{t(D-d)}).$$

Setting $f_y(t) := \lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\omega\Gamma)$ for $y \in \Sigma^{\mathbb{N}}$, $\chi := \mathbb{1}_{\Sigma^{\mathbb{N}}}$, $\eta := -d\xi$ and assuming the regularity conditions of Sec. 3, we can apply Thm. 3.2 and, if ξ is non-lattice, obtain

$$\sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathrm{e}^{-dS_n\xi(u\omega x)} \lambda_d(F_{\mathrm{e}^{-t+S_n\xi(u\omega x)}} \cap \phi_\omega\Gamma) = N(t,\omega x) \sim \mathrm{e}^{t\delta} \frac{h_{-(d+\delta)\xi}(\omega x)}{\int \xi \mathrm{d}\mu_{-(d+\delta)\xi}} \int_{-\infty}^{\infty} \mathrm{e}^{-T\delta} \lambda_d(F_{\mathrm{e}^{-T}} \cap \phi_\omega\Gamma)\mathrm{d}T,$$

where $\delta > 0$ is the unique value for which $P(-(d+\delta)\xi) = 0$, see Sec. 2.3. It is proven in [Bed88] that the Minkowski dimension D of F is the unique solution to $P(-D\xi) = 0$ thus, $d + \delta = D$. Using the bounded distortion property [MU96, Lem. 2.3.1] we all in all obtain, in the non-lattice situation, that

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap O) \sim \mathrm{e}^{t(D-d)} \lim_{m\to\infty} \sum_{\omega\in\Sigma^m} \frac{h_{-D\xi}(\omega x)}{\int \xi \mathrm{d}\mu_{-D\xi}} \int_{-\infty}^{\infty} \mathrm{e}^{-T(D-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap \phi_\omega\Gamma)\mathrm{d}T.$$

This shows that ξ being non-lattice implies existence of the Minkowski content which can be determined explicitly. The lattice and the average cases can be treated similarly.

In the special case that F is self-similar, the approximation arguments are not needed and we can assume $m = 0$. Further, $h_{-D\xi} = \mathbb{1}_{\Sigma_A}$ and ξ is constant on cylinder sets of length one, taking the values $-\log(r_i)$ with $r_i := \|\phi_i'\|_\infty$ and $\mu_{-D\xi}([i]) = r_i^D$, where $\|\cdot\|_\infty$ denotes the supremum-norm. Therefore, for self-similar sets F we obtain

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap O) \sim \mathrm{e}^{t(D-d)} \frac{1}{-\sum_{i=1}^{M} \log(r_i) r_i^D} \int_{-\infty}^{\infty} \mathrm{e}^{-T(D-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap \Gamma)\mathrm{d}T$$

if ξ is non-lattice. If ξ is lattice then the respective formula is more involved but we can deduce from Thm. 3.2(iii) that

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(D-d)} \lambda_d(F_{\mathrm{e}^{-t}} \cap O)\mathrm{d}T = \frac{\int_{-\infty}^{\infty} \mathrm{e}^{-T(D-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap \Gamma)\mathrm{d}T}{-\sum_{i=1}^{M} \log(r_i) r_i^D}. \tag{4.4}$$

Example 4.1. The following self-conformal sets are depicted in Figs. 1 and 2. We will discuss the Minkowski content for the first set (a) and restrict ourselves to the definitions and verification of the main conditions for (b) and (c). In all the examples, $d = 2$.

(a) The Sierpinski gasket F is the self-similar set associated with the IFS $\Phi := \{\phi_1, \phi_2, \phi_3 \colon \mathbb{R}^2 \to \mathbb{R}^2\}$ given by $\phi_1(x) = x/2$, $\phi_2(x) = x/2 + (1/2, 0)$ and $\phi_3(x) = x/2 + (1/4, \sqrt{3}/4)$. Its Minkowski dimension is $D = \log(3)/\log(2)$. The open triangle O with vertices $(0,0)$, $(1,0)$, $(1/2, \sqrt{3}/2)$ is a feasible open set for Φ. The set Γ is the equilateral triangle with vertices $(1/2, 0)$, $(1/4, \sqrt{3}/4)$, $(3/4, \sqrt{3}/4)$, and

$$\lambda_2(F_{\mathrm{e}^{-t}} \cap \Gamma) = \begin{cases} \frac{3}{2}\mathrm{e}^{-t} - 3\sqrt{3}\mathrm{e}^{-2t} & : t > -\log(\sqrt{3}/12) \\ \frac{\sqrt{3}}{16} & : t \leq -\log(\sqrt{3}/12). \end{cases}$$

Thus, $\lambda_2(F_{\mathrm{e}^{-t}} \cap \Gamma) \in \mathfrak{o}(\mathrm{e}^{t(D-2)})$. We have $r_i := \|\phi_i'\|_\infty = 1/2$ for $i \in \{1,2,3\}$. Hence, the system is lattice and (4.4) yields that

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(D-2)} \lambda_2(F_{\mathrm{e}^{-T}} \cap O) \mathrm{d}T = \frac{\sqrt{3}^{-(D+1)}}{\log(2)} \left[\frac{1}{2-D} + \frac{2}{D-1} - \frac{1}{D} \right] \approx 1.8126.$$

(b) The set F depicted in Fig. 1(b) is the image of the Sierpinski gasket under the complex Möbius transform $g_1 \colon \mathbb{C} \to \mathbb{C}$, $z \mapsto -\mathbf{i}z/(\sqrt{3}z - \sqrt{3} - \mathbf{i})$. An associated IFS is $\{g_1 \circ \phi_i \circ g_1^{-1} \mid i \in \{1,2,3\}\}$ with ϕ_1, ϕ_2, ϕ_3 as in (a). The maps $g_1 \circ \phi_i \circ g_1^{-1}$ are contractive conformal $\mathcal{C}^{1+\alpha}$-diffeomorphisms, but not similitudes. Indeed, F is self-conformal but not self-similar. It is known that the Minkowski dimension is stable under bi-Lipschitz maps, see [Fal03, Ch. 3.2]. Thus, $D = \log(3)/\log(2)$. Since $\partial\Gamma \subseteq F$ is one-dimensional we have that $\lambda_2(F_{\mathrm{e}^{-t}} \cap \Gamma) \in \mathfrak{O}(\mathrm{e}^{-t})$ with the big Landau symbol $\mathfrak{O}$, whence $\lambda_2(F_{\mathrm{e}^{-t}} \cap \Gamma) \in \mathfrak{o}(\mathrm{e}^{t(D-d)})$.

(c) In Fig. 1(c) the image of the Sierpinski gasket under the complex Möbius transform $g_2 \colon \mathbb{C} \to \mathbb{C}$, $z \mapsto (1 - \sqrt{3}\mathbf{i})z/((12 + 10\sqrt{3}\mathbf{i})z - 11 - 11\sqrt{3}\mathbf{i})$ is depicted. Analogous to (b) an associated IFS is given by $\{g_2 \circ \phi_i \circ g_2^{-1} \mid i \in \{1,2,3\}\}$ and $\lambda_2(F_{\mathrm{e}^{-t}} \cap \Gamma) \in \mathfrak{O}(\mathrm{e}^{-t})$.

Remark 4.2. The setting of self-similar sets has been studied e. g. in [LvF06, Fal95, Gat00, Win15]. Self-conformal subsets of $\mathbb{R}$ and limit sets of graph-directed systems in $\mathbb{R}$, including Fuchsian groups of Schottky type, are treated in [KK12] and [KK15], where results of [Lal89] were applied. However, the setting of [Lal89] is too restrictive for higher dimensional Euclidean spaces $\mathbb{R}^d$, making necessary the renewal theorems developed in the present article. Notice, Lalley's renewal theorems have already been applied in [Lal89] to problems in hyperbolic geometry.

5. Proofs of the renewal theorems and their corollaries

To prove the renewal theorems (Thm. 3.2 and 3.3) we pursue an analytic approach using Fourier-Laplace transforms. In the present setting, the Fourer-Laplace transforms naturally lead to the Ruelle-Perron-Frobenius operator. Thus, in Sec. 5.1 we present essentials from complex Ruelle-Perron-Frobenius theory. These are used to prove the renewal theorems in Sec. 5.2.

5.1. **Analytic Properties of the Ruelle-Perron-Frobenius Operator.** Here, we collect results from [Pol84, Lal89, PP90] concerning analytic properties of the operator-valued function $z \mapsto (I - \mathcal{L}_{\eta+z\xi})^{-1}$, where $\xi, \eta \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$ are fixed and $z \in \mathbb{C}$.

Let $\mathcal{B}(\mathcal{F}_\alpha(\Sigma_A))$ denote the set of bounded linear operators on $\mathcal{F}_\alpha(\Sigma_A)$ to $\mathcal{F}_\alpha(\Sigma_A)$. Since $\mathcal{F}_\alpha(\Sigma_A)$ endowed with $\|\cdot\|_\alpha := |\cdot|_\alpha + \|\cdot\|_\infty$ is a Banach space, also $(\mathcal{B}(\mathcal{F}_\alpha(\Sigma_A)), \|\cdot\|_{\mathrm{op}})$ is a Banach space, [Kat95, p.150]. Here, $\|B\|_{\mathrm{op}} := \sup_{g \in \mathcal{F}_\alpha(\Sigma_A),\, \|g\|_\alpha = 1} \|Bg\|_\alpha$ for $B \in \mathcal{B}(\mathcal{F}_\alpha(\Sigma_A))$ denotes the *operator norm*. A function $f \colon D \to \mathcal{B}(\mathcal{F}_\alpha(\Sigma_A))$ defined on an open domain $D \subset \mathbb{C}$ is called *holomorphic* if, for all $z \in D$, there exists $l(z) \in \mathcal{B}(\mathcal{F}_\alpha(\Sigma_A))$ such that $\lim_{h\to 0} \|h^{-1}(f(z+h) - f(z)) - l(z)\|_{\mathrm{op}} = 0$. Following convention, we interchangeably use the terms holomorphic and *analytic*. For clarity we write $\varrho := \eta + z\xi$ and sometimes $\varrho(z) := \eta + z\xi$ if we want to stress dependence on z. The operator $\mathcal{L}_{\varrho(z)}$ is a bounded linear operator on the Banach space $(\mathcal{F}_\alpha(\Sigma_A), \|\cdot\|_\alpha)$. We write $\mathrm{spec}(\mathcal{L}_{\varrho(z)}) := \{\lambda \in \mathbb{C} \mid \mathcal{L}_{\varrho(z)} - \lambda I \text{ is not invertible}\}$ for its *spectrum* and $\mathrm{spr}(\mathcal{L}_{\varrho(z)})$ for its *spectral radius*, i. e. the radius of the smallest closed disc centred at the origin which contains $\mathrm{spec}(\mathcal{L}_{\varrho(z)})$. By the *spectral radius formula*, $\lim_{n\to\infty} \|\mathcal{L}^n_{\varrho(z)}\|^{1/n}_{\mathrm{op}} = \mathrm{spr}(\mathcal{L}_{\varrho(z)})$. For the following statement let $\Re(z)$ and $\Im(z)$ respectively denote the real and imaginary parts of $z \in \mathbb{C}$.

Theorem 5.1 ([Pol84])**.** *Let $\alpha \in (0,1)$ and $\xi, \eta \in \mathcal{F}_\alpha(\Sigma_A)$. Suppose that $z \in \mathbb{C} \setminus \mathbb{R}$.*

(i) If for some $b \in \mathbb{R}$ the function $(\Im(z)\xi - b)/(2\pi)$ is co-homologous to an integer-valued function, then $\mathrm{e}^{\mathbf{i}b}\gamma_{\eta+\Re(z)\xi}$ is a simple eigenvalue of $\mathcal{L}_{\eta+z\xi}$, and the rest of the spectrum is contained in a disc centred at zero of radius strictly less than $\gamma_{\eta+\Re(z)\xi}$.

(ii) Otherwise, the entire spectrum of $\mathcal{L}_{\eta+z\xi}$ is contained in a disc centred at zero of radius strictly less than $\gamma_{\eta+\Re(z)\xi}$.

Below, we present useful results that follow from [Lal89] and Thm. 5.1. Note that in [Lal89] only the special case that $\eta \equiv 0$ is covered. However, the proofs work in the same way when $\eta \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$ is arbitrary. Therefore, we omit the proofs and refer the reader to the respective proofs in [Lal89].

Results in regular perturbation theory [Kat95, Sec. 7.1 and 4.3] imply that $z \mapsto \gamma_{\varrho(z)}$, $z \mapsto h_{\varrho(z)}$ and $z \mapsto \nu_{\varrho(z)}$ extend to holomorphic functions in a neighbourhood of $\mathbb{R}$ such that $\gamma_{\varrho(z)} \neq 0$, $\mathcal{L}_{\varrho(z)} h_{\varrho(z)} = \gamma_{\varrho(z)} h_{\varrho(z)}$, $\mathcal{L}^*_{\varrho(z)} \nu_{\varrho(z)} = \gamma_{\varrho(z)} \nu_{\varrho(z)}$ and $\nu_{\varrho(z)}(h_{\varrho(z)}) = \nu_0(h_{\varrho(z)}) = 1$ [Lal89, p. 27].

Proposition 5.2 ([Lal89, Props. 7.1 and 7.2]). *Fix $\alpha \in (0,1)$. Let $\xi, \eta \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$ and let $-\delta$ be the unique real zero of $t \mapsto P(\eta + t\xi)$. Then*

(i) $z \mapsto (I - \mathcal{L}_{\varrho(z)})^{-1}$ is holomorphic in the half-plane $\Re(z) < -\delta$.

(ii) $z \mapsto (I - \mathcal{L}_{\varrho(z)})^{-1}$ has a simple pole at $z = -\delta$ and for $\chi \in \mathcal{F}_\alpha(\Sigma_A)$,

$$(I - \mathcal{L}_{\varrho(z)})^{-1}\chi = \gamma_{\varrho(z)}(1 - \gamma_{\varrho(z)})^{-1} \int \chi \mathrm{d}\nu_{\varrho(z)} \cdot h_{\varrho(z)} + (I - \mathcal{L}''_{\varrho(z)})^{-1}\chi, \tag{5.1}$$

for z in some punctured neighbourhood of $z = -\delta$, where

$$\mathcal{L}''_{\varrho(z)} := \mathcal{L}_{\varrho(z)} - \mathcal{L}'_{\varrho(z)} \qquad \text{with} \qquad \mathcal{L}'_{\varrho(z)}\chi := \gamma_{\varrho(z)} \int \chi \mathrm{d}\nu_{\varrho(z)} \cdot h_{\varrho(z)}.$$

Moreover, $z \mapsto (I - \mathcal{L}''_{\varrho(z)})^{-1}$ is a holomorphic operator-valued function in a neighbourhood of $z = -\delta$.

The factor $\gamma_{\varrho(z)}$ of the first summand of (5.1) is missing in [Lal89]. However, the relevant z-value is $-\delta$, and $\gamma_{-\delta} = 1$.

We are interested in the residue of $z \mapsto (I - \mathcal{L}_{\varrho(z)})^{-1}$ at the simple pole $z = -\delta$. For this, we use that the topological pressure function $t \mapsto P(\eta + t\xi)$ is real-analytic for $t \in \mathbb{R}$ and $\xi, \eta \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$ and that it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\eta + t\xi) = \int \xi \mathrm{d}\mu_{\eta + t\xi}, \quad t \in \mathbb{R}. \tag{5.2}$$

The analyticity of $z \mapsto (I - \mathcal{L}_{\varrho(z)})^{-1}$ can be proved with methods of analytic perturbation theory as presented in [Kat95]. This method of proof is due to [Rue04]. Further, since $z \mapsto \gamma_{\varrho(z)}$ has an analytic continuation to a neighbourhood of $\mathbb{R}$ and $P(\xi) = \log(\gamma_\xi)$ for real-valued $\xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$, we can extend P analytically by setting $P(\varrho(z)) := \log(\gamma_{\varrho(z)})$. 'Formally this definition can only be made modulo $2\pi\mathbf{i}$ since log is multiple valued, although we shall ask that $P(\xi)$ be real-valued when ξ is real-valued' [PP90, p. 31]. In this way, (5.2) extends to a neighbourhood of $\mathbb{R}$. Combined with (5.2), Prop. 5.2(ii) yields the following corollary since $z \mapsto \gamma_{\varrho(z)}$, $z \mapsto \int \chi \mathrm{d}\nu_{\varrho(z)}$ and $z \mapsto h_{\varrho(z)}$ are continuous at $z = -\delta$.

Corollary 5.3. *Let $\xi \in \mathcal{C}(\Sigma_A, \mathbb{R})$ and $\chi \in \mathcal{F}_\alpha(\Sigma_A, \mathbb{R})$. Then, for $x \in \Sigma_A$, the residue of $(I - \mathcal{L}_{\eta + z\xi})^{-1}\chi(x)$ at $z = -\delta$ is*

$$-\frac{\int \chi \mathrm{d}\nu_{\eta - \delta\xi}}{\int \xi \mathrm{d}\mu_{\eta - \delta\xi}} h_{\eta - \delta\xi}(x).$$

The residue is given in [Lal89, p.27]; however, with a different sign.

Proposition 5.4 ([Lal89, Prop. 7.3]). *If ξ is non-lattice then $z \mapsto (I - \mathcal{L}_{\eta + z\xi})^{-1}$ is holomorphic in a neighbourhood of every z on the line $\Re(z) = -\delta$ except for $z = -\delta$.*

Proposition 5.5 ([Lal89, Prop. 7.4]). *If ξ is integer-valued but not co-homologous to any function valued in a proper subgroup of the integers, then $z \mapsto (I - \mathcal{L}_{\eta + z\xi})^{-1}$ is $2\pi\mathbf{i}$-periodic, and holomorphic at every z on the line $\Re(z) = -\delta$ such that $\Im(z)/(2\pi)$ is not an integer.*

5.2. **Proof of the renewal theorems.**

5.2.1. *Proof of the Renewal Thm. 3.2.* As the statement of Thm. 3.2 suggests, we need to distinguish between the cases of ξ being lattice or non-lattice. We start with the non-lattice situation, for which we use a standard smoothing argument for showing the desired asymptotic. For a probability density $\Pi \colon \mathbb{R} \to \mathbb{R}$ we consider its Fourier-Laplace transform (characteristic function) given by

$$\widehat{\Pi}(\mathbf{i}\theta) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathbf{i}\theta t} \Pi(t) \mathrm{d}t$$

and introduce the following class of probability densities.

$$\begin{aligned} \mathfrak{P} := \{\Pi \colon \mathbb{R} \to \mathbb{R} \mid \Pi \text{ is a probability density}, \Pi(t) = \Pi(-t) \text{ for } t \in \mathbb{R}. \\ \widehat{\Pi}(\mathbf{i}\theta) \text{ is non-negative, } \mathcal{C}^\infty \text{ and has compact support}\} \end{aligned}$$

Note that the function $\widehat{\Pi}\colon \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{\Pi}(\mathbf{i}\theta) := \begin{cases} \exp\left(\frac{-\theta^2}{1-\theta^2}\right) & : |\theta| \leq 1, \\ 0 & : \text{otherwise} \end{cases} \tag{5.3}$$

defines an even probability density $\Pi\colon \mathbb{R} \to \mathbb{R}$ which lies in $\mathfrak{P}$. That Π is a probability density is due to Bochner's theorem, see e. g. [Kle08, Satz 15.29]. Thus, $\mathfrak{P} \neq \varnothing$. For the following, fix Π as such. As Π is an even probability density we know that for all $\varepsilon > 0$ there exists $\tau > 0$ such that $\int_{-\tau}^{\tau} \Pi(t)\mathrm{d}t \geq 1 - \varepsilon$. For each $\varepsilon > 0$ fix such $\tau = \tau(\varepsilon)$. Thus, Π_ε which for $\varepsilon > 0$ is defined by

$$\Pi_\varepsilon(t) := \tfrac{\tau(\varepsilon)}{\varepsilon}\Pi\left(t\tfrac{\tau(\varepsilon)}{\varepsilon}\right) \tag{5.4}$$

satisfies $\int_{-\varepsilon}^{\varepsilon} \Pi_\varepsilon(t)\mathrm{d}t = \int_{-\tau(\varepsilon)}^{\tau(\varepsilon)} \Pi(t)\mathrm{d}t \geq 1 - \varepsilon$. Moreover, it can be verified that $\Pi_\varepsilon \in \mathfrak{P}$ for all $\varepsilon > 0$. The smoothing argument is as follows.

Lemma 5.6. *If for each sufficiently small $\varepsilon > 0$*

$$\lim_{r\to\infty} \int_{-\infty}^{\infty} \Pi_\varepsilon(r - T)\mathrm{e}^{-T\delta} N(T, x)\mathrm{d}T = G(x) \tag{5.5}$$

uniformly for $x \in \Sigma_A$, then the statement of Thm. 3.2(i) holds.

Proof. In the proof we distinguish between the different cases of (D).

Case (Da): This statement follows from the proof of [Lal89, Lemma 8.2].

Case (Dc): For $r \in \mathbb{R}$ and $\varepsilon > 0$, Condition (B) implies that

$$\left| \int_{-\infty}^{\infty} \Pi_\varepsilon(r - T)\mathrm{e}^{-T\delta} N(T, x)\mathrm{d}T - \int_{r-\varepsilon}^{r+\varepsilon} \Pi_\varepsilon(r - T)\mathrm{e}^{-T\delta} N(T, x)\mathrm{d}T \right| \leq \mathfrak{C}\varepsilon, \tag{5.6}$$

which tends to 0 as $\varepsilon \to 0$ uniformly for $x \in \Sigma_A$. Moreover, we observe that

$$\begin{aligned} \inf_{\widetilde{\varepsilon}\in[0,2\varepsilon)} \mathrm{e}^{-(r-\widetilde{\varepsilon})\delta} N(r - \widetilde{\varepsilon}, x)(1 - \varepsilon) &\leq \int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r - \varepsilon - T)\mathrm{e}^{-T\delta} N(T, x)\mathrm{d}T \\ &\leq \sup_{\widetilde{\varepsilon}\in[0,2\varepsilon)} \mathrm{e}^{-(r-\widetilde{\varepsilon})\delta} N(r - \widetilde{\varepsilon}, x). \end{aligned} \tag{5.7}$$

These observations imply that

$$\begin{aligned} G(x) &\overset{(5.5)}{=} \lim_{r\to\infty} \int_{-\infty}^{\infty} \Pi_\varepsilon(r - T)\mathrm{e}^{-T\delta} N(T, x)\mathrm{d}T = \liminf_{\varepsilon\searrow 0} \limsup_{r\to\infty} \int_{-\infty}^{\infty} \Pi_\varepsilon(r - T)\mathrm{e}^{-T\delta} N(T, x)\mathrm{d}T \\ &\overset{(5.6)}{\leq} \liminf_{\varepsilon\searrow 0} \limsup_{r\to\infty} \int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r - \varepsilon - T)\mathrm{e}^{-T\delta} N(T, x)\mathrm{d}T \overset{(5.7)}{\leq} \liminf_{\varepsilon\searrow 0} \limsup_{r\to\infty} \sup_{\widetilde{\varepsilon}\in[0,2\varepsilon)} \mathrm{e}^{-(r-\widetilde{\varepsilon})\delta} N(r - \widetilde{\varepsilon}, x) \end{aligned}$$

and likewise that

$$G(x) \geq \limsup_{\varepsilon\searrow 0} \liminf_{r\to\infty} \inf_{\widetilde{\varepsilon}\in[0,2\varepsilon)} \mathrm{e}^{-(r-\widetilde{\varepsilon})\delta} N(r - \widetilde{\varepsilon}, x).$$

Using (Dc) and the inequalities

$$\inf_{\widetilde{\varepsilon}\in[0,2\varepsilon)} \mathrm{e}^{-(r-\widetilde{\varepsilon})\delta} N(r - \widetilde{\varepsilon}, x) \leq \mathrm{e}^{-r\delta} N(r, x) \leq \sup_{\widetilde{\varepsilon}\in[0,2\varepsilon)} \mathrm{e}^{-(r-\widetilde{\varepsilon})\delta} N(r - \widetilde{\varepsilon}, x)$$

we conclude that $G(x) = \lim_{r\to\infty} \mathrm{e}^{-r\delta} N(r, x)$ uniformly for $x \in \Sigma_A$.

Case (Db): For $x \in \Sigma_A$ and $t \in \mathbb{R}$ set

$$g_x(t) := \mathrm{e}^{-t\delta} f_x(t).$$

Then $\{|g_x| \mid x \in \Sigma_A\}$ is equi d. R. i. by (Db). Moreover,

(5.8)

$$\mathrm{e}^{-t\delta} N(t, x) := \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} \chi(y) f_y(t - S_n\xi(y))\mathrm{e}^{S_n\eta(y)}\mathrm{e}^{-t\delta} = \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} \chi(y) g_y(t - S_n\xi(y))\mathrm{e}^{S_n(\eta-\delta\xi)(y)}.$$

For showing that (5.5) implies $\lim_{r\to\infty} \mathrm{e}^{-r\delta}N(r,x) = G(x)$, we consider

$$\begin{aligned}
&A_\varepsilon(r,x)\\
&:= \left|\int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta}N(T,x)\mathrm{d}T - \mathrm{e}^{-r\delta}N(r,x)\right|\\
&\overset{(5.6)}{\leq} \left|\int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r-\varepsilon-T)\mathrm{e}^{-T\delta}N(T,x)\mathrm{d}T - \int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r-\varepsilon-T)\mathrm{e}^{-r\delta}N(r,x)\mathrm{d}T\right| + \mathfrak{C}\varepsilon + \left|\mathrm{e}^{-r\delta}N(r,x)\right|\varepsilon\\
&\overset{(B),(5.8)}{\leq} \left|\int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r-\varepsilon-T)\sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x} \chi(y)\mathrm{e}^{S_n(\eta-\delta\xi)(y)}\left[g_y(T-S_n\xi(y)) - g_y(r-S_n\xi(y))\right]\mathrm{d}T\right| + 2\mathfrak{C}\varepsilon\\
&\leq \sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x} \chi(y)\mathrm{e}^{S_n(\eta-\delta\xi)(y)}\int_{r-2\varepsilon}^{r} \Pi_\varepsilon(r-\varepsilon-T)\left|g_y(T-S_n\xi(y)) - g_y(r-S_n\xi(y))\right|\mathrm{d}T + 2\mathfrak{C}\varepsilon.
\end{aligned}$$

For the last inequality we have used the monotone convergence theorem.

Set

$$d^{2\varepsilon}(t) := \sup_{y\in\Sigma_A}\left(\sup_{\widetilde{\varepsilon}\in[0,2\varepsilon)} g_y(t-\widetilde{\varepsilon}) - \inf_{\widetilde{\varepsilon}\in[0,2\varepsilon)} g_y(t-\widetilde{\varepsilon})\right).$$

Since $\{g_x \mid x\in\Sigma_A\}$ is equi d. R. i. we know, for sufficiently small $\varepsilon>0$, that

$$\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) \quad \text{exists and} \quad \lim_{\varepsilon\to 0}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\cdot 2\varepsilon = 0. \tag{5.9}$$

Therefore, we may deduce the following chain of inequalities.

$$\begin{aligned}
A_\varepsilon(r,x) - 2\mathfrak{C}\varepsilon &\leq \sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x} \chi(y)\mathrm{e}^{S_n(\eta-\delta\xi)(y)} d^{2\varepsilon}(r-S_n\xi(y))\\
&\leq \sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\|\chi\|_\infty \sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x} \mathrm{e}^{S_n(\eta-\delta\xi)(y)}\mathbb{1}_{((k-1)2\varepsilon,(k+1)2\varepsilon]}(r-S_n\xi(y))\\
&\leq c\|\chi\|_\infty \sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\underbrace{\sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x} \mu_{\eta-\delta\xi}([y|_n])\mathbb{1}_{((k-1)2\varepsilon,(k+1)2\varepsilon]}(r-S_n\xi(y))}_{=:a_{\varepsilon,k}(r,x)},
\end{aligned}$$

where for the last estimate we have used the Gibbs property (2.3) of $\mu_{\eta-\delta\xi}$ with constant c. For simplicity of presentation we first treat the case that ξ is strictly positive. In this case there exists $\kappa>0$ such that $\xi(x)\geq\kappa$ for all $x\in\Sigma_A$ since ξ is continuous and Σ_A is compact. As we will later be interested in the limiting behaviour when $\varepsilon\to 0$, we can freely assume that $4\varepsilon\leq\kappa$ and set $m := \lfloor \kappa/(2\varepsilon)\rfloor$. Note that $m\geq 2$. Let $\sigma^{-n}(x)$ denote the set of pre-images of x under σ^n, that is

$$\sigma^{-n}(x) := \{y\in\Sigma_A \mid \sigma^n y = x\}.$$

Let $y\in\sigma^{-n}(x)$ and assume that $r-S_n\xi(y)\in((k-1)2\varepsilon,(k+1)2\varepsilon] =: I_k(\varepsilon)$. Then $\sigma y\in\sigma^{-(n-1)}(x)$ and

$$r - S_{n-1}\xi(\sigma y) = r - S_n\xi(y) + \xi(y) \geq r - S_n\xi(y) + \kappa > (k-1)2\varepsilon + 2m\varepsilon = ((k-2+m)+1)2\varepsilon,$$

whence $r - S_{n-1}\xi(\sigma y)\notin I_{k+q}(\varepsilon)$ for $q\in\{0,\dots,m-2\}$. Now let $\widetilde{y}\in\sigma^{-(n+1)}(x)$ be such that $\sigma\widetilde{y} = y$. Then

$$r - S_{n+1}\xi(\widetilde{y}) = r - S_n\xi(y) - \xi(\widetilde{y}) \leq r - S_n\xi(y) - \kappa \leq (k+1)2\varepsilon - 2m\varepsilon = ((k-m+2)-1)2\varepsilon,$$

whence $r - S_{n+1}\xi(\widetilde{y})\notin I_{k-q}(\varepsilon)$ for $q\in\{0,\dots,m-2\}$. Therefore,

$$\sum_{q=0}^{m-2} a_{\varepsilon,k+q}(r,x) = \sum_{q=0}^{m-2}\sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x} \mu_{\eta-\delta\xi}([y|_n])\mathbb{1}_{((k+q-1)2\varepsilon,(k+q+1)2\varepsilon]}(r-S_n\xi(y)) \leq 1, \tag{5.10}$$

as $\mu_{\eta-\delta\xi}$ is a probability measure on Σ_A.

Next, we show existence of $q \in \{0, \dots, m-2\}$ such that, for all $\varepsilon < \kappa/8$,

$$\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) a_{\varepsilon,k+q}(r,x) \leq \frac{2}{\kappa}\cdot 2\varepsilon \sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon). \tag{5.11}$$

Assume that this was not the case. Since for $\varepsilon < \kappa/8$ we have that $1 < 2(\kappa - 4\varepsilon)/\kappa < 2\varepsilon(m-1)\cdot 2/\kappa$, it follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) < 2\varepsilon(m-1)\frac{2}{\kappa}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) &= \sum_{q=0}^{m-2} 2\varepsilon\frac{2}{\kappa}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\\ &\overset{\neg(5.11)}{<} \sum_{q=0}^{m-2}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) a_{\varepsilon,k+q}(r,x) \overset{(5.10)}{\leq} \sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon).\end{aligned}$$

As $\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)$ exists for sufficiently small ε, it follows that (5.11) must be true for some $q \in \{0,\dots,m-2\}$.

Notice $a_{\varepsilon,k}(r - 2q\varepsilon, x) = a_{\varepsilon,k+q}(r,x)$, and so for this q,

$$\begin{aligned}\limsup_{r\to\infty} A_\varepsilon(r,x) &= \limsup_{r\to\infty} A_\varepsilon(r-2q\varepsilon,x)\\ &\leq c\|\chi\|_\infty \limsup_{r\to\infty}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon)\underbrace{a_{\varepsilon,k}(r-2q\varepsilon,x)}_{=a_{\varepsilon,k+q}(r,x)} + 2\mathfrak{C}\varepsilon\\ &\overset{(5.11)}{\leq} c\|\chi\|_\infty \frac{2}{\kappa}2\varepsilon\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) + 2\mathfrak{C}\varepsilon.\end{aligned} \tag{5.12}$$

All in all we have for each sufficiently small $\varepsilon > 0$

$$\begin{aligned}&\limsup_{r\to\infty}\left|G(x) - \mathrm{e}^{-r\delta}N(r,x)\right|\\ &\leq \limsup_{r\to\infty}\left|G(x) - \int_{-\infty}^{\infty}\Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta}N(T,x)\mathrm{d}T\right| \limsup_{r\to\infty}\left|\int_{-\infty}^{\infty}\Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta}N(T,x)\mathrm{d}T - \mathrm{e}^{-r\delta}N(r,x)\right|\\ &\overset{(5.5)}{=} \limsup_{r\to\infty} A_\varepsilon(r,x)\\ &\overset{(5.12)}{\leq} c\|\chi\|_\infty\frac{2}{\kappa}2\varepsilon\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) + 2\mathfrak{C}\varepsilon.\end{aligned}$$

This implies

$$\limsup_{r\to\infty}\left|G(x) - \mathrm{e}^{-r\delta}N(r,x)\right| \leq \limsup_{\varepsilon\to 0}\left(c\|\chi\|_\infty 2\varepsilon\frac{2}{\kappa}\sum_{k\in\mathbb{Z}} d^{2\varepsilon}(k\cdot 2\varepsilon) + 2\mathfrak{C}\varepsilon\right) \overset{(5.9)}{=} 0.$$

Now we treat the case that ξ is not strictly positive. By assumption there exists $n \in \mathbb{N}$ for which $S_n\xi$ is strictly positive. Since ξ is continuous and Σ_A is compact, there exists a $\widetilde{\kappa} > 0$ for which $S_n\xi \geq \widetilde{\kappa}$. For $l \in \mathbb{N}$ and $j \in \{0,\dots,n-1\}$ this implies that

$$\begin{aligned}S_{ln+j}\xi(x) &= \sum_{i=0}^{ln+j-1}\xi(\sigma^i x) = \sum_{m=0}^{l-1}\sum_{i=mn}^{(m+1)n-1}\xi(\sigma^i x) + \sum_{i=ln}^{ln+j-1}\xi(\sigma^i x)\\ &= \sum_{m=0}^{l-1} S_n\xi(\sigma^i(\sigma^{mn}x)) + \sum_{i=ln}^{ln+j-1}\xi(\sigma^i x) \geq l\widetilde{\kappa} + \inf_{\substack{x\in\Sigma_A\\ 0\leq i\leq n-1}} S_i\xi(x)\end{aligned} \tag{5.13}$$

From this we can conclude that there exists $m^* \in \mathbb{N}$ such that $S_m\xi$ is strictly positive for all $m \geq m^*$. Thus, there exists $\kappa > 0$ with $S_m\xi \geq \kappa$ for all $m \geq m^*$. In the same way as in the case that ξ is strictly positive we can show that if $y \in \sigma^{-n}(x)$ with $r - S_n\xi(y) \in I_k(\varepsilon)$ then $r - S_{n-m^*}\xi(\sigma^{m^*}y) \notin I_{k+q}(\varepsilon)$ for $q \in \{0,\dots,\lfloor\kappa/(2\varepsilon)\rfloor - 2\}$ whenever $n \geq m^*$ and $r - S_{n+m^*}\xi(\widetilde{y}) \notin I_{k-q}(\varepsilon)$ for all $q \in \{0,\dots,\lfloor\kappa/(2\varepsilon)\rfloor - 2\}$

where $\widetilde{y} \in \sigma^{-(n+m^*)}(x)$ is such that $\sigma^{m^*}\widetilde{y} = y$. This implies that

$$\sum_{q=0}^{\lfloor \kappa/(2\varepsilon)\rfloor -2} \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} \mu_{\eta-\delta\xi}([y|_n]) \mathbb{1}_{((k+q-1)2\varepsilon,(k+q+1)2\varepsilon]}(r - S_n\xi(y)) \leq 2m^*.$$

The remainder of the proof follows in the same way as in the case that ξ is strictly positive. □

Proof of Thm. 3.2(i). For $x \in \Sigma_A$ consider the Fourier-Laplace transform of $t \mapsto \mathrm{e}^{-t\delta}N(t,x)$ at $z \in \mathbb{C}$:

$$L(z,x) := \int_{-\infty}^{\infty} \mathrm{e}^{zT}\mathrm{e}^{-T\delta}N(T,x)\mathrm{d}T. \tag{5.14}$$

This approach is inspired by [Lal89, Thm. 1] whose proof we extend by some technical twists which are necessary in our more general setting. Conditions (B) and (C) imply that $L(\cdot,x)\colon \mathbb{C}\to\mathbb{C}$, $z \mapsto L(z,x)$ is well-defined and analytic on $\{z \in \mathbb{C} \mid -s < \Re(z) < 0\}$. What is more, for small enough $\varepsilon > 0$, Conditions (B) and (C) imply that $L(\cdot,x)$ converges absolutely and uniformly on $\{z \in \mathbb{C} \mid -s+\varepsilon \leq \Re(z) \leq -\varepsilon\}$. Now, in every such region, using (B) as well as the monotone and dominated convergence theorems, we obtain

$$\begin{aligned} L(z,x) &= \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} \mathrm{e}^{S_n(\eta+(z-\delta)\xi)(y)}\chi(y) \int_{-\infty}^{\infty} \mathrm{e}^{(z-\delta)T} f_y(T)\mathrm{d}T \\ &= \sum_{n=0}^{\infty} \mathcal{L}^n_{\eta+(z-\delta)\xi}\left(\chi \int_{-\infty}^{\infty} \mathrm{e}^{(z-\delta)T} f_{\cdot}(T)\mathrm{d}T\right)(x), \end{aligned}$$

where $f_{\cdot}(T)\colon \Sigma_A \to \mathbb{R}$, $x \mapsto f_x(T)$. Note that Conditions (A) to (C) imply that $\int_{-\infty}^{\infty} \mathrm{e}^{(z-\delta)T} f_x(T)\mathrm{d}T$ exists if $-s < \Re(z) \leq 0$, since $\chi(x)|f_x(t)| \leq N^{\mathrm{abs}}(t,x)$. Thus, by Thm. 5.1(ii), the spectral radius formula (Sec. 5.1) and the fact that $\gamma_{\eta-\delta\xi} = 1$ (see Prop. 2.4), the above series converges for $-s < \Re(z) < 0$, and we obtain

$$L(z,x) = (I - \mathcal{L}_{\eta+(z-\delta)\xi})^{-1}\left(\chi \int_{-\infty}^{\infty} \mathrm{e}^{(z-\delta)T} f_{\cdot}(T)\mathrm{d}T\right)(x).$$

By Prop. 5.4, the operator-valued function $z \mapsto (I - \mathcal{L}_{\eta+(z-\delta)\xi})^{-1}$ is holomorphic at every z on the line $\Re(z) = 0$ except for $z = 0$, which is a simple pole by Prop. 5.2. Thus, according to Cor. 5.3 the residue of $z \mapsto L(z,x)$ at $z = 0$ is

$$-\frac{\int_{\Sigma_A}\chi(y)\int_{-\infty}^{\infty}\mathrm{e}^{-T\delta}f_y(T)\mathrm{d}T\mathrm{d}\nu_{\eta-\delta\xi}(y)}{\int \xi \mathrm{d}\mu_{\eta-\delta\xi}} h_{\eta-\delta\xi}(x) = -G(x), \tag{5.15}$$

where G is as in Thm. 3.2(i). Hence, $L(z,x)$ has the following representation.

$$L(z,x) = q(z,x) - \frac{G(x)}{z}, \tag{5.16}$$

where $q(\cdot,x)\colon \mathbb{C}\to\mathbb{C}$, $z \mapsto q(z,x)$ is holomorphic in a region containing the strip $\{z \in \mathbb{C} \mid -s+\varepsilon \leq \Re(z) \leq 0\}$ with sufficiently small $\varepsilon > 0$. Conditions (B), (C) and Lebesgue's dominated convergence theorem now imply for every $\varepsilon \in (0,1]$ that

$$\int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-T\delta}N(T,x)\mathrm{d}T = \lim_{\beta \searrow 0}\int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-T(\delta+\beta)}N(T,x)\mathrm{d}T. \tag{5.17}$$

Using the inverse Fourier-Laplace transform $\Pi_\varepsilon(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathbf{i}\theta t}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\mathrm{d}\theta/(2\pi)$ and that the integral on the left hand side of (5.17) exists, we can convert the integral from the right hand side of (5.17) for sufficiently small

$\beta > 0$ as follows.

$$
\begin{aligned}
\int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-\beta T - T^\delta} N(T,x)\mathrm{d}T &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathbf{i}\theta(r-T)}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-\beta T - T^\delta} N(T,x)\mathrm{d}T \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{(\mathbf{i}\theta-\beta)T}\mathrm{e}^{-T^\delta} N(T,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\mathrm{e}^{-\mathbf{i}\theta r}\frac{\mathrm{d}\theta}{2\pi}\mathrm{d}T \\
&= \int_{-\infty}^{\infty} L(\mathbf{i}\theta-\beta,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\mathrm{e}^{-\mathbf{i}\theta r}\frac{\mathrm{d}\theta}{2\pi} \\
&\overset{(5.16)}{=} \int_{-\infty}^{\infty} \left(q(\mathbf{i}\theta-\beta,x) + \frac{G(x)(\mathbf{i}\theta+\beta)}{\theta^2+\beta^2}\right)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\mathrm{e}^{-\mathbf{i}\theta r}\frac{\mathrm{d}\theta}{2\pi}.
\end{aligned} \tag{5.18}
$$

The measures given by $\frac{\beta}{\pi(\theta^2+\beta^2)}\mathrm{d}\theta$ converge weakly to the Dirac point-mass at zero as $\beta \to 0$ [Lal89, p. 31]. Moreover, the imaginary part on the right hand side of (5.18) can be ignored since the left hand side is real. Using that $\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)$ is real and that $\widehat{\Pi_\varepsilon}(0) = 1$ for all $\varepsilon \in (0,1]$, we obtain

$$
\begin{aligned}
&\lim_{\beta \searrow 0} \int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-\beta T - T^\delta} N(T,x)\mathrm{d}T \\
&\quad = \Re\left(\int_{-\infty}^{\infty} q(\mathbf{i}\theta,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\mathrm{e}^{-\mathbf{i}\theta r}\frac{\mathrm{d}\theta}{2\pi} + \frac{G(x)}{2} + G(x)\int_{-\infty}^{\infty}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\frac{\mathbf{i}\mathrm{e}^{-\mathbf{i}\theta r}}{\theta}\frac{\mathrm{d}\theta}{2\pi}\right).
\end{aligned} \tag{5.19}
$$

We separately treat the two integrals on the right hand side of (5.19) and begin with the first one. Recall that $\widehat{\Pi_\varepsilon}(\mathbf{i}\theta) = \widehat{\Pi}(\mathbf{i}\theta\varepsilon/\tau(\varepsilon))$ is $\mathcal{C}^\infty$ and has compact support, which is contained in $[-\tau(\varepsilon)/\varepsilon, \tau(\varepsilon)/\varepsilon] =: [-S,S]$. Also, recall that $q(\cdot,x)$ is analytic in a neighbourhood of $[-\mathbf{i}S,\mathbf{i}S]$ and continuous in x. As mentioned in [Lal89, p. 31f.], the Cauchy integral formula for derivatives implies that $\frac{\mathrm{d}}{\mathrm{d}z}q(z,x)|_{z=\mathbf{i}\theta}$ is uniformly continuous in θ, whence bounded on $[-S,S]\times\Sigma_A$. Thus, $\frac{\mathrm{d}}{\mathrm{d}z}q(z,x)$ is bounded on $[-\mathbf{i}S,\mathbf{i}S]\times\Sigma_A$. Integration by parts now implies that

$$
\int_{-S}^{S} q(\mathbf{i}\theta,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\mathrm{e}^{-\mathbf{i}\theta r}\frac{\mathrm{d}\theta}{2\pi} = \mathbf{i}q(\mathbf{i}\theta,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\frac{\mathrm{e}^{-\mathbf{i}\theta r}}{2\pi r}\Big|_{\theta=-S}^{S} + \mathbf{i}\int_{-S}^{S}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(q(\mathbf{i}\theta,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\right)\frac{\mathrm{e}^{-\mathbf{i}\theta r}}{2\pi r}\mathrm{d}\theta. \tag{5.20}
$$

As the support of $\widehat{\Pi_\varepsilon}$ is contained in $[-\mathbf{i}S,\mathbf{i}S]$ and $\widehat{\Pi_\varepsilon}$ is $\mathcal{C}^\infty$, the first term on the right hand side of (5.20) equals zero for all $r > 0$. For the second term on the right hand side of (5.20) we use that the definition of $\widehat{\Pi}$ given in (5.3) and the fact that $\widehat{\Pi_\varepsilon}(\mathbf{i}\theta) = \widehat{\Pi}(\mathbf{i}\theta\varepsilon/\tau(\varepsilon))$ imply that $\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)$ and $\frac{\mathrm{d}}{\mathrm{d}\theta}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)$ are uniformly bounded for $\varepsilon \in (0,1]$. This shows that the second term on the right hand side of (5.20) converges to zero uniformly for $\varepsilon \in (0,1]$ and $x \in \Sigma_A$ as $r \to \infty$. Thus,

$$
\lim_{r\to\infty}\Re\left(\int_{-\infty}^{\infty} q(\mathbf{i}\theta,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\mathrm{e}^{-\mathbf{i}\theta r}\frac{\mathrm{d}\theta}{2\pi}\right) = 0 \tag{5.21}
$$

uniformly for $x \in \Sigma_A$. Now, we consider the second integral on the right hand side of (5.19):

$$
\Re\left(\int_{-\infty}^{\infty}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\frac{\mathbf{i}\mathrm{e}^{-\mathbf{i}\theta r}}{\theta}\frac{\mathrm{d}\theta}{2\pi}\right) = \int_{-S}^{S}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\frac{\sin(\theta r)}{\theta}\frac{\mathrm{d}\theta}{2\pi} = \int_{0}^{S}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\frac{\sin(\theta r)}{\theta}\frac{\mathrm{d}\theta}{\pi}.
$$

Using the sine integral $\mathtt{Si}(t) := \int_0^t \frac{\sin(\theta)}{\theta}\mathrm{d}\theta$ and $\lim_{t\to\infty}\mathtt{Si}(t) = \pi/2$ we infer that

$$
\lim_{r\to\infty}\int_0^S \frac{\sin(\theta r)}{\theta}\frac{\mathrm{d}\theta}{\pi} = \lim_{r\to\infty}\int_0^{rS}\frac{\sin(\theta)}{\theta}\frac{\mathrm{d}\theta}{\pi} = \lim_{r\to\infty}\mathtt{Si}(rS)/\pi = 1/2
$$

and remark that inserting the factor $\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)$ in the above integrand does not change this limit, since $\widehat{\Pi_\varepsilon}$ is uniformly continuous and $\widehat{\Pi}_\varepsilon(0) = 1$. Thus, for the third term on the right hand side of (5.19) we obtain

$$
\lim_{r\to\infty}\Re\left(G(x)\int_{-\infty}^{\infty}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\frac{\mathbf{i}\mathrm{e}^{-\mathbf{i}\theta r}}{\theta}\frac{\mathrm{d}\theta}{2\pi}\right) = \frac{G(x)}{2} \tag{5.22}
$$

uniformly for $x \in \Sigma_A$. Combining (5.17), (5.19), (5.21) and (5.22) it follows that

$$
\lim_{r\to\infty}\int_{-\infty}^{\infty}\Pi_\varepsilon(r-T)\mathrm{e}^{-T^\delta}N(T,x)\mathrm{d}T = G(x)
$$

uniformly for $x \in \Sigma_A$. An application of Lem. 5.6 yields the desired result. □

Proof of Thm. 3.2(ii). In the lattice situation we work with discrete Fourier-Laplace transforms inspired by [Lal89, proof of Thm. 2] and again insert a few technical twists to fit our more general framework. Conditions (B) and (C) imply that for fixed $\beta \in [0, a)$ and $x \in \Sigma_A$, the function $\widehat{N}^\beta(\cdot, x)$ given by

$$\widehat{N}^\beta(z,x) := \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} N(al + \beta - \psi(x), x) \tag{5.23}$$

is well-defined and analytic on $\mathcal{Z} := \{z \in \mathbb{C} \mid -a(s+\delta) < \Re(z) < -a\delta\}$. Note that $S_n\xi = S_n\zeta + \psi - \psi \circ \sigma^n$ and recall that $S_n\zeta \in a\mathbb{Z}$ for all $n \in \mathbb{N}$. Thus, (B) and (C) allow for the following conversions for $z \in \mathcal{Z}$.

$$\begin{aligned}
\widehat{N}^\beta(z,x) &= \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} \chi(y) \mathrm{e}^{S_n\eta(y)} \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} f_y(al + \beta - \psi(x) - S_n\xi(y)) \\
&= \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} \chi(y) \mathrm{e}^{S_n\eta(y)} \sum_{l=-\infty}^{\infty} \mathrm{e}^{(l + a^{-1}S_n\zeta(y))z} f_y(al + \beta - \psi(y)) \\
&= \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} \chi(y) \mathrm{e}^{S_n(\eta + a^{-1}z\zeta)(y)} \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} f_y(al + \beta - \psi(y)) \\
&= \sum_{n=0}^{\infty} \mathcal{L}^n_{\eta + a^{-1}z\zeta} \left(\chi \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} f_\cdot(al + \beta - \psi) \right)(x)
\end{aligned}$$

with $f_\cdot(t)\colon \Sigma_A \to \mathbb{R}$, $x \mapsto f_x(t)$ as before. For $z \in \mathcal{Z}$ we have that $\Re(a^{-1}z) < -\delta$. Moreover, observe $h_{\eta-\delta\zeta} = \mathrm{e}^{-\delta\psi} h_{\eta-\delta\xi}$ and $\gamma_{\eta-\delta\zeta} = \gamma_{\eta-\delta\xi}$. Thus, Prop. 2.4 yields that $\gamma_{\eta+\Re(a^{-1}z)\zeta} < 1$. Thm. 5.1 (both parts) and the spectral radius formula (Sec. 5.1) now imply, for every $z \in \mathcal{Z}$, that

$$\widehat{N}^\beta(z,x) = (I - \mathcal{L}_{\eta + a^{-1}z\zeta})^{-1} \left(\chi \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} f_\cdot(al + \beta - \psi) \right)(x).$$

Note that $\|\chi \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} f_\cdot(al + \beta - \psi)\|_\infty$ is finite for all $z \in \mathcal{Z}$ because of Conditions (B) and (C).

Because $a^{-1}\zeta$ is integer-valued but not co-homologous to any function valued in a proper subgroup of the integers, we can apply Prop. 5.5. Thus, $z \mapsto (I - \mathcal{L}_{\eta+a^{-1}z\zeta})^{-1}$ is $2\pi\mathbf{i}$-periodic and holomorphic at every z on the line $\Re(z) = -a\delta$ such that $\Im(z)/(2\pi)$ is not an integer. Therefore, $z \mapsto (I - \mathcal{L}_{\eta+a^{-1}z\zeta})^{-1}$ has an isolated singularity at $z = -a\delta$ and is holomorphic at each $z = -a\delta + \mathbf{i}\theta$, for $0 < |\theta| \leq \pi$. By Prop. 5.2 the singularity of $\widehat{N}^\beta(z,x)$ at $z = -a\delta$ is

$$\frac{\gamma_{\eta+a^{-1}z\zeta}}{1 - \gamma_{\eta+a^{-1}z\zeta}} \underbrace{\int_{\Sigma_A} \chi(y) \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} f_y(al + \beta - \psi(y)) \mathrm{d}\nu_{\eta+a^{-1}z\zeta}(y) h_{\eta+a^{-1}z\zeta}(x)}_{=: E_{x,\beta}(z)}$$

Since the function $E_{x,\beta}$ is continuous at $-a\delta$, we deduce from (5.2) that the singularity of $\widehat{N}^\beta(z,x)$ at $z = -a\delta$ is a simple pole with residue

$$C_\beta(x) := -\frac{a}{\int \zeta \mathrm{d}\mu_{\eta-\delta\zeta}} E_{x,\beta}(-a\delta).$$

It follows that $\widehat{N}^\beta(\cdot, x)\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \widehat{N}^\beta(z,x)$ is meromorphic in

$$\widetilde{\mathcal{Z}}(\varepsilon) := \{z \in \mathbb{C} \mid -a(\delta+s) < \Re(z) < -a\delta + \varepsilon,\ 0 \leq \Im(z) \leq \pi\},$$

for some $\varepsilon > 0$, and that the only singularity in this region is a simple pole at $-a\delta$ with residue $C_\beta(x)$. Additionally, by (C), $\sum_{l=-\infty}^{-1} \mathrm{e}^{lz} N(al + \beta - \psi(x), x)$ is finite for $\Re(z) > -a(\delta+s)$. We conclude that there exists $\varepsilon > 0$ such that

$$\sum_{l=0}^{\infty} \mathrm{e}^{lz} N(al + \beta - \psi(x), x) - \frac{C_\beta(x)}{z + a\delta}$$

is holomorphic in $\widetilde{\mathcal{Z}}(\varepsilon)$. Also observe that $z \mapsto (\mathrm{e}^{z+a\delta}-1)/(z+a\delta)$ is holomorphic in $\mathbb{C}$. Making the change of variable $\widetilde{z} := \mathrm{e}^{z+a\delta}$ we obtain that

$$\sum_{l=0}^{\infty} \widetilde{z}^l \mathrm{e}^{-al\delta} N(al+\beta-\psi(x),x) - \frac{C_\beta(x)}{\widetilde{z}-1} \quad \text{and whence} \quad L(\widetilde{z},x) := \sum_{l=0}^{\infty} \widetilde{z}^l \left(\mathrm{e}^{-al\delta} N(al+\beta-\psi(x),x) + C_\beta(x)\right)$$

are respectively holomorphic in $\{\mathrm{e}^{z+a\delta} \mid z \in \widetilde{\mathcal{Z}}(\varepsilon)\}$ and $\{\widetilde{z} \mid |\widetilde{z}| < \mathrm{e}^\varepsilon\}$ (compare [Lal89, p. 27]). Since $\mathrm{e}^\varepsilon > 1$, the coefficient sequence of the power series of $L(\cdot,x)\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto L(z,x)$ converges to zero exponentially fast, more precisely,

$$\mathrm{e}^{-an\delta} N(an+\beta-\psi(x),x) + C_\beta(x) \in \mathfrak{o}((1+(\mathrm{e}^\varepsilon-1)/2)^{-n})$$

as $n \to \infty$ ($n \in \mathbb{N}$). Thus, for $x \in \Sigma_A$ we have

$$\begin{aligned}
&N(t,x)\\
&= N\Bigg(a \underbrace{\left\lfloor \frac{t+\psi(x)}{a} \right\rfloor}_{=:n} + a\underbrace{\left\{ \frac{t+\psi(x)}{a} \right\}}_{=:\beta} - \psi(x), x\Bigg)\\
&\sim -\mathrm{e}^{a\lfloor \frac{t+\psi(x)}{a} \rfloor \delta} C_{a\{(t+\psi(x))/a\}}(x)\\
&= \mathrm{e}^{t\delta} \mathrm{e}^{-a\left\{\frac{t+\psi(x)}{a}\right\}\delta} \mathrm{e}^{\delta\psi(x)} \frac{a}{\int \zeta \mathrm{d}\mu_{\eta-\delta\zeta}} h_{\eta-\delta\zeta}(x) \int_{\Sigma_A} \chi(y) \sum_{l=-\infty}^{\infty} \mathrm{e}^{-la\delta} f_y\left(al + a\left\{\frac{t+\psi(x)}{a}\right\} - \psi(y)\right) \mathrm{d}\nu_{\eta-\delta\zeta}(y)\\
&= \mathrm{e}^{t\delta} \widetilde{G}_x(t)
\end{aligned}$$

as $t \to \infty$. Since in all instances where t occurs only the fractional part is involved, it is clear that $\widetilde{G}_x$ is periodic with period a, which finishes the proof. □

Proof of Thm. 3.2(iii). First, consider the case that ξ is non-lattice. Since $\mathrm{e}^{-t\delta} N(t,x)$ is bounded in t by (B), the result from Thm. 3.2(i) implies Thm. 3.2(iii).

Second, consider the case that ξ is lattice. Thm. 3.2(ii) states that $\mathrm{e}^{-t\delta} N(t,x) \sim \widetilde{G}_x(t)$ as $t \to \infty$. Since $\mathrm{e}^{-t\delta} N(t,x)$ is bounded in t by (B), and $\widetilde{G}_x$ is periodic with period a we have

$$\begin{aligned}
\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T\delta} N(T,x)\mathrm{d}T &= \lim_{t\to\infty} t^{-1} \int_0^{a\lfloor t/a \rfloor} \mathrm{e}^{-T\delta} N(T,x)\mathrm{d}T = \lim_{t\to\infty} t^{-1} \int_0^{a\lfloor t/a\rfloor} \widetilde{G}_x(T)\mathrm{d}T\\
&= \lim_{t\to\infty} t^{-1} \left\lfloor \frac{t}{a} \right\rfloor \int_0^a \widetilde{G}_x(T)\mathrm{d}T = \frac{1}{a} \int_0^a \widetilde{G}_x(T)\mathrm{d}T.
\end{aligned}$$

The latter integral transforms as follows:

$$\begin{aligned}
&\int_0^a \widetilde{G}_x(T)\mathrm{d}T\\
&= \int_0^a \mathrm{e}^{-a\left\{\frac{T+\psi(x)}{a}\right\}\delta} \frac{a\mathrm{e}^{\delta\psi(x)}}{\int \zeta \mathrm{d}\mu_{\eta-\delta\zeta}} h_{\eta-\delta\zeta}(x) \int_{\Sigma_A} \chi(y) \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al\delta} f_y\left(al + a\left\{\tfrac{T+\psi(x)}{a}\right\} - \psi(y)\right) \mathrm{d}\nu_{\eta-\delta\zeta}(y)\mathrm{d}T\\
&= \frac{a\mathrm{e}^{\delta\psi(x)}}{\int \zeta \mathrm{d}\mu_{\eta-\delta\zeta}} h_{\eta-\delta\zeta}(x) \int_{\Sigma_A} \chi(y) \sum_{l=-\infty}^{\infty} \int_0^a \mathrm{e}^{-\left(al+a\left\{\frac{T+\psi(x)}{a}\right\}\right)\delta} f_y\left(al + a\left\{\tfrac{T+\psi(x)}{a}\right\} - \psi(y)\right) \mathrm{d}T\mathrm{d}\nu_{\eta-\delta\zeta}(y)\\
&= \frac{a\mathrm{e}^{\delta\psi(x)} h_{\eta-\delta\zeta}(x)}{\int \zeta \mathrm{d}\mu_{\eta-\delta\zeta}} \int_{\Sigma_A} \chi(y)\mathrm{e}^{-\psi(y)\delta} \sum_{l=-\infty}^{\infty} \int_{al-\psi(y)}^{a(l+1)-\psi(y)} \mathrm{e}^{-T\delta} f_y(T)\,\mathrm{d}T\mathrm{d}\nu_{\eta-\delta\zeta}(y)\\
&= \frac{a h_{\eta-\delta\xi}(x)}{\int \xi \mathrm{d}\mu_{\eta-\delta\xi}} \int_{\Sigma_A} \chi(y) \int_{-\infty}^{\infty} \mathrm{e}^{-T\delta} f_y(T)\,\mathrm{d}T\mathrm{d}\nu_{\eta-\delta\xi}(y)\\
&= aG(x),
\end{aligned}$$

where for the second to last equality we used that $\mathrm{e}^{-\delta\psi}\mathrm{d}\nu_{\eta-\delta\zeta} = \mathrm{d}\nu_{\eta-\delta\xi}$, $\mathrm{e}^{\delta\psi} h_{\eta-\delta\zeta} = h_{\eta-\delta\xi}$ and that $\int \zeta \mathrm{d}\mu_{\eta-\delta\zeta} = \int \xi \mathrm{d}\mu_{\eta-\delta\xi}$. □

5.2.2. *Proof of Thm. 3.3.* In the setting of Thm. 3.3, Condition (Db) is automatically satisfied. Thus, Thm. 3.3 is proved by combining the following three lemmas with Thm. 3.2.

Lemma 5.7. *If $\{t \mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x \in \Sigma_A\}$ is equi d. R. i. then (A) holds.*

Proof. This is clear since d. R. i. implies Lebesgue integrability. □

Lemma 5.8. *If $\{t \mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x \in \Sigma_A\}$ is equi d. R. i. and there exist $\mathfrak{C}', s > 0$ such that $\mathrm{e}^{-t\delta}|f_x(t)| \leq \mathfrak{C}'\mathrm{e}^{st}$, for $t < 0$ and $x \in \Sigma_A$ then (C) holds, i. e. there exist $\widetilde{\mathfrak{C}} > 0, t^* \leq 0$ such that $\mathrm{e}^{-\delta t}N^{abs}(t,x) \leq \widetilde{\mathfrak{C}}\mathrm{e}^{st}$ for $t \leq t^*$.*

Proof. In (5.13) we have seen that the existence of $n \in \mathbb{N}$ for which $S_n\xi$ is strictly positive implies existence of $m^* \in \mathbb{N}$ such that $S_m\xi$ is strictly positive for all $m \geq m^*$. Set

$$t^* := \min\left\{0, \inf_{x\in\Sigma_A,\, 0\leq m\leq m^*} S_m\xi(x)\right\}.$$

Then $t - S_m\xi < 0$ for all $t < t^*$ and $m \in \mathbb{N}$. Additionally using that there exist $\mathfrak{C}', s > 0$ such that $\mathrm{e}^{-t\delta}|f_x(t)| \leq \mathfrak{C}'\mathrm{e}^{st}$ for $t < 0$ and $x \in \Sigma_A$ we have for $t \leq t^*$,

$$\begin{aligned}
\mathrm{e}^{-\delta t}N^{\mathrm{abs}}(t,x) &= \sum_{n=0}^{\infty}\sum_{y:\sigma^n y = x} \mathrm{e}^{S_n(\eta-\delta\xi)(y)}|\chi(y)|\mathrm{e}^{-\delta(t-S_n\xi(y))}|f_y(t-S_n\xi(y))| \\
&\leq \mathfrak{C}'\mathrm{e}^{st}\sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x}\mathrm{e}^{S_n(\eta-(\delta+s)\xi)(y)}|\chi(y)| = \mathfrak{C}'\mathrm{e}^{st}\sum_{n=0}^{\infty}\mathcal{L}^n_{\eta-(\delta+s)\xi}|\chi|(x) \\
&\leq \mathfrak{C}'\mathrm{e}^{st}\sum_{n=0}^{\infty}\|\mathcal{L}^n_{\eta-(\delta+s)\xi}\|_{\mathrm{op}}\|\chi\|_\infty.
\end{aligned}$$

By Prop. 2.4 we know that $t \mapsto \gamma_{\eta+t\xi}$ is strictly monotonically increasing and $\gamma_{\eta-\delta\xi} = 1$. Hence, by the spectral radius formula (Sec. 5.1) the last series converges and the assertion follows. □

Lemma 5.9. *If $\{t \mapsto \mathrm{e}^{-t\delta}|f_x(t)| \mid x \in \Sigma_A\}$ is equi d. R. i. then (B) is satisfied, i. e. there exists $\mathfrak{C} > 0$ such that $\mathrm{e}^{-t\delta}N^{abs}(t,x) \leq \mathfrak{C}$ for all $t \in \mathbb{R}$.*

Proof. By assumption there exists $n \in \mathbb{N}$ for which $S_n\xi$ is strictly positive. Fix this n, choose $\kappa > 0$ such that $S_n\xi \geq \kappa$ and consider the n-th iterate of the renewal equation, namely,

$$N(t,x) = \sum_{y:\sigma^n y=x} N(t-S_n\xi(y),y)\mathrm{e}^{S_n\eta(y)} + \sum_{i=0}^{n-1}\sum_{y:\sigma^i y=x}\chi(y)f_y(t-S_i\xi(y))\mathrm{e}^{S_i\eta(y)}.$$

The function defined by $M(t,x) := \mathrm{e}^{-t\delta}N(t,x)/h_{\eta-\delta\xi}(x)$ satisfies

$$\begin{aligned}
M(t,x) = &\sum_{y:\sigma^n y=x} M(t-S_n\xi(y),y)\mathrm{e}^{S_n(\eta-\delta\xi)(y)}\frac{h_{\eta-\delta\xi}(y)}{h_{\eta-\delta\xi}(x)} \\
&+\sum_{i=0}^{n-1}\sum_{y:\sigma^i y=x}\chi(y)\mathrm{e}^{-\delta(t-S_i\xi(y))}f_y(t-S_i\xi(y))\mathrm{e}^{S_i(\eta-\delta\xi)(y)}\frac{1}{h_{\eta-\delta\xi}(x)}.
\end{aligned} \tag{5.24}$$

Set $g_x(t) := \mathrm{e}^{-t\delta}f_x(t)$ and $g_{i,x}(t) := g_x(t-S_i\xi(x))$ for $x \in \Sigma_A$. Then (5.24) becomes

$$M(t,x) = \sum_{y:\sigma^n y=x} M(t-S_n\xi(y),y)\mathrm{e}^{S_n(\eta-\delta\xi)(y)}\frac{h_{\eta-\delta\xi}(y)}{h_{\eta-\delta\xi}(x)} + \sum_{i=0}^{n-1}\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t))(x)/h_{\eta-\delta\xi}(x) \tag{5.25}$$

with $g_{i,\cdot}(t)\colon \Sigma_A \to \mathbb{R}$, $x \mapsto g_{i,x}(t)$. Define

$$\overline{M}(t) := \sup_{\substack{t'\in(t-\kappa,t]\\ x\in\Sigma_A}} M(t',x). \tag{5.26}$$

Then (5.25) implies that

$$\overline{M}(t) \leq \sup_{y\in\Sigma_A}\overline{M}(t-S_n\xi(y)) + \sup_{t'\in(t-\kappa,t]}\sum_{i=0}^{n-1}\|\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))/h_{\eta-\delta\xi}\|_\infty, \tag{5.27}$$

as $\sum_{y:\sigma^n y=x} \mathrm{e}^{S_n(\eta-\delta\xi)(y)} h_{\eta-\delta\xi}(y)/h_{\eta-\delta\xi}(x) = 1$. Iterating (5.27) k times and using the abbreviation $I_\kappa(t, x^1, \ldots, x^m) := (t - \sum_{j=1}^m S_n\xi(x^j) - \kappa, t - \sum_{j=1}^m S_n\xi(x^j)]$ yields that

$$\overline{M}(t) \le \sup_{x^1,\ldots,x^k \in \Sigma_A} \left[\overline{M}\Big(t - \sum_{j=1}^k S_n\xi(x^j)\Big) + \sum_{i=0}^{n-1} \sum_{m=0}^{k-1} \sup_{t' \in I_\kappa(t,x^1,\ldots,x^m)} \|\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t')/h_{\eta-\delta\xi}\|_\infty \right],$$

where $\sum_{j=1}^0 S_n\xi(x^j)$ shall be understood to be zero.

Fix $t \in \mathbb{R}$. As $S_n\xi \ge \kappa > 0$ we can find a $k^* \in \mathbb{N}$ such that for all $k \ge k^*$ and $x^1, \ldots, x^k \in \Sigma_A$ we have $t - \sum_{j=1}^k S_n\xi(x^j) \le t^* \le 0$ with t^* as in Lem. 5.8. Hence by Lem. 5.8 there exists a constant $\mathfrak{C}''$ such that

$$\overline{M}(t) \le \mathfrak{C}'' + \sup_{x^1,x^2,\ldots \in \Sigma_A} \sum_{i=0}^{n-1} \sum_{m=0}^{\infty} \sup_{t' \in I_\kappa(t,x^1,\ldots,x^m)} \|\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))/h_{\eta-\delta\xi}\|_\infty. \tag{5.28}$$

If $t' \mapsto \|\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))/h_{\eta-\delta\xi}\|_\infty$ is d. R. i. then $t - \sum_{j=1}^{m+1} S_n\xi(x^j) \le t - \sum_{j=1}^m S_n\xi(x^j) - \kappa$ for $x^1, \ldots, x^{m+1} \in \Sigma_A$ implies that the series in (5.28) converges for any constellation $x^1, x^2, \ldots \in \Sigma_A$ and thus is uniformly bounded for $x^1, x^2, \ldots \in \Sigma_A$. This proves the assertion. Hence all that remains to be shown is that $t' \mapsto \|\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))/h_{\eta-\delta\xi}\|_\infty$ is d. R. i. For this, note that using the terminology of Defn. 3.1

$$\underline{m}_k\left(\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot})(x), h\right) \ge \sum_{y:\sigma^i y = x} \chi(y) \mathrm{e}^{S_i(\eta-\delta\xi)(y)} \underline{m}_{k - S_i\xi(y)/h}(g_y, h) \quad \text{and}$$

$$\overline{m}_k\left(\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot})(x), h\right) \le \sum_{y:\sigma^i y = x} \chi(y) \mathrm{e}^{S_i(\eta-\delta\xi)(y)} \overline{m}_{k - S_i\xi(y)/h}(g_y, h).$$

Since $\sum_{y:\sigma^i y=x} \chi(y)\mathrm{e}^{S_i(\eta-\delta\xi)(y)} = \mathcal{L}^i_{\eta-\delta\xi}\chi(x)$ is finite, the hypothesis of $\{g_x \mid x \in \Sigma_A\}$ being equi d. R. i. implies that $t' \mapsto \|\mathcal{L}^i_{\eta-\delta\xi}(\chi g_{i,\cdot}(t'))/h_{\eta-\delta\xi}\|_\infty$ is d. R. i. which finishes the proof. □

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