# EQUIVARIANT LOGIC AND APPLICATIONS TO C*-DYNAMICS

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Abstract. We introduce a model-theoretic framework for the study actions of a locally compact second countable group on metric structures. In this setting, we prove analogs of fundamental model-theoretic results, such as Łos' theorem and countable saturation of ultrapowers. We then present applications to C*-dynamics. In particular, we prove that the continuous part of the central sequence algebra of a strongly self-absorbing action is indistinguishable from the continuous part of the sequence algebra, and in fact equivariantly isomorphic under the Continuum Hypothesis. Furthermore, strongly self-absorbing actions are classified up to cocycle conjugacy by their existential theory. It follows that the classification problem for strongly self-absorbing actions of a given locally compact group is smooth in the sense of Borel complexity theory. In stark contrast, the classification problem for $\mathbb{Z}/2\mathbb{Z}$-actions on the Cuntz algebra $\mathcal{O}_2$ with approximately $\mathbb{Z}/2\mathbb{Z}$-inner half-flip is not smooth, and in fact conjugacy and cocycle conjugacy for such actions are complete analytic sets. These statements provide the equivariant generalizations of results of Farah–Hart–Rørdam–Tikuisis. We also present, within the framework of equivariant logic, a unified approach to preservation results for actions with finite Rokhlin dimension. To this purpose, we introduce a notion of order zero dimension for $G$-equivariant *-homomorphisms between $G$-C*-algebras. We prove that for $G$-equivariant *-homomorphisms with finite order zero dimension, many regularity properties—including finite decomposition rank, finite nuclear dimension, finite Rokhlin dimension, and $G$-equivariant absorption of a given strongly self-absorbing $G$-C*-algebra—pass from the target C*-algebra to the domain C*-algebra. As a consequence, we recover and extend several dimensional inequalities for nuclear dimension, decomposition rank, and Rokhlin dimension. Finally, we use our general results to show that, if $D$ is a strongly self-absorbing C*-algebra and $G$ is a compact group, then any $D$-stable $G$-C*-algebra with finite Rokhlin dimension with commuting towers $G$-equivariantly absorbs the trivial $G$-action on $D$, and in particular $D$-stability passes to the crossed product and the fixed point algebra.

## 1. Introduction

The model-theoretic study of C*-algebras and von Neumann algebras has been initiated in [32–34] and continued in [18,23–25,28,31,48–53]. Remarkably, in [7,12,27,29,30,35–37] many properties and notions in C*-algebra theory have been proved to be captured by model-theoretic methods. We will observe below that, among these notions, one can list the sequentially split *-homomorphisms, as introduced in [4]. As remarked in [4, Subsection 4.3], any *existential* *-homomorphism is sequentially split. We will show that, in fact, the sequentially split *-homomorphisms are precisely the *positively existential* *-homomorphisms, that is, those that preserve the value of existential positive primitive formulas. (This is true at least for separable C*-algebras. For larger C*-algebras the notion of sequentially split *-homomorphism as defined in [4] is more restrictive.) In order to generalize this fact to the equivariant setting, we are lead to regard C*-algebras as structures in equivariant logic.

Let $G$ be a second countable locally compact topological group. In this paper, we develop a general framework, which we call $G$-equivariant logic (for metric structures), where continuous $G$-actions on metric structures can be regarded themselves as structures. We then prove some fundamental model-theoretic results, including analogs of Łos' theorem and countable saturation of ultrapowers. The general notions of ultraproduct and reduced product in this setting recover various notions of reduced product and ultraproduct of $G$-actions that have appeared in the literature: for measure-preserving actions on probability spaces [13,21], for trace-preserving actions on tracial von Neumann algebra and, particularly, $\mathrm{II}_1$ factors [62], and actions on C*-algebras [4,44,56,57,75,78].

When the group $G$ is discrete, one can regard a $G$-action as a usual metric structure in the obvious way, by adding a function symbol for every element of the group. This does not work in the nondiscrete case, even if the group is

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compact, since in general the canonical action on the ultrapower that one obtains in this way is not continuous; see Example 4.2 below. On the other hand, adding a sort for the group and enforcing uniform bounds on the continuity moduli of an action would not capture the notion of ultrapower of $G$-actions. Here we take the route of considering $G$ as an additional sort, which is however not on the same footing as a usual metric sort. Some special care in the definition of reduced product is required, so as to ensure that it still gives a continuous $G$-action. In this setting, we prove analogs of Łos' theorem and countable saturation of ultraproducts with respect to quantifier-free formulas, which subsume various results that have appeared in the literature. These results are then used to prove that the space of existential theories of $G$-actions on $\mathcal{L}$-structures has a canonical compact topology, which is metrizable whenever the language $\mathcal{L}$ is separable. This recovers, in the particular case of continuous measure-preserving $G$-actions on probability spaces, a result from [13], which in turn generalizes a result from [1] in the case of discrete groups.

In the case of $G$-C*-algebras, we also consider the notions of *commutant* (positive) existential theory, and *commutant* (positive) weak containment, as well as its higher dimensional generalization to commutant $d$-containment (with commuting towers). The latter one is obtained by replacing $G$-equivariant *-homomorphisms with $(d+1)$-tuples of equivariant completely positive contractive order zero bimodule maps with unital sum (and commuting ranges). In the case of $G$-C*-algebras with Rokhlin dimension with commuting towers at most $d$, we show that the underlying C*-algebra is commutant $d$-contained with commuting towers in the fixed point algebra. This fact, together with the observation that the commutant existential theory of a C*-algebra captures the property of $D$-absorbing a given strongly self-absorbing C*-algebra $D$, yields that $D$-absorption is preserved by taking fixed point algebras and crossed products by actions with finite Rokhlin dimension with commuting towers. This represents a significant generalization of previous results of Hirshberg–Winter–Zacharias [57] (for finite groups) and the first-named author [44] (for compact groups), which only considered the case when $D = \mathcal{Z}$ is the Jiang-Su algebra and the C*-algebra is unital.

We will also show that the $G$-equivariant sequentially split *-homomorphisms between separable C*-algebras, as defined in [4], coincide with the positively existential embeddings between $G$-C*-algebras regarded as structures as described above. In particular, this observation allows one to recover and extend many preservation results from [4], once one observes that the relevant classes of C*-algebras and $G$-C*-algebras are captured by formulas of a certain complexity. We take this approach further, and introduce a notion of $G$-equivariant order zero dimension (with or without commuting towers) for $G$-equivariant *-homomorphism, in such a way that order zero dimension equal to zero corresponds to being positively existential. As an example, for a compact group $G$, the Rokhlin dimension of a $G$-C*-algebra $A$ is equal to the $G$-equivariant order zero dimension of the canonical *-homomorphism $\theta\colon A \to C(G, A)$. We then observe that the preservation results for nuclear dimension and decomposition rank under crossed products by actions with finite Rokhlin dimension is a consequence of the syntactic characterization of $G$-equivariant order zero dimension, together with the syntactic description of nuclear dimension and decomposition rank from [30]. This provides new information in many cases of interest.

We use results from the literature to give many other examples of *-homomorphisms with finite order zero dimension which do not come from group actions. Notable examples are the unital inclusions $\mathcal{O}_\infty \to \mathcal{O}_2$ and $\mathcal{Z} \to U$, where $U$ is any UHF-algebra of infinite type. As a consequence of our general results for order zero dimension, we recover and extend useful inequalities relating the nuclear dimensions, decomposition ranks, and Rokhlin dimension of the $\mathcal{Z}$- and $U$-stabilization of an arbitrary ($G$-)C*-algebra. Similar statements hold for the $\mathcal{O}_\infty$- and $\mathcal{O}_2$-stabilizations, and this allows us to recover a result from [68]: the nuclear dimension of a Kirchberg algebra is at most 3. (The actual dimension of Kirchberg algebras has been computed in [11]: it is 1.) The nuclear dimensional estimates that we derive here from our general results have also been observed in [3], while the estimates involving the decomposition rank seem to be new.

As a further application of the general framework introduced here, we extend the results about strongly self-absorbing C*-algebras obtained in [31] to the equivariant setting. The notion of strongly self-absorbing action has been recently introduced and studied by Szabó in [75, 78], where it is shown that many familiar properties of strongly self-absorbing C*-algebras have natural analogues for strongly self-absorbing actions. Our results show that this also applies to all the properties considered in [31]. In particular, that the continuous part of the central sequence algebra of a strongly self-absorbing action is indistinguishable from the continuous part of the sequence algebra, and in fact equivariantly isomorphic under the Continuum Hypothesis. We take the occasion to remove an unnecessary hypothesis present in [31], and observe that all the results hold for reduced products according to an arbitrary countably incomplete filter, even without the assumption that the corresponding reduced product be countably saturated. We also show that strongly self-absorbing actions on separable C*-algebras are classified, up to cocycle conjugacy, by their existential theory. As a consequence, the classification problem for strongly self-absorbing actions of a fixed locally compact second countable group on separable C*-algebras is smooth in the sense of Borel complexity theory. This is no longer the case for actions with approximately inner half-flip, even if one restricts to actions on the Cuntz algebra $\mathcal{O}_2$. Indeed we observe below that the relations of conjugacy and cocycle conjugacy for $\mathbb{Z}/2\mathbb{Z}$-actions on $\mathcal{O}_2$ with approximately $\mathbb{Z}/2\mathbb{Z}$-inner half-flip are complete analytic sets.

Finally, we prove that, for a compact group $G$ and a unitarily regular strongly self-absorbing $G$-C*-algebra $D$, any $G$-C*-bundle whose fibers are $G$-isomorphic to $D$ must be a trivial $G$-C*-bundle, as long as the base space has finite covering dimension; see Theorem 5.26. This can be seen as an equivariant generalization of the main result from [22]. We then deduce from this and our general results about commutant containment that any continuous action of $G$ with finite Rokhlin dimension with commuting towers on a $D$-absorbing C*-algebra $G$-equivariantly absorbs the trivial a $G$-action on $D$. In particular, this implies that under the same assumptions the fixed point algebra and the crossed product are $D$-absorbing as well.

The rest of the paper is organized as follows. In Section 2 we present the general framework of equivariant logic for metric structures: its syntax and semantic, together with some examples from the literature. In Section 3 we consider various languages for C*-algebras, corresponding to different notions of morphisms: completely positive contractive (nuclear) maps, completely positive contractive order zero (nuclear) maps, (nuclear) *-homomorphisms, and bimodule maps. Section 4 contains the applications of the general framework to strongly self-absorbing actions. The notion of $G$-equivariant order zero dimension (with commuting towers) for $G$-equivariant *-homomorphism is presented in Section 5, together with applications to the dimensions of the $\mathcal{Z}$- and $U$-stabilizations of a C*-algebra, and to crossed products by actions with finite Rokhlin dimension. The notion of commutant $d$-containment (with commuting towers) for group actions on C*-algebras is also introduced in Section 5. It is then proved therein that $D$-absorption for a given strongly self-absorbing C*-algebra $D$ is preserved by commutant containment with commuting towers, and that for a compact group action with finite Rokhlin dimension with commuting towers, the whole C*-algebra is commutant $d$-contained with commuting towers for some $d \in \mathbb{N}$ in the fixed point algebra. These results extend various preservation theorems for Jiang-Su absorption for crossed products from [44, 57].

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## 2. Equivariant logic for metric structures

### 2.1. **Syntax.**

2.1.1. *Structures.* Throughout this section, we fix a metric signature $\mathcal{L}$ and a locally compact second countable group $G$ with unit $1_G$. We will assume that $\mathcal{L}$ contains a distinguished binary relation symbol to be interpreted as the metric. We will say that $\mathcal{L}$ has density character smaller than or equal to $\kappa$ if there exists a collection $\mathcal{C}$ of $\mathcal{L}$-formulas of size at most $\kappa$ such that any $\mathcal{L}$-formula is a uniform limit of $\mathcal{L}$-formulas from $\mathcal{C}$ (uniformly over all $\mathcal{L}$-structures).

An $\mathcal{L}_G$-structure is an $\mathcal{L}$-structure endowed with a continuous action of $G$ by automorphisms. If $M$ is an $\mathcal{L}_G$-structure, then we denote the action $G \times M \to M$ by $(g, a) \mapsto g^M a$. A thorough introduction to the logic for metric structures can be found in [5] and, as it pertains to C*-algebras, in [30, 33].

2.1.2. *Terms and formulas.* We consider variables $x_0, x_1, \ldots$ for elements of the structure, and variables $\gamma_0, \gamma_1, \ldots$ for elements of the group. The notion of $\mathcal{L}_G$-*term* is defined recursively. The variables $x_0, x_1, \ldots$ are $\mathcal{L}_G$-terms. If $t_1, \ldots, t_n$ are $\mathcal{L}_G$-terms and $f$ is an $n$-ary function symbol in $\mathcal{L}$, then $f(t_1, \ldots, t_n)$ is an $\mathcal{L}_G$-term. Finally if $t$ is an $\mathcal{L}_G$-term, then $\gamma t$ is an $\mathcal{L}_G$-term.

**Definition 2.1.** An *atomic $\mathcal{L}_G$-formula* is an expression of the form $R(t_1, \ldots, t_n)$, where $R$ is an $n$-ary relation symbol in $\mathcal{L}$ and $t_1, \ldots, t_n$ are $\mathcal{L}_G$-terms, or a continuous function $\lambda\colon G \to \mathbb{R}$. We define a *quantifier-free $\mathcal{L}_G$-formula* by recursion on the complexity:

(1) any atomic $\mathcal{L}_G$-formula is a quantifier-free $\mathcal{L}_G$-formula;
(2) if $\varphi_1, \ldots, \varphi_n$ are quantifier-free $\mathcal{L}_G$-formulas and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous function (also called a *connective*), then $f(\varphi_1, \ldots, \varphi_n)$ is a quantifier-free $\mathcal{L}_G$-formula;
(3) if $\varphi$ is a quantifier free $\mathcal{L}_G$-formula, and $V$ is an open precompact subset of $G$, then $\inf_{\gamma\in V} \varphi$ and $\sup_{\gamma\in V} \varphi$ are quantifier-free $\mathcal{L}_G$-formulas.

A quantifier-free formula is said to be *positive primitive* if it is of the form

$$\max \{u_1(\varphi_1), \ldots, u_n(\varphi_n)\}$$

where $\varphi_1, \ldots, \varphi_n$ are atomic, and $u_1, \ldots, u_n : \mathbb{R} \to \mathbb{R}$ are continuous nondecreasing functions.

The interpretation of a quantifier-free $\mathcal{L}_G$-formula in an $\mathcal{L}_G$-structure is defined in the usual way by induction on the complexity. If $\varphi(x_1, \ldots, x_m, \gamma_1, \ldots, \gamma_n)$ is a quantifier-free $\mathcal{L}_G$-formula and $M$ is an $\mathcal{L}_G$-structure, then the interpretation $\varphi^M\colon M^m \times G^n \to \mathbb{R}$ is a continuous function. Furthermore, the collection of functions $M^m \to \mathbb{R}$ given by $(x_1, \ldots, x_m) \mapsto \varphi^M(x_1, \ldots, x_m, g_1, \ldots, g_n)$ is uniformly bounded and uniformly equicontinuous, uniformly over $g_1, \ldots, g_n \in G$ and the $\mathcal{L}_G$-structure $M$. This can be easily proved by induction on the complexity of $\varphi$.

If $M$ is a structure, $\varphi$ is a sentence, and $r \in \mathbb{R}$, we write $M \models \varphi \leq r$ if the interpretation $\varphi^M$ of $\varphi$ in $M$ is smaller than or equal to $r$. A collection of formulas is *uniformly equicontinuous* if their interpretations are uniformly equicontinuous functions, uniformly over all $\mathcal{L}_G$-structures.

Let $n \in \mathbb{N}$, and fix a countable collection $\mathcal{H}$ of continuous functions $\mathbb{R}^n \to \mathbb{R}$ that is dense in the space $C(\mathbb{R}^n, \mathbb{R})$ of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ endowed with the compact-open topology. As in the nonequivariant setting, any quantifier-free $\mathcal{L}_G$-formula is the uniform limit (uniformly over all $\mathcal{L}_G$-structures) of quantifier-free $\mathcal{L}_G$-formulas with connectives from $\mathcal{H}$. One can therefore assume that all the quantifier-free $\mathcal{L}_G$-formulas only involve connectives from $\mathcal{H}$. One can also assume, without loss of generality, that any quantifier-free $\mathcal{L}_G$-formula is of the form

$$\sup_{\rho_1\in W_1} \inf_{\gamma_1\in V_1} \cdots \sup_{\rho_m\in W_m} \inf_{\gamma_m\in V_m} f(\varphi_1(\overline{x}, \overline{\rho}, \overline{\gamma}), \ldots, \varphi_k(\overline{x}, \overline{\rho}, \overline{\gamma}))$$

for some $m \in \mathbb{N}$, some open precompact subsets $V_1, \ldots, V_n, W_1, \ldots, W_n$ of $G$, some continuous function $f \in \mathcal{H}$, some $k \in \mathbb{N}$, and some atomic $\mathcal{L}_G$-formulas $\varphi_j(\overline{x}, \overline{\rho}, \overline{\gamma})$ for $j = 1, 2, \ldots, k$. The fact that any quantifier-free $\mathcal{L}_G$-formula is uniformly approximated by an $\mathcal{L}_G$-formula of this form can be shown as in the proof of the prenex normal form for the logic for metric structures [5, Proposition 6.9].

An *$\mathcal{L}_G$-embedding* between $\mathcal{L}_G$-structures is a function $\theta\colon M \to N$ such that $\varphi(\theta(\overline{a})) = \varphi(\overline{a})$ for any quantifier-free $\mathcal{L}_G$-formula $\theta(\overline{x})$ and tuple $\overline{a}$ in $M$. Similarly, an *$\mathcal{L}_G$-morphism* between $\mathcal{L}_G$-structures is a function $\theta\colon M \to N$ such that $\varphi(\theta(\overline{a})) \leq \varphi(\overline{a})$ for any atomic $\mathcal{L}_G$-formula $\theta(\overline{x})$ and any tuple $\overline{a}$ in $M$.

2.1.3. *Quantifier-free types.* A *closed quantifier-free $\mathcal{L}_G$-condition* in the variables $x_1, \ldots, x_n$ is an expression of the form $\varphi(x_1, \ldots, x_n) \leq r$, where $r \in \mathbb{R}$ and $\varphi(x_1, \ldots, x_n)$ is a quantifier-free $\mathcal{L}_G$-formula. A *quantifier-free $\mathcal{L}_G$-type* $t(x_1, \ldots, x_n)$ in the variables $x_1, \ldots, x_n$ is a collection of closed quantifier-free $\mathcal{L}_G$-conditions, with the property that for any $\varepsilon > 0$ there exists a precompact open neighborhood $V$ of the identity such that the condition

$$\sup_{\gamma\in V} \max_{1\leq i\leq n} d(\gamma x_i, x_i) \leq \varepsilon$$

belongs to $t(x_1, \ldots, x_n)$. Given a collection of quantifier-free $\mathcal{L}_G$-conditions $t(\overline{x})$, we let $t^+(\overline{x})$ be the collection of all $\mathcal{L}_G$-conditions of the form $\varphi(\overline{x}) \leq r + \varepsilon$, where $\varphi(\overline{x}) \leq r$ is an $\mathcal{L}_G$-condition in $t(\overline{x})$ and $\varepsilon > 0$. A quantifier-free $\mathcal{L}_G$-type is said to be *positive* if its conditions only involve quantifier-free positive primitive formulas.

The notion of *realization* of an $\mathcal{L}_G$-condition and of an $\mathcal{L}_G$-type is defined as follows: a quantifier-free $\mathcal{L}_G$-type $t(\overline{x})$ is approximately realized in an $\mathcal{L}_G$-structure $M$ if and only if any finite set of conditions in $t^+(\overline{x})$ is realized in $M$.

If $\varphi(\overline{x}) \leq r$ and $\psi(\overline{x}) \leq s$ are two quantifier-free $\mathcal{L}_G$-conditions, we write $\varphi(\overline{x}) \leq r \models \psi(\overline{x}) \leq s$ if for any $\mathcal{L}_G$-structure $M$ and tuple $\overline{a}$ in $M$, the fact that $M \models \varphi(\overline{a}) \leq r$ implies $M \models \psi(\overline{a}) \leq s$. Without loss of generality, we can assume that any quantifier-free $\mathcal{L}_G$-type has the property that whenever it contains $\varphi(\overline{x}) \leq r$ and $\varphi(\overline{x}) \leq r \models \psi(\overline{x}) \leq s$, then it also contains $\psi(\overline{x}) \leq s$.

2.1.4. *Quantifiers.* As in the nonequivariant case, we consider quantifiers of the form $\inf_x$ and $\sup_x$, where the variable $x$ ranges among the elements of the given $\mathcal{L}_G$-structure. We will mostly consider *existential $\mathcal{L}_G$-formulas*, of the form $\inf_{\overline{y}} \varphi(\overline{x}, \overline{y})$, and *universal $\mathcal{L}_G$-formulas*, of the form $\sup_{\overline{y}} \varphi(\overline{x}, \overline{y})$, where in both cases $\varphi$ is a quantifier-free $\mathcal{L}_G$-formula. An *existential positive primitive formula* is of the form $\inf_{\overline{y}} \varphi(\overline{x}, \overline{y})$, where $\varphi(\overline{x}, \overline{y})$ is a quantifier-free positive primitive formula. One can show that the interpretation of an $\mathcal{L}_G$-formula in an $\mathcal{L}_G$-structure $M$ is uniformly continuous, and that its modulus of continuity is independent from $M$.

More generally, we consider *type quantifiers* defined as follows. Suppose that $t(\overline{y})$ is an $\mathcal{L}_G$-type and $\varphi(\overline{x}, \overline{y})$ is an $\mathcal{L}_G$-formula. Then $\inf^t_{\overline{y}} \varphi(\overline{x}, \overline{y})$ is interpreted in an $\mathcal{L}_G$-structure $M$ by declaring that $M \models \inf^t_{\overline{y}} \varphi(\overline{a}, \overline{y}) \leq r$ for some tuple $\overline{a}$ if and only if the $\mathcal{L}(M)_G$-type $t(\overline{y}) \cup \{\varphi(\overline{a}, \overline{y}) \leq r\}$ is approximately realized in $M$. One can define $\sup^t_{\overline{y}} \varphi(\overline{x}, \overline{y})$ similarly.

2.1.5. *Bases.* Let us fix a countable basis $\mathcal{B}$ of precompact open subsets of $G$ that is closed under finite unions and intersections. We say that an $\mathcal{L}_G$-formula $\varphi$ is *over* $\mathcal{B}$ if all the expressions of the form $\inf_{\gamma \in V}$ and $\sup_{\gamma \in V}$ that appear in $\varphi$ satisfy $V \in \mathcal{B}$. Similarly, we say that an $\mathcal{L}_G$-type $t(\overline{y})$ is over $\mathcal{B}$ if the conditions in $t$ are of the form $\varphi(\overline{a}) \leq r$, where $r \in \mathbb{Q}$ and $\varphi$ is a quantifier-free $\mathcal{L}_G$-formula over $\mathcal{B}$.

**Remark 2.2.** Denote by $\psi(\overline{x})$ the quantifier-free $\mathcal{L}_G$-formula

$$\sup_{\rho_1 \in W_1} \inf_{\gamma_1 \in V_1} \cdots \sup_{\rho_n \in W_n} \inf_{\gamma_n \in V_n} \varphi(\overline{x}, \overline{\rho}, \overline{\gamma}).$$

Fix $\varepsilon > 0$. Then there exist $\delta > 0$ and open precompact sets $W'_j, V'_j, U \in \mathcal{B}$, for $j = 1, 2, \ldots, n$, such that $W_j \subset W'_j$, $V_j \subset V'_j$, and $U$ is a neighborhood of the identity of $G$, with the following property: if $\psi'(\overline{x})$ denotes the quantifier-free $\mathcal{L}_G$-formula

$$\sup_{\rho_1 \in W'_1} \inf_{\gamma_1 \in V'_1} \cdots \sup_{\rho_n \in W'_n} \inf_{\gamma_n \in V'_n} \varphi(\overline{x}, \overline{\rho}, \overline{\gamma}),$$

and $\overline{a}$ is a tuple in an $\mathcal{L}_G$-structure satisfying $\sup_{g \in U} d(ga_j, a_j) \leq \delta$ for $j = 1, \ldots, n$, then $\psi(\overline{a}) \leq \psi'(\overline{a}) \leq \psi(\overline{a}) + \varepsilon$.

The above observation can be used to conclude that one can just consider without loss of generality $\mathcal{L}_G$-types over $\mathcal{B}$.

## 2.2. **Semantic.**

2.2.1. *Ultrapowers.* Let $I$ be an index set and let $(M_i)_{i \in I}$ be a collection of $\mathcal{L}_G$-structures. Fix a filter $\mathcal{F}$ over $I$. The notion of *reduced product* $\prod_{\mathcal{F}} M_i$ of $(M_i)_{i \in I}$ can be defined as in [47]. In the following, we will denote by $\boldsymbol{a}$ or $[a_i]_{i \in I}$ the element of $\prod_{\mathcal{F}} M_i$ with representative collection $(a_i)_{i \in I}$, and similarly for other letters. The support of $\prod_{\mathcal{F}} M_i$ is the Hausdorff quotient of $\prod_i M_i$ endowed with the metric $d(\boldsymbol{a}, \boldsymbol{b}) = \limsup_{i \to \mathcal{F}} d^{M_i}(a_i, b_i)$. The interpretation of an $n$-ary relation symbol $R$ from $\mathcal{L}_G$ in $\prod_{\mathcal{F}} M_i$ is defined by $R^{M_{\mathcal{F}}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \limsup_{i \to \mathcal{F}} R^{M_i}(a_{i,1}, \ldots, a_{i,n})$.

The reduced product $\prod_{\mathcal{F}} M_i$ is naturally endowed with a $G$-action by automorphisms, defined by $g^{M_{\mathcal{F}}} \boldsymbol{a} = \left[g^{M_i} a_i\right]_{i \in I}$. However, since such a $G$-action is in general not continuous, $\prod_{\mathcal{F}} M_i$ is not necessarily an $\mathcal{L}_G$-structure.

**Definition 2.3.** The *reduced $G$-product* of $(M_i)_{i \in I}$ is the $\mathcal{L}_G$-structure

$$\prod\nolimits^G_{\mathcal{F}} M_i = \left\{ \boldsymbol{a} \in \prod\nolimits_{\mathcal{F}} M_i \colon \text{ for every } \varepsilon > 0 \text{ there is } 1_G \in V \subset G \text{ open, with } \limsup_{i \to \mathcal{F}} \sup_{h \in V} d(h^{M_i} a_i, a_i) \leq \varepsilon \right\}.$$

When $\mathcal{F}$ is in fact an ultrafilter, we call the corresponding reduced $G$-product the *$G$-ultraproduct*. The *reduced $G$-power* of an $\mathcal{L}_G$-structure $M$ is the reduced $G$-product of the collection $(M_i)$ where $M_i$ is equal to $M$ for every $i \in I$. The canonical inclusion of $M$ into $\prod^G_{\mathcal{F}} M$ mapping $a \in M$ to $[a]_{i \in I}$ will be called the *diagonal* $\mathcal{L}_G$-morphism, and denoted by $\Delta_M$.

**Remark 2.4.** The argument in [54, Lemma 1.8] shows that, when $\mathcal{F}$ is a *countably generated* filter, then $\prod^G_{\mathcal{F}} M_i$ is the set of elements of $\prod_{\mathcal{F}} M_i$ where the canonical $G$-action is continuous. It is not clear whether this holds for more general filters.

One can easily prove, by induction on the complexity of $\mathcal{L}_G$-terms, that for any $n, m \in \mathbb{N}$, for any $\mathcal{L}_G$-term $t(x_1, \ldots, x_n, \gamma_1, \ldots, \gamma_m)$, for any $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \prod_{\mathcal{F}}^G M_i$, and any $g_1, \ldots, g_m \in G$, there exists a neighborhood $V$ of the identity in $G$ such that

$$\limsup_{i \to \mathcal{F}} \sup_{h_1, \ldots, h_m \in V} \left\| t^{M_i}(a_{i,1}, \ldots, a_{i,n}, g_1, \ldots, g_m) - t^{M_i}(a_{i,1}, \ldots, a_{i,n}, h_1 g_1, \ldots, h_m g_m) \right\| \leq \varepsilon.$$

It follows, again by induction on the complexity, that for any $n, m \in \mathbb{N}$, for any quantifier-free $\mathcal{L}_G$-formula $\varphi(x_1, \ldots, x_n, \gamma_1, \ldots, \gamma_m)$, for any $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \prod_{\mathcal{F}}^G M_i$, and any $g_1, \ldots, g_m \in G$, there exists a neighborhood $V$ of the identity in $G$ such that

$$\limsup_{i \to \mathcal{F}} \sup_{h_1, \ldots, h_m \in V} \left| \varphi^{M_i}(a_{i,1}, \ldots, a_{i,n}, g_1, \ldots, g_m) - \varphi^{M_i}(a_{i,1}, \ldots, a_{i,n}, h_1 g_1, \ldots, h_m g_m) \right| \leq \varepsilon.$$

When $\mathcal{F}$ is an ultrafilter, the same conclusion holds for an arbitrary (not necessarily positive primitive) quantifier-free $\mathcal{L}_G$-formula. The following result, which can be seen as an analog of Łos' theorem in this context, is then an easy consequence of these observations.

**Theorem 2.5.** *Fix $m, n \in \mathbb{N}$. Let $\varphi(x_1, \ldots, x_n, \gamma_1, \ldots, \gamma_m)$ be a quantifier-free $\mathcal{L}_G$-formula, let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \prod_{\mathcal{F}}^G M_i$, let $g_1, \ldots, g_m \in G$, and let $\mathcal{F}$ be a filter over $I$. If either $\varphi$ is positive primitive or $\mathcal{F}$ is an ultrafilter, then*

$$\varphi^{\prod_{\mathcal{F}}^G M_i}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n, g_1, \ldots, g_m) = \limsup_{i \to \mathcal{F}} \varphi^{M_i}(a_{i,1}, \ldots, a_{i,n}, g_1, \ldots, g_m).$$

**Corollary 2.6.** Let $M$ be an $\mathcal{L}_G$-structure, let $\overline{a}$ be a tuple in $M$, let $\varphi(\overline{x}, \overline{y})$ be a quantifier-free $\mathcal{L}_G$-formula, and let $t(\overline{y})$ be a quantifier-free $\mathcal{L}_G$-type. If either $\mathcal{F}$ is an ultrafilter or $\varphi$ is positive primitive and $t$ is positive, then

$$\prod_{\mathcal{F}}^G M \models \inf_{\overline{y}}^t \varphi(\overline{a}, \overline{y}) \leq r \text{ if and only if } M \models \inf_{\overline{y}}^t \varphi(\overline{a}, \overline{y}) \leq r.$$

We say that an $\mathcal{L}_G$-structure $M$ is *countably quantifier-free $\mathcal{L}_G$-saturated* if for any separable substructure $A \subset M$, every quantifier-free $\mathcal{L}_G$-type over $A$ that is approximately realized in $M$ is in fact realized in $M$. The notion of countably positively quantifier-free $\mathcal{L}_G$-saturated is defined similarly, where only positive quantifier-free $\mathcal{L}_G$-types are considered. Recall that a filter $\mathcal{F}$ is *countably incomplete* if it contains a countable collection of elements with empty intersection. The same proof as in the nonequivariant setting [5, Proposition 7.6] shows the following.

**Proposition 2.7.** *Let $\mathcal{L}$ be a countable language, and let $\mathcal{U}$ be a countably incomplete ultrafilter. Then $\prod_{\mathcal{U}}^G M_i$ is countably quantifier-free $\mathcal{L}_G$-saturated. If $\mathcal{F}$ is a countably incomplete filter, then $\prod_{\mathcal{F}}^G M_i$ is countably positively quantifier-free $\mathcal{L}_G$-saturated.*

We say that a quantifier-free $\mathcal{L}_G$-type $t$ is *consistent* if every finite subset of $t^+$ is realized in some $\mathcal{L}_G$-structure. The following *compactness* theorem is an easy consequence of Theorem 2.5 and Proposition 2.7.

**Corollary 2.8.** A quantifier-free $\mathcal{L}_G$-type is consistent if and only if it is realized in some $\mathcal{L}_G$-structure.

Fix a indexing set $I$. It is well known that if $M$ is a *compact* $\mathcal{L}$-structure and $\mathcal{U}$ is any ultrafilter, then the diagonal $\mathcal{L}$-embedding $\Delta_M \colon M \to \prod_{\mathcal{U}} M$ is surjective. The following proposition gives the $G$-equivariant analog of this fact.

**Proposition 2.9.** *Fix an ultrafilter $\mathcal{U}$ over $I$. Suppose that $M$ is a $G$-structure such that for any $\varepsilon > 0$ there exist $\delta > 0$ and a neighborhood $V$ of the identity in $G$ such that the set*

$$M_{V,\delta} = \left\{ a \in M \colon \sup_{g \in V} d(ga, a) < \delta \right\}$$

*admits a finite $\varepsilon$-dense set. Then the diagonal $\mathcal{L}_G$-embedding $\Delta_M \colon M \to \prod_{\mathcal{U}}^G M$ is surjective.*

*Proof.* Fix $n \in \mathbb{N}$ and an element $\boldsymbol{a}$ of $\prod_{\mathcal{U}}^G M$ with representative $(a_i)_{i \in I}$. Choose an open neighborhood $V$ of $M$ and $\delta > 0$ such that $M_{V,\delta}$ admits a finite $2^{-n}$-dense set. Then $\{i \in I \colon a_i \in M_{V,\delta}\}$ belongs to $\mathcal{U}$. Therefore, there exists a subset $U$ of $M_{V,\delta}$ of diameter of most $2^{-n}$ such that $\{i \in I \colon a_i \in U\}$ belong to $\mathcal{U}$.

The above observation allows one to define recursively a chain $(F_n)_{n \in \mathbb{N}}$ of closed subsets of $M$ such that the diameter of $F_n$ is at most $2^{-n}$ and $\{i \in I \colon a_i \in F_n\}$ belongs to $\mathcal{U}$. If $a$ is the unique element of $\bigcap_{n \in \mathbb{N}} F_n$, then $a = \lim_{n \to \mathcal{U}} a_n$. This shows that $\boldsymbol{a}$ belongs to the image of $\Delta_M$, and the proof is finished. ☐

2.2.2. *Existential theories of structures.* Suppose that $M$ is an $\mathcal{L}_G$-structure. The *existential $\mathcal{L}_G$-theory* $\mathrm{Th}_{\exists}^{\mathcal{L}_G}(M)$ of $M$ is the function $\varphi \mapsto \varphi^M$ that assigns to an existential $\mathcal{L}_G$-sentence $\varphi$ its value $\varphi^M$ in $M$. We say that $M$ is *weakly $\mathcal{L}_G$-contained* in $N$ if $\mathrm{Th}_{\exists}^{\mathcal{L}_G}(M) \leq \mathrm{Th}_{\exists}^{\mathcal{L}_G}(N)$, and *weakly $\mathcal{L}_G$-equivalent* to $N$ if $M$ and $N$ have the same existential $\mathcal{L}_G$-theory. We will identify the existential $\mathcal{L}_G$-theory of an $\mathcal{L}_G$-structure with its weak $\mathcal{L}_G$-equivalence class. It follows from Proposition 2.7 and Theorem 2.5 that $M$ is weakly $\mathcal{L}_G$-contained in $N$ if and only if for some (equivalently, any) countably incomplete ultrafilter $\mathcal{U}$, every separable substructure of $M$ admits an $\mathcal{L}_G$-embedding into $\prod_{\mathcal{U}}^G N$. This is equivalent to the assertion that if a quantifier-free $\mathcal{L}_G$-type is approximately realized in $M$, then it is approximately realized in $N$.

Given a quantifier-free $\mathcal{L}_G$-type $t(\overline{x})$, we write $\mathrm{Omit}(t)$ for the set of existential theories $\mathrm{Th}_{\exists}^{\mathcal{L}_G}(M)$ such that $t$ is *not* approximately realized in $M$. Denote by $\mathcal{E}(\mathcal{L}_G)$ the space of existential $\mathcal{L}_G$-theories of $\mathcal{L}_G$-structures, and write $\mathcal{A}$ for the family of all subsets of $\mathcal{E}(\mathcal{L}_G)$ of the form $\mathrm{Omit}(t)$, for some quantifier-free $\mathcal{L}_G$-type $t(\overline{x})$.

**Proposition 2.10.** *The family $\mathcal{A}$ is a basis for a topology on $\mathcal{E}(\mathcal{L}_G)$, which is compact and Hausdorff. Moreover, when $\mathcal{L}$ is separable, then this topology is also metrizable.*

*Proof.* We first show that $\mathcal{A}$ is a basis for a topology on $\mathcal{E}(\mathcal{L}_G)$. Suppose that $t_0$ and $t_1$ are quantifier-free $\mathcal{L}_G$-types $t(\overline{x})$ such that $\mathrm{Th}_{\exists}^{\mathcal{L}_G}(M) \in \mathrm{Omit}(t_0) \cap \mathrm{Omit}(t_1)$. We can assume, in view of Proposition 2.7 and Theorem 2.5, that $M$ is countably quantifier-free $\mathcal{L}_G$-saturated. Furthermore, we can assume that the types $t_0$ and $t_1$ are closed under logical implication. Thus there exist quantifier-free $\mathcal{L}_G$-conditions $\varphi_0(\overline{x}) \leq 0$ in $t_0$ and $\varphi_1(\overline{x}) \leq 0$ in $t_1$ such that

$$M \models \inf_{\overline{x}} \min\{\varphi_0(\overline{x}), \varphi_1(\overline{x})\} \geq \varepsilon.$$

Set $\varphi = \min\{\varphi_0, \varphi_1\}$. Then $t_0$ and $t_1$ contain the condition $\varphi(\overline{x}) \leq 0$, and that $M \models \inf_{\overline{x}} \varphi(\overline{x}) \geq \varepsilon$. If $t = t_0 \cap t_1$, then $\mathrm{Th}_{\exists}^{\mathcal{L}_G}(M) \in \mathrm{Omit}(t) \subset \mathrm{Omit}(t_0) \cap \mathrm{Omit}(t_1)$, as desired.

It is clear that the topology generated by $\mathcal{A}$ is Hausdorff. We proceed to show that it is compact. A closed set in $\mathcal{E}(\mathcal{L}_G)$ is of the form $[t]$ for some quantifier-free $\mathcal{L}_G$-type $t(\overline{x})$, where $[t]$ denotes the set of existential theories of $\mathcal{L}_G$-structures in which $t$ is (approximately) realized. If $(t_i)_{i\in I}$ is a collection of types such that the family of closed sets $\{[t_i] \colon i \in I\}$ has the finite intersection property, then $t = \bigcup_{i\in I} t_i$ is a consistent quantifier-free $\mathcal{L}_G$-type. Therefore, by Corollary 2.8, $t$ is realized in some $\mathcal{L}_G$-structure $M$. The existential theory of $M$ witnesses that the family of closed sets $\{[t_i] : i \in I\}$ has nonempty intersection, so $\mathcal{E}(\mathcal{L}_G)$ is compact.

Fix a countable basis $\mathcal{B}$ of precompact open subsets of $G$ that is closed under finite unions and intersections. Then Remark 2.2 shows that in the definition of the topology in the space of existential $\mathcal{L}_G$-theories, it is enough to consider existential $\mathcal{L}_G$-formulas over $\mathcal{B}$. This shows that, when $\mathcal{L}$ is separable, the space $\mathcal{E}(\mathcal{L}_G)$ is metrizable. □

A class $\mathfrak{C}$ of structures is said to be *existentially $\mathcal{L}_G$-axiomatizable* if the collection of weak $\mathcal{L}_G$-equivalence classes of elements of $\mathfrak{C}$ is a closed subset of $\mathcal{E}(\mathcal{L}_G)$. More generally, we consider the following notion, which has been introduced in the nonequivariant setting in [30, Definition 5.7.1].

**Definition 2.11.** A class $\mathfrak{C}$ of (separable) structures is said to be *definable by a uniform family of existential $\mathcal{L}_G$-formulas* if, for every $k \in \mathbb{N}$, there exist $n_k \in \mathbb{N}$ and an uniformly equicontinuous collections $\mathcal{F}_k(x_1, \ldots, x_{n_k})$ of existential $\mathcal{L}_G$-formulas, such that a (separable) $\mathcal{L}_G$-structure $M$ belongs to $\mathfrak{C}$ if and only if for every $k \in \mathbb{N}$ and every $\overline{a} \in M^{n_k}$ there exists $\varphi \in \mathcal{F}_k$, such that $M \models \varphi(\overline{a}) \leq 1/k$.

Observe that if $\mathfrak{C}$ is a class of (separable) structures definable by a uniform family of existential $\mathcal{L}_G$-formulas, then $\mathfrak{C}$ is closed under (countable) direct limits. We say that a property is definable by a uniform family of existential $\mathcal{L}_G$-formulas if the class of $\mathcal{L}_G$-structures satisfying that property is.

The notions of existential positive primitive $\mathcal{L}_G$-theory, positive weak $\mathcal{L}_G$-containment, positive weak $\mathcal{L}_G$-equivalence, positively existentially $\mathcal{L}_G$-axiomatizable class, and class definable by a uniform family of existential positive primitive $\mathcal{L}_G$-formulas are defined as above, by only considering existential positive primitive $\mathcal{L}_G$-formulas. It follows from Theorem 2.5 and Proposition 2.7 that $M$ is positively weakly $\mathcal{L}_G$-contained in $N$ if and only if for some (equivalently, any) countably incomplete filter $\mathcal{F}$, every separable substructure of $M$ admits an $\mathcal{L}_G$-morphism to $\prod_{\mathcal{F}}^G N$.

Suppose that $M$ and $N$ are $\mathcal{L}_G$-structures that are isomorphic to the same separable $\mathcal{L}_G$-structure $A$. Denote by $\mathrm{Act}_G(A)$ the set of continuous actions of $G$ on $A$, endowed with the compact-open topology, which makes it a Polish space. If $\mathrm{Aut}_{\mathcal{L}}(A)$ denotes the group of $\mathcal{L}$-automorphisms of $A$, then $\mathrm{Act}_G(A)$ is canonically a Polish

$\mathrm{Aut}_{\mathcal{L}}(A)$-space. One can identify $M$ and $N$ as elements of $\mathrm{Act}_G(A)$. Assume furthermore that any $\mathcal{L}$-embedding $\eta\colon A \to \prod_{\mathcal{U}} A$, where $\mathcal{U}$ is an ultrafilter over $\mathbb{N}$, admits a lift $(\alpha_n)_{n\in\mathbb{N}}$ consisting of $\mathcal{L}$-automorphisms. One can characterize weak $\mathcal{L}_G$-containment for the $\mathcal{L}_G$-structures $M$ and $N$ in terms of the $\mathrm{Aut}_{\mathcal{L}}(A)$-orbits of $M$ and $N$ inside $\mathrm{Act}_G(A)$.

**Proposition 2.12.** *Suppose that $M, N$ are continuous actions of $G$ on $A$. The following statements are equivalent:*

1. *$M$ is weakly $\mathcal{L}_G$-contained in $N$;*
2. *$M$ belongs to the closure of the $\mathrm{Aut}_{\mathcal{L}}(A)$-orbit of $N$.*

2.2.3. *Existential theories of embeddings.* Suppose that $A, M$ are $\mathcal{L}_G$-structures and $\theta\colon A \to M$ is an $\mathcal{L}_G$-embedding. We can regard $(M, \theta_M)$ as a structure in the language $\mathcal{L}(A)$ obtained by adding a constant symbol $c_a$ for any element $a \in A$. The interpretation of $c_a$ in $(M, \theta)$ is the image $\theta(a)$ of $a$ under $\theta$. One can then define the notions of quantifier-free $\mathcal{L}(A)_G$-formula and quantifier-free $\mathcal{L}(A)_G$-type. The same definition as in Subsubsection 2.2.2 gives the notion of weak $\mathcal{L}_G$-containment, weak $\mathcal{L}_G$-equivalence, and existential $\mathcal{L}_G$-theory for embeddings $\theta_M\colon A \to M$ and $\theta_N\colon A \to N$. As in the case of $\mathcal{L}_G$-structures, one can say that $\theta_M$ is weakly $\mathcal{L}_G$-contained in $\theta_N$ if and only if for any separable substructures $A_0 \subset A$ and $M_0 \subset M$ such that $\theta_M(A_0) \subset M_0$, and for some (equivalently, any) countably incomplete ultrafilter $\mathcal{U}$, there exists an $\mathcal{L}_G$-embedding $\eta\colon M_0 \to \prod_{\mathcal{U}}^G N$ such that $\Delta_N \circ \theta_N|_{A_0} = \eta \circ \theta_M|_{A_0}$.

**Definition 2.13.** An $\mathcal{L}_G$-embedding $\theta_M\colon A \to M$ is said to be $\mathcal{L}_G$-*existential* if for any quantifier-free $\mathcal{L}_G$-formula $\varphi(\overline{x}, \overline{y})$ and any tuple $\overline{a} \in A$, the value of $\inf_{\overline{y}} \varphi(\overline{a}, \overline{y})$ in $A$ is the same as the value of $\inf_{\overline{y}} \varphi(\theta_M(\overline{a}), \overline{y})$ in $M$.

It is easy to see that $\theta_M\colon A \to M$ is $\mathcal{L}_G$-existential if and only if $\theta_M$ is weakly $\mathcal{L}_G$-contained in the identity embedding $\mathrm{id}_A\colon A \to A$.

Similarly, one can define the notion of positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding $\theta_M\colon A \to M$, by only considering existential positive primitive $\mathcal{L}_G$-formulas. The following fact is an easy consequence of the definition of positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding.

**Proposition 2.14.** *Suppose that $\mathfrak{C}$ is a class of structures that is definable by a uniform family of existential positive primitive $\mathcal{L}_G$-formulas. If $\theta_M\colon A \to M$ is a positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding and $M \in \mathfrak{C}$, then $A \in \mathfrak{C}$.*

**Proposition 2.15.** *Let $\Lambda$ be a directed set. The following properties follow easily from the definition.*

1. *The composition of positive $\mathcal{L}_G$-existential $\mathcal{L}_G$-embeddings is a positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding.*
2. *Let $(\{M_\lambda\}_{\lambda\in\Lambda}, \{\theta_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda,\lambda<\mu})$ be a direct system of $\mathcal{L}_G$-structures with positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embeddings $\theta_{\lambda\mu}\colon M_\lambda \to M_\mu$ for $\lambda < \mu$. If $M$ is the corresponding direct limit, then the canonical $\mathcal{L}_G$-embedding of $M_\lambda$ into $M$, for $\lambda \in \Lambda$, is positively $\mathcal{L}_G$-existential.*
3. *For $j = 0, 1$, let $(\{M_\lambda^{(j)}\}_{\lambda\in\Lambda}, \{\theta_{\lambda\mu}^{(j)}\}_{\lambda,\mu\in\Lambda,\lambda<\mu})$ be a direct system of $\mathcal{L}_G$-structures. Let $\{\eta_\lambda\colon M_\lambda^{(0)} \to M_\lambda^{(1)}\}_{\lambda\in\Lambda}$ be a family of intertwining positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embeddings. Then*
$$\varinjlim \eta_\lambda\colon \varinjlim_\lambda M_\lambda^{(0)} \to \varinjlim_\lambda M_\lambda^{(1)}$$
*is a positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding.*

The analogue of Remark 2.15 holds for $\mathcal{L}_G$-existential $\mathcal{L}_G$-embeddings as well.

2.2.4. *Saturation.* Suppose that $M$ is an $\mathcal{L}_G$-structure. We say that an $\mathcal{L}_G$-structure $N \supset M$ is a quantifier-free *enlargement* of $M$ if every quantifier-free $\mathcal{L}_G(M)$-type that is approximately realized in $N$, is also realized in $M$. We say that $N$ is a *positive quantifier-free enlargement* if the same holds for positive quantifier-free $\mathcal{L}_G(M)$-types. The same proof as [5, Lemma 7.9] using Theorem 2.5 shows the following.

**Proposition 2.16.** *Suppose that $M$ is an $\mathcal{L}_G$-structure. Let $\kappa$ be a cardinal larger than the density character of $M$ and the density character of $\mathcal{L}$. Denote by $\Lambda$ the set of finite subsets of $\kappa$ ordered by inclusion. Let $\mathcal{F}$ be a filter on $\Lambda$ that contains the filer of upper cones in $\Lambda$. Then $\prod_{\mathcal{F}}^G M$ is a positive quantifier-free enlargement of $M$. If $\mathcal{F}$ is an ultrafilter, then $\prod_{\mathcal{F}}^G M$ is a quantifier-free enlargement of $M$.*

Suppose that $\kappa$ is an uncountable cardinal. We say that an $\mathcal{L}_G$-structure is *quantifier-free $\mathcal{L}_G$-$\kappa$-saturated* if for every substructure $A \subset M$ of density character less than $\kappa$ and any quantifier-free $\mathcal{L}(A)_G$-type $t$ over $A$, whenever

$t$ is approximately realized in $M$, then $t$ is realized in $M$. Starting from Proposition 2.16 and Remark 2.15, one can prove the following proposition similarly as [5, Proposition 7.10].

**Proposition 2.17.** *Suppose that $M$ is an $\mathcal{L}_G$-structure, and $\kappa$ is an uncountable cardinal. Then $M$ admits an $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding into a quantifier-free $\mathcal{L}_G$-$\kappa$-saturated $\mathcal{L}_G$-structure.*

An alternative approach to the proof of Proposition 2.17 consists in using $\kappa$-good ultrafilters as defined in [19, Section 6.1]. The notion of $\kappa$-good filter can be defined exactly as for ultrafilters. Theorem 6.1.4 of [19] shows that countably incomplete $\kappa$-good ultrafilters exist for any cardinal $\kappa$. Every countably incomplete ultrafilter is $\aleph_1$-good; see [19, Exercise 6.1.2]. The same proof as [19, Theorem 6.1.8] shows the following.

**Proposition 2.18.** *Suppose that $\kappa$ is a cardinal larger than the density character of $\mathcal{L}$. Suppose that $M$ is an $\mathcal{L}$-structure and $\mathcal{U}$ is a countably incomplete $\kappa$-good ultrafilter. Then $\prod_{\mathcal{U}} M$ is $\mathcal{L}$-$\kappa$-saturated.*

*Suppose now that $M$ is an $\mathcal{L}_G$-structure. In this case, $\prod_{\mathcal{U}}^G M$ is quantifier free $\mathcal{L}_G$-$\kappa$-saturated. If $\mathcal{F}$ is a countably incomplete $\kappa$-good filter, then $\prod_{\mathcal{F}}^G M$ is positively $\mathcal{L}_G$-$\kappa$-saturated.*

Using Proposition 2.17 one can easily deduce the following characterization of $\mathcal{L}_G$-existential $\mathcal{L}_G$-embeddings.

**Theorem 2.19.** *Let $A$ and $M$ be $\mathcal{L}_G$-structures, and let $\theta\colon A \to M$ be an $\mathcal{L}_G$-embedding. Let $\kappa$ be a cardinal greater than the density character of $M$ and the density character of $\mathcal{L}$. The following assertions are equivalent:*

1. *$\theta$ is an $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding;*
2. *there exist an $\mathcal{L}_G$-structure $N$ and an $\mathcal{L}_G$-embedding $\eta\colon M \to N$ such that $\eta\circ\theta\colon A \to N$ is an $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding;*
3. *if $N$ is a quantifier-free $\mathcal{L}_G$-$\kappa$-saturated $\mathcal{L}_G$-structure, and $\theta_N\colon A \to N$ is an $\mathcal{L}_G$-embedding, then there exists an $\mathcal{L}_G$-embedding $\eta\colon M \to N$ such that $\eta\circ\theta = \theta_N$;*
4. *for some (equivalently, any) countably incomplete ultrafilter $\mathcal{U}$, and for every separable $A_0 \subset A$ and $M_0 \subset M$ such that $\theta_M(A_0) \subset M_0$, there exists an $\mathcal{L}_G$-embedding $\eta\colon M_0 \to \prod_{\mathcal{U}}^G A$ such that $\eta\circ\theta_M|_{A_0} = \Delta_A|_{A_0}$.*

A similar characterization can be given for positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embeddings.

**Theorem 2.20.** *Let $A$ and $M$ be $\mathcal{L}_G$-structures, and let $\theta\colon A \to M$ be an $\mathcal{L}_G$-embedding. Let $\kappa$ be a cardinal larger than the density character of $M$ and the density character of $\mathcal{L}$. The following assertions are equivalent:*

1. *$\theta$ is a positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding;*
2. *there exist an $\mathcal{L}_G$-structure $N$ and an $\mathcal{L}_G$-morphism $\eta\colon M \to N$ such that $\eta\circ\theta\colon A \to N$ is an $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding;*
3. *if $N$ is a quantifier-free positively $\mathcal{L}_G$-$\kappa$-saturated $\mathcal{L}_G$-structure, and $\theta_N\colon A \to N$ is an $\mathcal{L}_G$-embedding, then there exists an $\mathcal{L}_G$-morphism $\eta\colon M \to N$ such that $\eta\circ\theta = \theta_N$;*
4. *for some (equivalently, any) countably incomplete ultrafilter $\mathcal{F}$, and for every separable $A_0 \subset A$ and $M_0 \subset M$ such that $\theta_M(A_0) \subset M_0$, there exists an $\mathcal{L}_G$-morphism $\eta\colon M_0 \to \prod_{\mathcal{F}}^G A$ such that $\eta\circ\theta_M|_{A_0} = \Delta_A|_{A_0}$.*

We isolate the following fact, which is an immediate consequence of the semantic characterization of positive $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding. If $F$ is a functor between two categories, we denote by $F(\theta)$ the image of a morphism $\theta$ under $F$. We regard the class of $\mathcal{L}_G$-structures as a category with $\mathcal{L}_G$-morphisms as morphisms.

**Proposition 2.21.** *Suppose that $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ are languages, and that $G_0$ and $G_1$ are locally compact second countable groups. Let $F$ be a functor from the category of $\mathcal{L}^{(0)}_{G_0}$-structures to the category of $\mathcal{L}^{(1)}_{G_1}$-structures. Assume that $F$ preserves direct limits and that for any separable $\mathcal{L}^{(0)}_{G_0}$-structure $M$ and nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, there exists an $\mathcal{L}^{(1)}_{G_1}$-morphism $\rho_M\colon F(\prod_{\mathcal{U}}^{G_0} M) \to \prod_{\mathcal{U}}^{G_1} F(M)$ such that $\rho_M \circ F(\Delta_M) = \Delta_{F(M)}$. If $A$ and $M$ are $\mathcal{L}^{(0)}_{G_0}$-structures in $\mathfrak{C}$ and $\theta_M\colon A \to M$ is a positive $\mathcal{L}^{(0)}_{G_0}$-existential $\mathcal{L}^{(0)}_{G_0}$-embedding, then $F(\theta_M)$ is a positive $\mathcal{L}^{(1)}_{G_1}$-existential $\mathcal{L}^{(1)}_{G_1}$–embedding.*

*Proof.* Since $F$ preserves direct limits, it is enough to consider the case when $M$ is separable. In this case, the conclusion is the consequence of the first assumption on the functor $F$ and Condition (3) of Theorem 2.19. ☐

In particular, Proposition 2.21 applies when the functor $F$ preserves both direct limits and ultraproducts.

2.2.5. *Definability.* The notion of an $\mathcal{L}_G$-definable set and $\mathcal{L}_G$-definable predicate can be defined as in the nonequivariant setting [5, Section 9]. If $P$ is an $\mathbb{R}_+$-valued $\mathcal{L}_G$-definable predicate, then we say that $P$ is *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any tuple $\overline{a}$ in an $\mathcal{L}_G$-structure $M$, if $P^M(\overline{a}) \leq \varepsilon$, then there exists a tuple $\overline{b}$ in the zero-set of $P^M$ at distance at most $\varepsilon$ from $\overline{a}$.

The $\mathcal{L}_G$-definable sets are precisely the zero-set of stable $\mathcal{L}_G$-definable predicates. For instance, an averaging argument using the Haar measure shows that, when the group $G$ is compact, the $\mathcal{L}_G$-formula $\sup_{\gamma \in G} \|\gamma x - x\|$ is stable. Its zero-set in an $\mathcal{L}_G$-structure $M$ is the fixed point substructure $M^G = \{x \in M \colon \forall g \in G, g^M x = x\}$.

We say that a predicate is positively existentially $\mathcal{L}_G$-definable if it is the uniform limit of existential positive primitive $\mathcal{L}_G$-formulas. Similar terminology applies to definable sets. Note that if $D$ is a (positively existentially) $\mathcal{L}_G$-definable set and $P(\overline{x}, y)$ is a (positively existentially) $\mathcal{L}_G$-definable predicate, then $\inf_{y \in D} P(\overline{x}, y)$ is a (positively existentially) $\mathcal{L}_G$-definable predicate; see [5, Theorem 9.17].

**Remark 2.22.** If $D$ is a positively existentially $\mathcal{L}_G$-definable set, and $\theta \colon A \to M$ is a positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding, then $\theta|_D \colon D^A \to D^B$ is a positively $\mathcal{L}_G$-existential $\mathcal{L}_G$-embedding.

2.3. **Examples.** In this subsection, we point the reader to some examples coming from measurable dynamics. More applications and motivation coming from C*-dynamics will be the subject of the rest of the present article.

Let $(X, \mu)$ be a measure space and let $G$ be a countable discrete group. The notion of weak containment for measure-preserving actions of $G$ on Borel probability spaces has first been considered in [64, Section 10] and then studied in [1,15,17,21,80]. A measure preserving action $\alpha$ of $G$ on $(X, \mu)$ is said to be *weakly contained* in another action $\beta$ on $(Y, \nu)$ if for any finite subset $F \subset G$, for any $\varepsilon > 0$, and for any Borel partition $\{P_1, \ldots, P_n\}$ of $X$, there exists a Borel partition $\{Q_1, \ldots, Q_n\}$ of $Y$ such that

$$|\nu(Q_i \cap \beta_g(Q_j)) - \mu(P_i \cap \alpha_g(P_j))| < \varepsilon$$

for $1 \leq i, j \leq n$ and for $g \in F$. The definition was later extended to measure-preserving continuous actions of an arbitrary locally compact metrizable group in [13], by replacing finite subsets of the group with compact subsets.

One can identify, at least for actions on standard Borel probability spaces, a continuous measure-preserving action with the corresponding continuous isometric action on the measure algebra. In this way, continuous measure-preserving actions of $G$ can be regarded as $\mathcal{L}_G$-structures, where $\mathcal{L}$ is the language of probability algebras; see [5,6]. It is not difficult to verify that an action $\alpha$ is weakly contained in another action $\beta$ if and only if $\mathrm{Th}_{\exists}^{\mathcal{L}_G}(\alpha) \leq \mathrm{Th}_{\exists}^{\mathcal{L}_G}(\beta)$. Thus, the notion of weak containment introduced in Subsection 2.2.2 extends the corresponding notion for continuous measure-preserving actions. Proposition 2.10 generalizes the result on compactness of the space of measure preserving actions from [1, 13]. Furthermore, the characterization of weak equivalence from Proposition 2.12 can be seen as a generalization of [64, Proposition 10.1]. This perspective, together with the omitting types theorem for the logic for metric structures, has been applied in [16], in the more general context of stationary actions, to the Furstenberg entropy realization problem.

The general framework described above applies equally well in the noncommutative setting to continuous trace-preserving actions of $G$ on tracial von Neumann algebras and, in particular, $\mathrm{II}_1$-factors. Ultrapowers of continuous $\mathbb{R}$-actions (*flows*) on the hyperfinite $\mathrm{II}_1$-factor have been considered in [62, Definition 2.2]. Our general definition of ultrapower agrees with [62, Definition 2.2] in this particular case, when $\mathrm{II}_1$-factors and more generally tracial von Neumann algebras are regarded as metric structures in the language considered in [33].

2.4. **A more general framework.** In this subsection, we consider a slightly more general framework, that will later allow us to deal with not necessarily unital C*-algebras. Consider the language $\mathcal{L}$ that contains

- a collection of function symbols,
- a collection of relation symbols,
- a collection $\mathcal{D}$ of *pseudometric* symbols, and
- a distinguished collection $p(x)$ of quantifier-free positive primitive $\mathcal{L}$-conditions (which are defined in the usual way).

The elements of $\mathcal{D}$ are to be interpreted as pseudometrics in a given $\mathcal{L}$-structure. Recall that $p^+(x)$ denotes the collection of conditions $\varphi(x) \leq r + \varepsilon$ whenever $\varphi(x) \leq r$ is a condition in $p$. We let $\wp_{\mathrm{fin}}(p^+)$ be the collection of finite subsets of $p^+$. We will assume that if $d_0(t_0(\overline{x}), t_1(\overline{x})) \leq r$ is a condition in $p$ for some $d_0 \in \mathcal{D}$ and $\mathcal{L}$-terms $t_0, t_1$, then the condition $d(t_0(\overline{x}), t_1(\overline{x})) \leq r$ also belongs to $p$ for every $d \in \mathcal{D}$. Furthermore, we assume that for any relation symbol $B$ in $\mathcal{L}$ and function symbol $f$ in $\mathcal{L}$, the language $\mathcal{L}$ contains functions $\varpi_B\colon \mathbb{R} \to \mathbb{R} \times \mathcal{D} \times \wp_{\mathrm{fin}}(p^+)$ and $\varpi_f\colon \mathbb{R} \times \mathcal{D} \to \mathbb{R} \times \mathcal{D} \times \wp_{\mathrm{fin}}(p^+)$.

**Definition 2.23.** An $\mathcal{L}$-structure is a set $M$ endowed with an interpretation $B^M$ of any function or relation symbol in $\mathcal{L}$ such that:

(1) the pseudometric symbols in $\mathcal{D}$ are interpreted as pseudometrics on $M$;
(2) for any $n$-ary relation symbol $B$ and any $\varepsilon_0 > 0$, if $\varpi_B(\varepsilon_0) = (\varepsilon_1, d_1, q_1)$, then for any realizations $\overline{a}, \overline{b}$ of $q_1$ in $M$ with $\max_i d_1^M(a_i, b_i) \le \varepsilon_1$, one has $\left|B(\overline{a}) - B(\overline{b})\right| \le \varepsilon_0$;
(3) for any $n$-ary function symbol $f$, any $\varepsilon > 0$, and any $d_0 \in \mathcal{D}$, if $\varpi_f(\varepsilon_0, d_0, q_0) = (\varepsilon_1, d_1, q_1)$, then for any realizations $\overline{a}, \overline{b}$ of $q_1$ in $M$ with $\max_i d_1^M(a_i, b_i) \le \varepsilon_1$, then $f(\overline{a})$ is a realization of $q_0$ and $d_0^M(f(\overline{a}), f(\overline{b})) \le \varepsilon_0$.

The notions of $\mathcal{L}$-formulas and $\mathcal{L}$-types in this setting are defined in the usual way.

Suppose that $(M_i)_{i\in I}$ is a collection of $\mathcal{L}$-structures, and $\mathcal{F}$ is a filter over $I$. We let $M = \prod_{i\in I} M_i$ be the cartesian product. For every $i \in I$ and $d \in \mathcal{D}$, define a pseudometric $d^M$ on $M$ by

$$d^M((a_i)_{i\in I}, (b_i)_{i\in I}) = \limsup_{i\to\mathcal{F}} d^{M_i}(a_i, b_i).$$

Let now $M_{\mathcal{F}}$ be the quotient of $M$ by the equivalence relation $(a_i)_{i\in I} \sim (b_i)_{i\in I}$ if and only if $d^M((a_i)_{i\in I}, (b_i)_{i\in I}) = 0$ for every $d \in \mathcal{D}$. As before, we denote by $\boldsymbol{a}$ the equivalence class of the collection $(a_i)_{i\in I}$. Set

$$\prod\nolimits_{\mathcal{F}} M_i = \left\{\boldsymbol{a} \in M_{\mathcal{F}}\colon \text{ for every } q \in \wp_{\mathrm{fin}}(p^+), \text{ the set } \{i \in I\colon a_i \text{ is a realization of } q\} \text{ belongs to } \mathcal{F}\right\}.$$

The interpretation in $\prod_{\mathcal{F}} M_i$ of function and relation symbols from $\mathcal{L}$ is also defined in the usual way. For instance, if $B$ is an $n$-ary relation symbol from $\mathcal{L}$ and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \prod_{\mathcal{F}} M_i$, then we let

$$B^{\prod_{\mathcal{F}} M_i}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \limsup_{i\to\mathcal{F}} B(a_{1,i}, \ldots, a_{n,i}).$$

Similarly, if $f$ is an $n$-ary function symbol from $\mathcal{L}$ and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \prod_{\mathcal{F}} M_i$, we let $f^{\prod_{\mathcal{F}} M_i}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ be the element with representative $(f(a_{1,i}, \ldots, a_{n,i}))_{i\in I}$. The assumptions on the notion of $\mathcal{L}$-structure guarantee that these definitions do not depend on the representatives, and define an $\mathcal{L}$-structure $\prod_{\mathcal{F}} M_i$, which we call the *reduced product*.

**Remark 2.24.** Let $M$ be an $\mathcal{L}$-structure, let $t(\overline{x})$ be a positive quantifier free type. Let $\kappa$ be a cardinal larger than the density character of $M$ and the density character of $\mathcal{L}$, and let $\mathcal{F}$ be a countably incomplete $\kappa$-good filter. It follows from Proposition 2.18 that the following statements are equivalent:

(1) $t(\overline{x})$ is realized in $\prod_{\mathcal{F}} M$
(2) $t(\overline{x})$ is approximately realized in $\prod_{\mathcal{F}} M$
(3) $t(\overline{x}) \cup p(x_1) \cup \cdots \cup p(x_n)$ is approximately realized in $M$.

By expanding the language to include constants to name elements of $\prod_{\mathcal{F}} M$, one can deduce that $\prod_{\mathcal{F}} M$ is positively quantifier-free $\mathcal{L}_G$-$\kappa$-saturated. One may also consider countably incomplete $\kappa$-good *ultrafilters* (rather than filters) and arbitrary quantifier-free types.

We now discuss the equivariant analog of the observations above.

**Definition 2.25.** Fix a locally compact second countable group $G$ and an action $(g, d) \mapsto d_g$ of $G$ on the set $\mathcal{D}$ of pseudometrics in $\mathcal{L}$. An *$\mathcal{L}_G$-structure* is an $\mathcal{L}$-structure $M$ endowed with a continuous $G$-action by uniform automorphisms. In other words,

(1) for any $g \in G$, the function $M \to M$, given by $x \mapsto g^M x$, commutes with the interpretations of all the function and relation symbols;
(2) $d(g^M a, g^M b) = d_g(a, b)$ for every $a, b \in M$ and $g \in G$;
(3) the action $G \times M \to M$ is continuous when $M$ is endowed with the topology induced by $\left\{d^M\colon d \in \mathcal{D}\right\}$.

Let $(M_i)_{i\in I}$ be a collection of $\mathcal{L}$-structures. Endow $\prod_{\mathcal{F}} M_i$ with the canonical (not necessarily continuous) action of $G$. We set

$$\prod\nolimits_{\mathcal{F}}^{G} M_i = \left\{[a_i]_{i\in I} \in \prod\nolimits_{\mathcal{F}}^{G} M_i \colon \forall\, d \in \mathcal{D}, \forall\, \varepsilon > 0, \exists\, 1_G \in V \subset G \text{ open precompact}\colon \limsup_{i\to\mathcal{F}} \sup_{g\in V} d(g^{M_i} a_i, a_i) \le \varepsilon\right\}.$$

It is easy to check that $\prod_{\mathcal{F}}^{G} M_i$ is indeed an $\mathcal{L}_G$-structure.

By a *positive quantifier-free $\mathcal{L}_G$-type* we mean a collection $t(x_1, \ldots, x_n)$ of quantifier-free positive primitive $\mathcal{L}_G$-conditions with the property that, for any $\varepsilon > 0$ and any $d \in \mathcal{D}$, there exists an open precompact neighborhood $V$ of the identity in $G$ such that the condition $\sup_{g \in V} d(gx, x) \leq \varepsilon$ belongs to $t$. The characterization from Remark 2.24 holds in this context as well.

## 3. Languages for C*-algebras

In this section we consider some natural languages for C*-algebras. By *a degree* 1 *matrix $G$-*-polynomial* we mean an expression of the form

$$\alpha_1^* g_1 x_1 \beta_1 + \cdots + \alpha_n^* g_n x_n \beta_n + \gamma_1^* h_1 x_1^* \delta_1 + \cdots + \gamma_n^* h_n x_n^* \delta_n$$

where $n$ is a positive integer, $\alpha_j, \beta_j, \gamma_j, \delta_j$, for $1 \leq j \leq n$, are scalar matrices, and $g_j, h_j \in G$. A degree 1 matrix *-polynomial *with constant term* is an expression as above with an additional constant term. A *matrix $G$-*-polynomial* is a linear combination of expressions of the form $X_1 \cdots X_n$ where $X_j$, for $j = 1, \ldots, n$, is either a scalar matrix, or $gx$, or $hy^*$ for some variables $x, y$ and some group elements $g, h \in G$.

### 3.1. The ordered selfadjoint operator space language.

An *ordered selfadjoint operator space*, as defined in [9, 73], is a matricially normed and matricially ordered $*$-vector space that admits a selfadjoint completely isometric complete order embedding into a C*-algebra. Concretely, one can defined an ordered selfadjoint operator space as a selfadjoint closed subspace of $B(H)$ with the inherited matrical norms, matrical positive cones, and involution. Ordered selfadjoint operator spaces have been abstractly characterized in [73, 74, 81], and further studied in [9, 10, 60, 61, 69, 82]. For ordered operator spaces $X$ and $Y$, we denote by $\mathrm{CPC}(X, Y)$ the set of all selfadjoint completely positive completely contractive linear maps $X \to Y$. (Observe that in an ordered selfadjoint operator space the matrix positive cones are not necessarily spanning. Therefore a completely positive linear map on an ordered operator space is not necessarily selfadjoint.)

An ordered selfadjoint operator space $X$ can be naturally seen as a structure in the language $\mathcal{L}^{\mathrm{osos}}$ that contains

- sorts $M_n(X)$, with $n \in \mathbb{N}$, for (the unit balls of) the matrix amplifications of the space $X$;
- a sort for each finite-dimensional C*-algebra $F$;
- function symbols for the vector space operations and the involution in $X$ and $F$;
- predicate symbols for the norms in $M_n(X)$ and in $F$;
- predicate symbols for the distance function from the cone of positive elements in $M_n(X)$ and in $F$;
- predicate symbols for the function $F^k \times X^k \to \mathbb{R}$ given by

$$(\overline{y}, \overline{z}) \mapsto \inf_{t \in \mathrm{CPC}(F,X)} \max_{j=1,\ldots,k} \|t(y_j) - z_j\| .$$

We call such a language the *ordered selfadjoint operator space language* $\mathcal{L}^{\mathrm{osos}}$. Observe that the $\mathcal{L}^{\mathrm{osos}}$-terms can be seen as degree 1 matrix *-polynomials without constant terms. It is clear that a function between ordered selfadjoint operator spaces is an $\mathcal{L}^{\mathrm{osos}}$-morphism if and only if it is *selfadjoint*, *completely positive* and *completely contractive*, and it is an $\mathcal{L}^{\mathrm{osos}}$-embedding if and only if it is a *selfadjoint completely isometric complete order embedding*. In particular, any C*-algebra can be seen as an $\mathcal{L}^{\mathrm{osos}}$-structure in the obvious way, by considering its canonical matrix norms and matrix positive cones. It is observed in [53, Appendix C], [30, Section 3 and Section 5] that all the predicates above are definable in the usual language of C*-algebras as considered in [30, 33].

An operator system is a closed, selfadjoint subspace $X \subset A$ of a *unital* C*-algebra that contains its unit. The *operator system language* $\mathcal{L}^{\mathrm{osy}}$ is obtained from the ordered operator space language by adding a constant symbol for the unit in $X$. The $\mathcal{L}^{\mathrm{osy}}$-terms can be seen as degree 1 matrix *-polynomials with constant term.

### 3.2. The order zero language.

If $A, B$ are C*-algebras, we denote by $\mathrm{OZ}(A, B)$ the space of completely positive contractive order zero maps $A \to B$. The *order zero language* $\mathcal{L}^{\mathrm{oz}}$ for C*-algebras is obtained from $\mathcal{L}^{\mathrm{osos}}$ by adding, for any finite-dimensional C*-algebra $F$ and any $k \in \mathbb{N}$, a predicate symbol to be interpreted as the function $F^k \times A^k \to \mathbb{R}$, given by

$$(\overline{y}, \overline{z}) \mapsto \inf_{t \in \mathrm{OZ}(F,X)} \max_{j=1,\ldots,k} \|t(y_j) - z_j\| .$$

It is proved in [30, Section 5.2] that such functions are definable in the usual language of C*-algebras as considered in [30,33]. This follows from the structure theorem for completely positive contractive order zero maps [84, Corollary 4.1] and stability of the relations defining cones of finite-dimensional C*-algebras [67, Section 3.3].

A C*-algebras can be seen as an $\mathcal{L}^{\mathrm{oz}}$-structure in the obvious way. Let $A$ and $B$ be C*-algebras and let $f\colon A \to B$ be a function. Then $f$ is an $\mathcal{L}^{\mathrm{oz}}$-morphism if and only if $f$ is a completely positive contractive order zero map.

**Remark 3.1.** In the order-zero language one can express the fact that a pair $(a_1, a_2)$ of elements of a C*-algebra $A$ are (almost) orthogonal. Indeed one can consider the canonical basis elements $(e_1, e_2)$ of $\mathbb{C} \oplus \mathbb{C}$ and the formula $\varphi(e_1, e_2, x_1, x_2)$ defined by

$$\inf_{t \in \mathrm{OZ}(\mathbb{C}\oplus\mathbb{C}, A)} \max \left\{ \|t(e_1) - x_1\|, \|t(e_2) - x_2\| \right\}.$$

We have that if $\varphi(e_1, e_2, a_1, a_2) < \varepsilon$, then $\|a_1 a_2\| < 2\varepsilon$. Conversely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $a_1, a_2$ are positive contractions such that $\|a_1 a_2\| < \delta$, then $\varphi(e_1, e_2, a_1, a_2) < \varepsilon$.

3.3. **The C*-algebra language.** The *C*-algebra language* $\mathcal{L}^{\mathrm{C}^*}$ is obtained from $\mathcal{L}^{\mathrm{osos}}$ by adding a function symbol for the multiplication operation in $M_n(A)$, for every $n \in \mathbb{N}$. Similarly, the *unital C*-algebra language* $\mathcal{L}^{1,\mathrm{C}^*}$ is obtained from $\mathcal{L}^{\mathrm{C}^*}$ by adding a constant symbol for the unit. Observe that the terms in $\mathcal{L}^{1,\mathrm{C}^*}$ (respectively, $\mathcal{L}^{\mathrm{C}^*}$) can be canonically identified with matrix *-polynomials with (respectively, without) constant term. A function between C*-algebras is an $\mathcal{L}^{\mathrm{C}^*}$-morphism ($\mathcal{L}^{1,\mathrm{C}^*}$-morphism) if and only if it is a (unital) *-homomorphism, and an $\mathcal{L}^{\mathrm{C}^*}$-embedding ($\mathcal{L}^{1,\mathrm{C}^*}$-embedding) if and only if it is a (unital) injective *-homomorphism.

**Remark 3.2.** The following properties of C*-algebras have been proved to be definable by a uniform family of existential positive primitive $\mathcal{L}^{\mathrm{C}^*}$-formulas in [30, Theorem 2.5.1 and Theorem 5.7.3]: real rank zero, stable rank at most $n$, quasidiagonality, simplicity, being simple and purely infinite, being simple and TAF, being abelian of real rank at most $n$. Considering unital C*-algebras and positive existential $\mathcal{L}^{1,\mathrm{C}^*}$-formulas gives approximate divisibility.

**Remark 3.3.** The following properties of C*-algebras have been proved to be definable by a uniform family of existential positive primitive $\mathcal{L}^{\mathrm{C}^*}$-formulas among *separable* C*-algebras in [30, Theorem 2.5.1 and Theorem 5.7.3]: being UHF; being AF; being $D$-absorbing for a given strongly self-absorbing C*-algebra $D$; and being $\mathcal{K}$-absorbing—also called *stable*—where $\mathcal{K}$ is the algebra of compact operators.

3.4. **The nuclear languages.** The *nuclear ordered selfadjoint operator space language* $\mathcal{L}^{\text{osos-nuc}}$ is obtained from $\mathcal{L}^{\mathrm{osos}}$ by adding, for every $k \in \mathbb{N}$ and every finite-dimensional C*-algebra $F$, a predicate symbol for the function $X^k \times F^k \to \mathbb{R}$

$$(\overline{x}, \overline{y}) \mapsto \inf_{s \in \mathrm{CPC}(X,F)} \max_{j=1,\ldots,k} \|s(x_j) - y_j\|.$$

It is proved in [30, Section 5] that such a function is definable in the language of C*-algebras considered in [30,33].

In the proof of Lemma 3.4, we will need the following version of the Choi-Effros lifting theorem: if $A, B$ are C*-algebras, $A$ is separable, $f\colon A \to B$ is a nuclear completely positive contractive map, $E \subset A$ is a finite-dimensional subspace, and $\varepsilon > 0$, then there exists a completely positive contractive map $\eta\colon \overline{f(A)} \to A$ such that $f \circ \eta$ is the identity map on $\overline{f(A)}$, and $\|(\eta \circ f)(x) - x\| < \varepsilon \|x\|$ for all $x \in E$. When $A$, $B$, and $f$ are unital, this is a consequence of the Choi-Effros lifting theorem for operator systems; see [20, Lemma 3.8 and Section 4.3]. The general case can be reduced to the unital one by taking unitizations; see [14, Proposition 2.2.1 and Proposition 2.2.4].

**Lemma 3.4.** Let $A$ and $B$ be C*-algebras, and let $f\colon A \to B$ be a function. Consider the following assertions:

(1) $f$ is a nuclear completely positive contractive map;
(2) $f$ is an $\mathcal{L}^{\text{osos-nuc}}$-morphism;
(3) $f$ is a completely positive contractive map.

Then (1)⇒(2)⇒(3), and they are all equivalent if either $A$ or $B$ is nuclear.

*Proof.* The implication (1)⇒(2) uses the Choi-Effros lifting theorem as stated above. The proof is the same as the proof that nuclearity passes to quotients and that decomposition rank and nuclear dimension are nonincreasing under quotients; see [83, §2.9], [66, Section 3], and [85, Proposition 2.3]. The implication (2)⇒(3) is obvious. Finally, if either $A$ or $B$ are nuclear, then any completely positive contractive map $f\colon A \to B$ is nuclear, which gives (3)⇒(1). □

The *nuclear order zero language* $\mathcal{L}^{\text{oz-nuc}}$ and the *nuclear C*-algebra language* $\mathcal{L}^{\text{C*-nuc}}$, are defined as above starting from $\mathcal{L}^{\text{oz}}$ and $\mathcal{L}^{\text{C*}}$, respectively. It follows from Lemma 3.4 that any nuclear completely positive contractive order zero map (nuclear *-homomorphism) between C*-algebras is an $\mathcal{L}^{\text{oz-nuc}}$-morphism ($\mathcal{L}^{\text{C*-nuc}}$-morphism), and the converse holds for nuclear C*-algebras.

It is proved in [30, Section 5] that any predicate that is definable in $\mathcal{L}^{\text{C*-nuc}}$ is also definable in $\mathcal{L}^{\text{C*}}$. However, considering the larger language $\mathcal{L}^{\text{C*-nuc}}$ gives a more generous notion of (positive) existential formula.

**Remark 3.5.** The following properties of C*-algebras are definable by a uniform family of existential positive primitive $\mathcal{L}^{\text{C*-nuc}}$-formulas (see [30, Section 5]): nuclearity, having nuclear dimension at most $n$, and having decomposition rank at most $n$.

3.5. **Languages for $A$-bimodules.** Let $A$ and $B$ be $G$-C*-algebras. We say that $B$ is a *$G$-equivariant $A$-bimodule* if it is an $A$-bimodule with bimodule actions $(a,b)\mapsto a\cdot b$ and $(b,a)\mapsto b\cdot a$ satisfying $\|a\cdot b\|\le\|a\|\,\|b\|$, $\|b\cdot a\|\le\|b\|\,\|a\|$, $(g^A a)\cdot(g^B b)=g^B(a\cdot b)$, and $(g^B b)\cdot(g^A a)=g^B(b\cdot a)$ for all $a\in A$, $b\in B$ and $g\in G$. If $f\colon A\to B$ is a $G$-equivariant *-homomorphism, then it induces a canonical $G$-equivariant $A$-bimodule structure on $B$, defined by $a\cdot b:=f(a)b$ and $b\cdot a=bf(a)$ for $a\in A$ and $b\in B$.

If $B$ is a $G$-equivariant $A$-bimodule, then $\prod_{\mathcal{F}}^{G}B$ has a natural $G$-equivariant $A$-bimodule structure. We let $\mathcal{L}^{\text{C*},A\text{-}A}$ be the language obtained from $\mathcal{L}^{\text{C*}}$ by adding symbols for the $A$-bimodule structure. Similar definitions apply to the other languages for C*-algebras considered above. The interpretation of an $\mathcal{L}^{\text{C*},A\text{-}A}$-formula on a $G$-equivariant $A$-bimodule is defined in the obvious way.

3.6. **The Kirchberg language.** Fix a C*-algebra $A$. In this subsection, we define a language $\mathcal{L}^{K}(A)$, which we call the *Kirchberg language*, that fits into the more flexible setting described in Subsection 2.4. This language is obtained from $\mathcal{L}^{\text{C*}}$ by replacing the symbols for the matrix norms with pseudometric symbols $d_F$ for every finite set $F$ in the unit ball of $A$. The distinguished collection $t_A^c(x)$ of positive quantifier-free conditions that is part of the language $\mathcal{L}^{K}(A)$ consists of the conditions $\max_{a\in F}\|ax-xa\|=0$ for every finite set $F$ of elements in the unit ball of $A$.

One can regard $A$ as an $\mathcal{L}^{K}(A)$-structure by interpreting $d_F$ on $M_n(A)$ as the pseudometric

$$(x,y)\mapsto\max_{a\in M_n(F)}\|a(x-y)\|.$$

Suppose that $\mathcal{U}$ is an ultrafilter. Then the reduced power of $A$ as an $\mathcal{L}^{K}(A)$-structure is equal to the Kirchberg invariant $F_{\mathcal{U}}(A)$ as introduced by Kirchberg in [65]; see also [2]. Considering reduced powers instead of ultrapowers yields the generalization of the Kirchberg invariant to arbitrary filters considered in [4, 75]. In the following, we denote by $t_A^c(x_1,\ldots,x_n)$ the type $t_A^c(x_1)\cup\cdots\cup t_A^c(x_n)$. If $A$ is *unital*, then $F_{\mathcal{F}}(A)$ is equal to $A'\cap\prod_{\mathcal{F}}A$.

Let $\kappa$ be an infinite cardinal that is larger than the density character of $A$, and let $\mathcal{F}$ be a countably incomplete $\kappa$-good filter. (When $A$ is separable, one can take any countably incomplete ultrafilter.) Considering an approximate unit for $A$ shows that $F_{\mathcal{F}}(A)$ is unital. Let $t(x_1,\ldots,x_n)$ be a positive quantifier free $\mathcal{L}^{1,\text{C*}}$-type. The corresponding *multiplier $\mathcal{L}^{K}(A)$-type* $t_A^m(x_1,\ldots,x_n)$ is defined as follows. Any condition in $t(\overline{x})$ should be replaced with all the conditions obtained by substituting every occurrence of a basic formula of the form $\|\mathfrak{p}(\overline{x})\|$, for some *-polynomial $\mathfrak{p}$ with constant term, with the basic formula $\|b\,\mathfrak{p}(\overline{x})\|$, where $b$ is an arbitrary element of the unit ball of $A$.

**Remark 3.6.** It follows from Remark 2.24 that the following statements are equivalent:

(1) $t(\overline{x})$ is realized in $F_{\mathcal{F}}(A)$,
(2) $t^m(\overline{x})$ is realized in $F_{\mathcal{F}}(A)$,
(3) $t^m(\overline{x})$ is approximately realized in $F_{\mathcal{F}}(A)$,
(4) $t^m(\overline{x})\cup t_A^c(\overline{x})$ is approximately realized in $A$.

Furthermore $F_{\mathcal{F}}(A)$ is positively quantifier-free $\mathcal{L}^{1,\text{C*}}$-$\kappa$-saturated. When $\mathcal{U}$ is an ultrafilter, $F_{\mathcal{U}}(A)$ is quantifier-free $\mathcal{L}^{1,\text{C*}}$-$\kappa$-saturated.

Various results from [65] can be seen as consequences of Remark 3.6.

Suppose now that $A$ is a $G$-C*-algebra. Then one can consider $A$ as an $\mathcal{L}_G^{K}(A)$-structure. In this case, the reduced power of $A$ as an $\mathcal{L}_G^{K}(A)$-structure with respect to a filter $\mathcal{F}$—which we denote by $F_{\mathcal{F}}^{G}(A)$—recovers the equivariant version of the Kirchberg invariant considered in [4,75]. Again, the following proposition follows from the general remarks of Subsection 2.4. If $t(\overline{x})$ is a positive quantifier free $\mathcal{L}_G^{1,\mathrm{C}^*}$-type, then the corresponding multiplier $\mathcal{L}_G^K(A)$-type $t_A^m(\overline{x})$ can be defined as above.

**Proposition 3.7.** *Suppose that $A$ is a $G$-C*-algebra, $\kappa$ is a cardinal larger than the density character of $A$, $\mathcal{F}$ is a countably incomplete $\kappa$-good filter, and $t(\overline{x})$ is a positive quantifier-free $\mathcal{L}_G^{1,C^*}$-type. Then $F_{\mathcal{F}}^G(A)$ is a unital $G$-C*-algebra, and the following statements are equivalent:*

1. *$t(\overline{x})$ is realized in $F_{\mathcal{F}}^G(A)$,*
2. *$t^m(\overline{x})$ is realized in $F_{\mathcal{F}}^G(A)$*
3. *$t^m(\overline{x})$ is approximately realized in $F_{\mathcal{F}}^G(A)$,*
4. *$t(\overline{x}) \cup t_A^c(\overline{x})$ is approximately realized in $A$.*

*Furthermore $F_{\mathcal{F}}(A)$ is positively quantifier-free $\mathcal{L}_G^{1,C^*}$-$\kappa$-saturated.*

Similar conclusions hold if one replaces filters with ultrafilters, and positive quantifier free types with arbitrary quantifier free types.

Suppose now that $B$ is another C*-algebra, and $f : A \to B$ is a $G$-equivariant *-homomorphism. As discussed in Subsection 3.5, this defines a $G$-equivariant $A$-bimodule structure on $B$. We can also regard $B$ as a structure in the Kirchberg language $\mathcal{L}_G^K(A)$. One can prove in this setting the analog of Proposition 3.7 where $F_{\mathcal{F}}^G(A)$ is replaced with the reduced product of $B$ regarded as an $\mathcal{L}_G^K(A)$-structure, and where $t(\overline{x})$ is a positive quantifier-free $\mathcal{L}_G^{\mathrm{C}^*,A\text{-}A}$-type.

## 4. Strongly self-absorbing $G$-C*-algebras

In this section, we exhibit some applications of the framework described in Section 2 to strongly self-absorbing actions on C*-algebras, as introduced and studied in [75, 78].

4.1. **$G$-C*-algebras.** In this section, we regard $G$-C*-algebras as structures in the language of $G$-C*-algebras $\mathcal{L}_G^{\mathrm{C}^*}$. An $\mathcal{L}_G^{\mathrm{C}^*}$-morphism between $G$-C*-algebras is a $G$-equivariant *-homomorphism, and an $\mathcal{L}_G^{\mathrm{C}^*}$-embedding is an injective $G$-equivariant *-homomorphism. If $A$ and $B$ are $G$-C*-algebras, then we denote by $A \otimes B$ the *minimal* tensor product of $A$ and $B$ endowed with the continuous $G$-action defined by $g^{A\otimes B}(a \otimes b) = (g^A a) \otimes (g^B b)$. We also let $A^G$ be the *fixed point subalgebra*

$$A^G = \left\{ a \in A \colon g^A a = a \text{ for every } g \in G \right\}.$$

Definition 2.3 yields, in particular, the notion of reduced $G$-product and $G$-ultraproduct of $G$-C*-algebras. The argument from [54, Lemma 1.8] shows that when $\mathcal{F}$ is the filter of cofinite subsets of $\mathbb{N}$ (or, more generally, a countably generated filter), then the reduced $G$-product as in Definition 2.3 coincides with the continuous part of the central sequence algebra as considered in the C*-algebra literature; see for example [4, Lemma 3.4]. It remains unclear whether this also holds under weaker assumptions on $\mathcal{F}$.

4.2. **Ultrapowers of $G$-C*-algebras.** We point out here that, given a $G$-C*-algebra $A$ and a nonprincipal ultrafilter $\mathcal{U}$ over a set $I$, then the $G$-ultrapower $\prod_{\mathcal{U}}^G A$ is in general different from the ultrapower $\prod_{\mathcal{U}} A$ of $A$ as a C*-algebra. For instance, we have the following, which is a particular instance of the general Proposition 2.9. We present a direct proof of this particular case, modeled after the proof of the Arzelá-Ascoli theorem.

**Proposition 4.1.** *Let $G$ be a compact second countable group, and let $C(G)$ be the C*-algebra of continuous complex-valued functions on $G$, endowed with the canonical* (left) translation action*. Fix an ultrafilter $\mathcal{U}$ over an index set $I$. Then the $G$-ultrapower $\prod_{\mathcal{U}}^G C(G)$ is equal to $C(G)$.*

*Proof.* Fix an element $\boldsymbol{a} \in \prod_{\mathcal{U}}^G C(G)$ with representative $(a_i)_{i\in I}$. For every $x \in G$, set $a(x) := \lim_{i\to\mathcal{U}} a_i(x)$. We claim that $\lim_{i\to\mathcal{U}} \sup_{x\in G} |a_i(x) - a(x)| = 0$. To see this, fix $\varepsilon > 0$. Since $\boldsymbol{a}$ belongs to $\prod_{\mathcal{U}}^G C(G)$, there exist a neighborhood $V$ of the identity in $G$ and an element $J$ of $\mathcal{U}$ such that

$$\sup_{x\in G} \sup_{g\in V} |a_n(gx) - a_i(x)| \leq \varepsilon$$

for every $i \in J$. Therefore, by compactness of $G$, there exists a finite subset $F$ of $G$ such that

$$\sup_{x\in G} |a_i(x) - a_j(x)| \leq 2\varepsilon + \max_{x\in F} |a_i(x) - a_j(x)|$$

for every $i, j \in J$. Observe that

$$\{i \in I \colon \max_{x \in F} |a_i(x) - a(x)| \leq \varepsilon\} \in \mathcal{U}.$$

Therefore there exists a subset $J'$ of $J$, that belongs to $\mathcal{U}$, such that $\sup_{x \in G} |a_i(x) - a_j(x)| \leq 4\varepsilon$ for $i, j \in J'$.

Fix $x \in G$ and $j \in J'$. Taking the limit for $i \to \mathcal{U}$ shows that $|a_i(x) - a(x)| \leq 4\varepsilon$. Since this holds for every $x \in G$ and $i \in J'$, we deduce that $\lim_{i \to \mathcal{U}} \sup_{x \in G} |a_i(x) - a(x)| \leq 4\varepsilon$. Since this is true for every $\varepsilon > 0$, we conclude that $\boldsymbol{a} = a \in C(G)$. This concludes the proof that $\prod_{\mathcal{U}}^{G} C(G) = C(G)$. ☐

**Example 4.2.** The inclusion of $C(G)$ in the (nonequivariant) C*-algebra ultrapower $\prod_{\mathcal{U}} C(G)$ is in general strict. For example, if $G = \mathbb{T}$ and $u \in C(\mathbb{T})$ is the canonical unitary generator, then the element $[u^n]$ of $\prod_{\mathcal{U}} C(\mathbb{T})$ with representative sequence $(u^n)_{n \in \mathbb{N}}$ does not belong to $C(\mathbb{T})$. Indeed, the canonical action of $\mathbb{T}$ on $\prod_{\mathcal{U}} C(\mathbb{T})$ is not continuous at $[u^n]_{n \in \mathbb{N}}$. It follows that $C(\mathbb{T}) = \prod_{\mathcal{U}}^{\mathbb{T}} C(\mathbb{T})$ is properly contained in $\prod_{\mathcal{U}} C(\mathbb{T})$.

4.3. **Positively $\mathcal{L}_G^{\mathrm{C}^*}$-existential injective *-homomorphisms.** An injective *-homomorphism $\theta \colon A \to M$ between *separable* $G$-C*-algebras is $G$-equivariantly sequentially split, in the sense of [4, Definition 3.3], if and only if it is positively $\mathcal{L}_G^{\mathrm{C}^*}$-existential, as defined in Subsubsection 2.2.2. For arbitrary $G$-C*-algebras, the notion of positively $\mathcal{L}_G^{\mathrm{C}^*}$-existential injective *-homomorphism is more generous than being $G$-equivariantly sequentially split.

Theorem 2.20, in the particular case of $G$-C*-algebras, recovers [4, Lemma 3.6 and Corollary 3.7]. Lemma 2.3, Corollary 2.4, Proposition 2.5, Proposition 2.9, Proposition 3.8 and Corollary 3.17 of [4] are then an immediate consequence of the definition of positively $\mathcal{L}_G^{\mathrm{C}^*}$-existential injective *-homomorphism; see Proposition 2.15. Proposition 3.11 of [4] is a particular instance of Remark 2.22, observing that when the group $G$ is compact, the fixed point algebra $A^G$ of a $G$-C*-algebra is positively existentially definable. By appropriately choosing the functor, one can also see that Proposition 2.21 has as particular instances the following results from [4]: (I), (II), (IV) of Theorem 2.10, Proposition 3.9, Proposition 3.12, Corollary 3.14, Corollary 3.15, Proposition 3.16.

It follows from Proposition 2.14 that if $A, B$ are C*-algebras and $f \colon A \to B$ is a positively $\mathcal{L}^{\mathrm{C}^*}$-existential injective *-homomorphism, then $A$ has any of the properties listed in Remark 3.2 or Remark 3.3, whenever $B$ does. The same assertion holds for any of the properties listed in Remark 3.5 when $B$ is nuclear. In particular, this observation recovers (1), (2), (3), (4), (5), (7), (11), the first part of (12), the first half of (14), and (16) of [4, Theorem 2.11]. Other preservation results have been obtained in [42, 43, 77].

4.4. **Commutant existential theories.** Suppose that $A, B$ are $G$-C*-algebras. We say that $A$ is *commutant positively weakly contained* in $B$ if for some (equivalently, any) cardinal $\kappa$ larger than the density character of $A$ and $B$ and for some (equivalently, any) countably incomplete $\kappa$-good filter $\mathcal{F}$ on has that every $\mathcal{L}_G^{1,\mathrm{C}^*}$-type $t$ that is realized in $F_{\mathcal{F}}^{G}(A)$ is also realized in $F_{\mathcal{F}}^{G}(B)$. Equivalently, for any unital $G$-C*-subalgebra $C$ of $F_{\mathcal{F}}(A)$ of density character less than $\kappa$ there exists a $G$-equivariant injective unital *-homomorphism from $C$ to $F_{\mathcal{F}}^{G}(B)$. If $A$ is unital, then $A$ is commutant positively weakly contained in $B$ if and only if there exists a unital *-homomorphism from $A$ to $F_{\mathcal{F}}^{G}(B)$ for any filter $\mathcal{F}$ as above. A syntactic characterization of commutant positively weak containment can be obtained using Proposition 3.7.

Suppose that $A$ is a $G$-C*-algebra, and let $t_A^c(\overline{x})$ be the collection of conditions $\max_{j=1,\ldots,n} \|y_j a - a y_j\| \leq 0$, for $a \in A$. Recall that if $t(\overline{x})$ is a positive quantifier-free $\mathcal{L}_G^{1,\mathrm{C}^*}$-type, then $t_A^m(\overline{x})$ denotes the positive quantifier-free $\mathcal{L}_G^{K}(A)$-type obtained from $t(\overline{x})$ by replacing every occurrence of $\|\mathfrak{p}(\overline{x})\|$ for some $G$-*-polynomial $\mathfrak{p}$ with $\|b\mathfrak{p}(\overline{x})\|$ where $b$ is an arbitrary element of $A$ of norm at most 1. We then have that $A$ is commutant positively weakly contained in $B$ if and only if, for any positive quantifier-free $\mathcal{L}_G^{1,\mathrm{C}^*}$-type $t(\overline{x})$, $t_A^m(\overline{x}) \cup t_A^c(\overline{x})$ is approximately satisfied in $A$ if and only if $t_B^m(\overline{x}) \cup t_B^c(\overline{x})$ is approximately satisfied in $B$.

Two C*-algebras are *commutant positively weakly equivalent* if they are one commutant positively weakly contained in the other. The *commutant positive existential theory* of a C*-algebra is its commutant positively weak equivalence class. The space of commutant positive existential theories has a natural compact metrizable topology, with basis given by sets of the form $\mathrm{Omit}^c(t)$, where $t(\overline{x})$ is a quantifier-free $\mathcal{L}_G^{1,\mathrm{C}^*}$-type and $\mathrm{Omit}^c(t)$ is the set of commutant existential theories of $G$-C*-algebras $A$ such that $t_A^m(\overline{x}) \cup t_A^c(\overline{x})$ is *not* approximately realized in $A$. For unital nuclear $G$-C*-algebras, the following characterization of commutant weak $\mathcal{L}_G^{1,\mathrm{C}^*}$-containment follows from the Choi-Effros lifting theorem.

**Proposition 4.3.** *Suppose that $A$ is a unital nuclear $G$-C*-algebra, and $B$ is a $G$-C*-algebra. Then $A$ is commutant positively weakly contained in $B$ if and only if for any separable nuclear $G$-invariant unital C*-subalgebra $A_0 \subset A$ and separable $G$-C*-subalgebra $B_0 \subset B$, there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of completely positive contractive maps $\phi_n\colon A_0 \to B$ such that, for every compact subset $K \subset G$, for every $x, y \in A_0$, and for every $b \in B_0$, we have*

$$\lim_{n\to+\infty} \max\left\{ \|b(\phi(x)\phi(y) - \phi(xy))\|, \max_{g\in K} \left\|b(\phi(g^A x) - g^B\phi(x))\right\|, \|\phi(x)b - b\phi(x)\|, \|b\phi(1) - b\|\right\} = 0.$$

The notions of *commutant weak containment* and commutant existential theory are defined analogously, considering arbitrary (not necessarily positive) quantifier-free $\mathcal{L}_G^{1,\mathrm{C}^*}$-types.

4.5. **Space of separable nuclear $G$-C*-algebras and smooth classification.** We now observe that, for a second countable locally compact group $G$, there exists a natural standard Borel space of separable *nuclear* $G$-C*-algebras. Indeed, by Kirchberg's nuclear embedding theorem [71, Theorem 6.3.12], any separable nuclear C*-algebra is *-isomorphic to the range of a conditional expectation on $\mathcal{O}_2$. Given such a conditional expectation $E$, set $A = E(\mathcal{O}_2)$. Write $\mathrm{CPC}(\mathcal{O}_2)$ for the semigroup of all completely positive contractive maps of $\mathcal{O}_2$ into itself, endowed with the topology of pointwise convergence in norm. Given an action $\alpha\colon G \to \mathrm{Aut}(A)$, one can define a continuous function $\rho_\alpha\colon G \to \mathrm{CPC}(\mathcal{O}_2)$ by $\rho_\alpha(g) := \alpha_g \circ E$. Then

(1) $\rho_\alpha(gh) = \rho_\alpha(g) \circ \rho_\alpha(h)$ for every $g, h \in G$, and
(2) $\rho_\alpha(1) = E$.

Conversely, any pair $(E, \rho)$, consisting of a conditional expectation $E\colon \mathcal{O}_2 \to \mathcal{O}_2$ and a continuous function $\rho\colon G \to \mathrm{CPC}(\mathcal{O}_2)$ satisfying (1) and (2), arises from a continuous action of $G$ on the range of $E$, as described above.

Observe that $\mathrm{CPC}(\mathcal{O}_2)$ is a Polish space when endowed with the topology of pointwise convergence. Similarly, the space $\mathrm{Exp}(\mathcal{O}_2)$ of conditional expectations defined on $\mathcal{O}_2$ is a Polish space when endowed with the topology of pointwise convergence. (This can be seen for instance by observing that conditional expectations onto a given C*-subalgebra are precisely the idempotent maps of norm 1 mapping onto that C*-subalgebra [8, Theorem II.6.10.2].)

The space $G$-$\mathrm{C}^*\mathrm{ALG}$ of pairs $(E, \rho)$ arising from a continuous action of $G$ on the image of $E$, is a $G_\delta$ subset of the space $\mathrm{Exp}(\mathcal{O}_2) \times \mathrm{CPC}(\mathcal{O}_2)$, hence a Polish space with the induced topology; see [63, Theorem 3.11]. We will regard $G$-$\mathrm{C}^*\mathrm{ALG}$ as the Polish space of separable nuclear $G$-C*-algebras. For an element $(E, \rho) \in G$-$\mathrm{C}^*\mathrm{ALG}$, we write $C^*(E, \rho)$ for the associated $G$-C*-algebra.

It is easy to see, by induction on the complexity, that any $\mathcal{L}_G^{\mathrm{C}^*}$-formula $\varphi(x_1, \ldots, x_n, \gamma_1, \ldots, \gamma_m)$ induces a Borel map $\hat{\varphi}\colon G\text{-}\mathrm{C}^*\mathrm{ALG} \times \mathcal{O}_2^n \times G^m \to \mathbb{R}$ given by

$$((E, \rho), (a_1, \ldots, a_n), (g_1, \ldots, g_m)) \mapsto \varphi^{C^*(E,\rho)}(E(a_1), \ldots, E(a_n), g_1, \ldots, g_m).$$

In other words, the $\mathcal{L}_G^{\mathrm{C}^*}$-theory of a separable nuclear $G$-C*-algebra can be computed in a Borel fashion in the parameterization $G$-$\mathrm{C}^*\mathrm{ALG}$ of $G$-C*-algebras. In particular, the canonical map sending a separable $G$-C*-algebra to its existential theory is a Borel map. This allows one to conclude the following.

**Theorem 4.4.** *Let $G$ be any locally compact second countable group. Then separable nuclear $G$-C*-algebras are smoothly classifiable, in the sense of Borel complexity theory, up to weak $\mathcal{L}_G^{C^*}$-equivalence and positive weak $\mathcal{L}_G^{C^*}$-equivalence.*

An introduction to the theory of Borel complexity of equivalence relations can be found in [38]. Similar conclusions hold for unital C*-algebras and (positive) weak $\mathcal{L}_G^{1,\mathrm{C}^*}$-equivalence.

4.6. **Strongly self-absorbing $G$-C*-algebras.** Let us recall some notions and terminology from [75]. Suppose that $A$ and $B$ are $G$-C*-algebras and $\eta_1, \eta_2\colon A \to B$ are unital $G$-*-homomorphisms. By $\mathcal{M}(A)$ we denote the multiplier algebra of $A$, which is endowed with a canonical *strictly continuous* $G$-action. Then $\eta_1$ and $\eta_2$ are said to be *approximately $G$-unitarily equivalent* if there exists a net $(u_i)_{i\in I}$ of unitaries in $\mathcal{M}(B)$ such that $(\mathrm{Ad}(u_i) \circ \eta_1)_{i\in I}$ converges pointwise to $\eta_2$, and $(g^B u_i - u_i)_{i\in I}$ converges strictly to 0, uniformly over compact subsets of $G$. Similarly, we say that $\eta_1$ and $\eta_2$ are *$G$-unitarily equivalent* if there exists a unitary element $u$ in $\mathcal{M}(B)$ such that $\mathrm{Ad}(u)\circ\eta_1 = \eta_2$ and $g^B u = u$ for every $g \in G$.

The $G$-C*-algebras $A$ and $B$ are said to be *cocycle conjugate* if there exist a *-isomorphism $\eta\colon A \to B$ and a strictly continuous map $v\colon G \to U(\mathcal{M}(A))$ such that $\eta(g^A a) = v_g g^B \eta(a) v_g^*$ for every $g \in G$ and every $a \in A$. If moreover one can find a sequence of unitaries $(w_n)$ such that $\left(w_n\left(g^A w_n\right)^*\right)$ converges strictly to $v_g$ for $g \in G$ uniformly over compact sets then the $G$-C*-algebras $A$ and $B$ are *strongly cocycle conjugate.* Finally, if $A$ and $D$ are $G$-C*-algebras, we say that $A$ is *$G$-equivariantly $D$-absorbing* if $A \otimes D$ is cocycle conjugate to $A$.

**Definition 4.5.** A $G$-C*-algebra $D$ is said to have *approximately $G$-inner half-flip,* if it is unital and the canonical $G$-equivariant injective unital *-homomorphisms $\mathrm{id}_D \otimes 1_D, 1_D \otimes \mathrm{id}_D \colon D \to D \otimes D$ are approximately $G$-unitarily equivalent. A $G$-C*-algebra $D$ is said to be a *strongly self absorbing* $G$-C*-algebra if it is unital and $\mathrm{id}_D \otimes 1_D$ is approximately $G$-unitarily equivalent to a $G$-equivariant *-isomorphism. Finally, $G$ is *semi-strongly self-absorbing* if it is strongly cocycle conjugate to a strongly self-absorbing $G$-C*-algebra.

Observe that if $D$ has approximately $G$-inner half-flip, then it has approximately inner half-flip as a C*-algebra. Similarly, if $D$ is a semi-strongly self-absorbing $G$-C*-algebra, then $D$ is strongly self-absorbing as a C*-algebra. Recall that any unital C*-algebra $D$ with approximately inner half-flip is automatically simple, nuclear, and monotracial; see [26]. The proof of the following theorem follows closely arguments from [31]. Proposition 4.8 [75] shows that cocycle conjugacy and strong cocycle conjugacy coincide when the group $G$ is compact. It follows that in this case strongly self-absorbing $G$-C*-algebra coincide with semi-strongly self-absorbing $G$-C*-algebras.

**Theorem 4.6.** *Suppose that $D$ is a separable $G$-C*-algebra with approximately $G$-inner half-flip, and that $C$ is a countably positively quantifier-free $\mathcal{L}_G^{1,\mathrm{C}^*}$-saturated unital $G$-C*-algebra. Suppose that $D$ is commutant weakly contained in $C$. Fix a $G$-equivariant unital *-homomorphism $\theta \colon D \to C$. The following statements hold:*

1. *Any two $G$-equivariant unital *-homomorphisms $D \to C$ are $G$-unitarily equivalent;*
2. *The inclusion $\theta(D)' \cap C \hookrightarrow C$ is an $\mathcal{L}_G^{1,\mathrm{C}^*}$-existential $G$-equivariant unital *-homomorphism;*
3. *$\theta(D)' \cap C$ is an elementary $\mathcal{L}_G^{1,\mathrm{C}^*}$-substructure of $C$;*
4. *If $C$ has density character $\aleph_1$, then the inclusion $\theta(D)' \cap C \hookrightarrow C$ is approximately $G$-unitarily equivalent to a $G$-isomorphism.*

*Proof.* Let $\theta_1 \colon D \to C$ be a $G$-equivariant unital *-homomorphism. Assume first that the ranges of $\theta$ and $\theta_1$ commute. Choose a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in $D \otimes D$ witnessing the fact that $\mathrm{id}_D \otimes 1_D, 1_D \otimes \mathrm{id}_D \colon D \to D \otimes D$ are approximately $G$-unitarily equivalent. Let $\Theta \colon D \otimes D \to C$ be the $G$-equivariant unital *-homomorphism given by $d_1 \otimes d_2 \mapsto \theta(d_1)\theta_1(d_2)$. Considering the unitaries $\Theta(u_n)$, for $n \in \mathbb{N}$, and applying the fact that $A$ is countably positively quantifier-free $\mathcal{L}_G^{1,\mathrm{C}^*}$-saturated, we obtain a unitary $u \in C^G$ satisfying $\mathrm{Ad}(u) \circ \theta = \theta_1$. In the general case, when the ranges of $\theta$ and $\theta_1$ do not necessarily commute, we may find a unital $G$-equivariant *-homomorphism $\theta_2 \colon D \to C$ whose range commutes with those of $\theta$ and $\theta_1$. By the argument above, it follows that $\theta_2$ is $G$-unitarily to both $\theta$ and $\theta_1$, so (1) follows.

We prove (2) and (3) simultaneously. Let us identify $D$ with its image under $\theta$. Suppose that $\overline{a}$ is a tuple in $D' \cap C$, $\overline{b}$ is a tuple in $C$, and $\varphi(\overline{x}, \overline{y})$ is an $\mathcal{L}_G^{1,\mathrm{C}^*}$-formula. Let $B$ be the $G$-C*-algebra generated by $D \cup \{\overline{a}, \overline{b}\}$ inside $C$. Observe that $B' \cap C$ satisfies the same assumptions as $C$. Particularly, by (1) there exists a unitary $u \in B' \cap C$ such that $g^C u = u$ for every $g \in G$ and $u^* \overline{b} u \in D' \cap C$. Hence we have $\varphi^C(\overline{a}, \overline{b}) = \varphi^{D' \cap C}(\overline{a}, u^* \overline{b} u)$, as desired.

The argument above shows that for any separable $G$-C*-subalgebra $B$ of $C$ and finite tuple $\overline{b}$ in $C$ there exists a unitary $u$ in the fixed point algebra of $C$ such that $u \in B' \cap C$ and $u^* \overline{b} u \in D' \cap C$. One can then apply the intertwining argument of [31, Theorem 2.11] to get (4). ☐

**Corollary 4.7.** Suppose that $D$ is a separable $G$-C*-algebra with approximately $G$-inner half-flip, and $\mathcal{F}$ is a countably incomplete filter. Let $A$ be a separable unital $G$-C*-algebra, and let $\theta \colon D \to \prod_{\mathcal{F}}^G A$ be a $G$-equivariant unital *-homomorphism. Then:

1. Any $G$-equivariant unital *-homomorphism $D \to \prod_{\mathcal{F}}^G A$ is $G$-unitarily equivalent to $\theta$;
2. The inclusion $\theta(D)' \cap \prod_{\mathcal{F}}^G A \hookrightarrow \prod_{\mathcal{F}}^G A$ is an $\mathcal{L}_G^{1,\mathrm{C}^*}$-existential $G$-equivariant unital *-homomorphism;
3. $\theta(D)' \cap \prod_{\mathcal{F}}^G A$ is an elementary $\mathcal{L}_G^{1,\mathrm{C}^*}$-substructure of $\prod_{\mathcal{F}}^G A$;
4. If $\mathcal{F}$ is a filter over $\mathbb{N}$ and the Continuum Hypothesis holds, then $\theta(D)' \cap \prod_{\mathcal{F}}^G A$ is $G$-equivariantly *-isomorphic to $\prod_{\mathcal{F}}^G A$.

**Remark 4.8.** Theorem 4.6 and Corollary 4.7 generalize [31, Theorem 1, Theorem 2, Corollary 2.12] in two ways: they extend them to the $G$-equivariant setting, and they remove the unnecessary assumption on the filter $\mathcal{F}$ that the corresponding reduced product be countably saturated. An example of a countable incomplete filter over $\mathbb{N}$ that does not satisfy such an assumption is provided in [31, Example 3.2].

Suppose now that $D$ is a semi-strongly self-absorbing $G$-C*-algebra. Observe that for any separable $G$-C*-algebra $A$ and any countably incomplete filter $\mathcal{F}$, the following assertions are equivalent:

(1) $D$ is commutant weakly contained in $A$
(2) $D$ is positively commutant weakly contained in $A$,
(3) $D$ embeds equivariantly into $F_{\mathcal{F}}^{G}(A)$.

If furthermore $A$ is unital, then by [75, Theorem 4.7] these conditions are in turn equivalent to:

(4) $A$ is $G$-equivariantly $D$-absorbing;
(5) $A$ and $A \otimes D$ are strongly cocycle conjugate;
(6) $D$ is weakly $\mathcal{L}_G^{1,\mathrm{C}^*}$-contained in $A$;
(7) $D$ is positively weakly $\mathcal{L}_G^{1,\mathrm{C}^*}$-contained in $A$.

We deduce the following rigidity result for semi-strongly self-absorbing $G$-C*-algebras.

**Proposition 4.9.** *Let $D$ and $E$ be semi-strongly self-absorbing $G$-C*-algebras. The following assertions are equivalent:*

(1) *$D$ and $E$ are cocycle conjugate;*
(2) *$D$ and $E$ are strongly cocycle conjugate;*
(3) *$D$ and $E$ are weakly $\mathcal{L}_G^{1,\mathrm{C}^*}$-equivalent;*
(4) *$D$ and $E$ are isomorphic as C*-algebras to the same strongly self-absorbing C*-algebra $B$, and the $\mathrm{Aut}(B)$-orbits of $D$ and $E$ inside the Polish space $\mathrm{Act}_G(B)$ of continuous actions of $G$ on $B$ have the same closure.*

*In particular, the classification problem for semi-strongly self-absorbing $G$-C*-algebras up to (strong) cocycle conjugacy is smooth.*

Proposition 4.9 can be seen as the equivariant analogue of [31, Theorem 2.16, Corollary 2.17]. We would like to remark, however, that Proposition 4.9 is in principle somewhat more surprising than its nonequivariant counterpart. Indeed, while there are only very few known strongly self-absorbing C*-algebras (and it is indeed currently known to be complete under additional regularity assumptions on the algebra), there seem to exist a greater variety of semi-strongly self-absorbing actions on C*-algebras. For instance, for a fixed group $G$ and a fixed strongly self-absorbing C*-algebra $D$, there may exist multiple (non cocycle equivalent) semi-strongly self-absorbing actions on $D$. In fact, a complete list of all semi-strongly self-absorbing actions is at the moment far out of reach.

The following consequence of Corollary 4.7 seems worth being isolated.

**Corollary 4.10.** Let $D$ be a semi-strongly self-absorbing $G$-C*-algebra, let $A$ be a separable unital $G$-equivariantly $D$-absorbing $G$-C*-algebra, let $\mathcal{F}$ be a countably incomplete filter, and let $\theta\colon D \to \prod_{\mathcal{F}}^{G} A$ be a $\mathcal{L}_G^{1,\mathrm{C}^*}$-embedding. Then:

(1) Any two $G$-equivariant unital *-homomorphisms of $D$ into $\prod_{\mathcal{F}}^{G} A$ are $G$-unitarily equivalent;
(2) The inclusion $\theta(D)' \cap \prod_{\mathcal{F}}^{G} A \hookrightarrow \prod_{\mathcal{F}}^{G} A$ is an $\mathcal{L}_G^{1,\mathrm{C}^*}$-existential $G$-equivariant unital *-homomorphism;
(3) $\theta(D)' \cap \prod_{\mathcal{F}}^{G} A$ is a $G$-elementary substructure of $\prod_{\mathcal{F}}^{G} A$;
(4) If $\mathcal{F}$ is a filter over $\mathbb{N}$ and the Continuum Hypothesis holds, then $\theta(D)' \cap \prod_{\mathcal{F}}^{G} A$ is $G$-equivariantly *-isomorphic to $\prod_{\mathcal{F}}^{G} A$.

Using the results above, one can provide the following model-theoretic characterization of semi-strongly self-absorbing $G$-C*-algebras, which in the nonequivariant setting is [31, Theorem 2.14]. (Recall that, then the group $G$ is compact, the notions of semi-strongly self-absorbing $G$-C*-algebra and strongly self-absorbing $G$-C*-algebra coincide.)

**Theorem 4.11.** *Let $D$ be a separable unital $G$-C*-algebra, and let $\mathcal{F}$ be a countably incomplete filter. Then $D$ is a semi-strongly self absorbing $G$-C*-algebra if and only if $D$ is weakly $\mathcal{L}_G^{1,\mathrm{C}^*}$-equivalent to $D \otimes D$, and all the $G$-equivariant unital *-homomorphisms $D \to \prod_{\mathcal{F}}^{G} D$ are $G$-unitarily equivalent.*

*Proof.* The "only if" implication is a consequence of the fact that $D$ is $G$-strongly cocycle conjugate to $D \otimes D$, and part (1) of Theorem 4.6. We prove the converse. Since $D$ is weakly $\mathcal{L}_G^{1,\mathrm{C}^*}$-equivalent to $D \otimes D$, we deduce that $D \otimes D$ is a $G$-elementary substructure of $\prod_{\mathcal{F}}^G D$, say via an embedding $\rho$. In particular, the $G$-equivariant unital *-homomorphisms $\rho_1, \rho_2 \colon D \to \prod_{\mathcal{F}}^G D$, given by $\rho_1(d) = \rho(d \otimes 1_D)$ and $\rho_2(d) = \rho(1_D \otimes d)$, for $d \in D$, are $G$-unitarily equivalent. It follows that $D$ has approximately $G$-inner half-flip. The conclusion now follows from the implication (ii)⇒(i) in [75, Theorem 4.6]. ☐

4.7. **Limiting examples.** We have shown in Proposition 4.9 that, for any second countable locally compact group $G$, the classification problem for semi-strongly self-absorbing $G$-actions on C*-algebras is smooth in the sense of Borel complexity theory. In this subsection, we observe that the same is not true for the broader class of $G$-actions with approximately $G$-inner half-flip, even if one only considers actions of $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ on the C*-algebra $\mathcal{O}_2$. The notion of complete analytic set can be found in [63, Section 22.9].

**Proposition 4.12.** *The relations of conjugacy and cocycle conjugacy for approximately representable $\mathbb{Z}_2$-actions on $\mathcal{O}_2$ with Rokhlin dimension* 1 *and approximately $\mathbb{Z}_2$-inner half-flip are complete analytic sets. Furthermore, the classification problem for such actions, up to conjugacy or cocycle conjugacy, is strictly more complicated than the classification problem for any class of countable structures with Borel isomorphism relation.*

*Proof.* Recall that in [59] Izumi constructed an action $\nu$ of $\mathbb{Z}_2$ on $\mathcal{O}_2$ with the property that the corresponding crossed product $D := \mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2$ is a Kichrberg algebra satisfying the Universal Coefficient Theorem, with trivial $K_1$-group, $K_0$-group isomorphic to $\mathbb{Z}\left[\frac{1}{2}\right]$, and zero element of $K_0(D)$ corresponding to the unit of $D$; see [59, Lemma 4.7].

Such an action was used in [46] to prove that the relations of conjugacy and cocycle conjugacy of $\mathbb{Z}_2$-actions on $\mathcal{O}_2$ are complete analytic sets, when regarded as subsets of $\mathrm{Act}_{\mathbb{Z}_2}(\mathcal{O}_2) \times \mathrm{Act}_{\mathbb{Z}_2}(\mathcal{O}_2)$. Precisely, it is proved in [46], relying on a construction of Rørdam from [70], that there exists a Borel map assigning to each uniquely 2-divisible torsion-free countable abelian group $\Gamma$ a Kirchberg algebra $A_\Gamma$ satisfying the Universal Coefficient Theorem, with trivial $K_1$-group, $K_0$-group isomorphic to $\Gamma$, and zero element of $K_0(A_\Gamma)$ corresponding to the unit of $A_\Gamma$. Denote by $\iota_{A_\Gamma}$ the trivial $\mathbb{Z}_2$-action on $A_\Gamma$. Then the function $\Gamma \mapsto \alpha_\Gamma := \nu \otimes \iota_{A_\Gamma}$ provides a Borel reduction from the relation $E$ of isomorphism of uniquely 2-divisible torsion-free countable abelian groups to the relations of conjugacy and cocycle conjugacy of $\mathbb{Z}_2$-actions on $\mathcal{O}_2$. It is furthermore shown in [46], modifying an argument of Hjorth from [58], that $E$ is a complete analytic set. Furthermore, if $F$ is the relation of isomorphism within a class of countable structures, and if $F$ is Borel, then $F$ is Borel reducible to $E$ (but not viceversa). It was furthermore observed in [46] that, for any uniquely 2-divisible torsion-free countable abelian group $\Gamma$, the action $\alpha_\Gamma$ has Rokhlin dimension 1, and is approximately representable.

We claim that $\alpha_\Gamma$ has approximately $\mathbb{Z}_2$-inner half-flip. To see this, it is enough to observe that the $\mathbb{Z}_2$-action $\nu$ on $\mathcal{O}_2$ (corresponding to the case when $\Gamma$ is trivial) is strongly self-absorbing. This follows from the fact that $\nu$ is constructed as the infinite tensor product $\bigotimes_{n\in\mathbb{N}} \mathrm{Ad}(u)$, where $u$ is a unitary element of $\mathcal{O}_\infty^{\mathrm{st}}$, using the identification $\mathcal{O}_2 \cong \bigotimes_{n\in\mathbb{N}} \mathcal{O}_\infty^{\mathrm{st}}$; see [59, Section 4]. Since $\mathcal{O}_\infty^{\mathrm{st}}$ is a C*-algebra with approximately inner half-flip, one can deduce from [78, Proposition 5.3] that $\nu$ is strongly self-absorbing. Since $\alpha_\Gamma$ is the tensor product of a strongly self-absorbing action (namely, $\nu$) with an action with approximately $\mathbb{Z}_2$-inner half-flip (namely, $\iota_{A_\Gamma}$), it follows that $\alpha_\Gamma$ has approximately $\mathbb{Z}_2$-inner half-flip. This proves the claim.

Using these observations, and considering the fact that the set of $\mathbb{Z}_2$-actions on $\mathcal{O}_2$ with approximately $\mathbb{Z}_2$-inner half-flip is analytic, the result follows. ☐

Clearly, similar conclusions hold for $G$-actions on $\mathcal{O}_2$ for any countable discrete group $G$ with a quotient of order 2, such as the group of integers. This can be seen by regarding a $\mathbb{Z}_2$-action as a $G$-action in the canonical way.

## 5. Order zero dimension and Rokhlin dimension

5.1. **Order zero dimension.** The notion of positive weak $\mathcal{L}_G$-containment between $\mathcal{L}_G$-morphisms has been considered in the general setting of equivariant logic for metric structures in Subsection 2.2.2. For $G$-C*-algebras, one has the following: a $G$-equivariant *-homomorphism $\theta \colon A \to B$ is positively weakly $\mathcal{L}_G^{\mathrm{C}^*}$-contained in another $G$-equivariant *-homomorphism $f \colon A \to C$ if for any separable subalgebras $A_0 \subset A$ and $B_0 \subset B$ such that $\theta(A_0) \subset B_0$, and for some (equivalently, any) countably incomplete filter $\mathcal{F}$, there exists a $G$-equivariant *-homomorphism $\gamma\colon B_0 \to \prod_{\mathcal{F}}^G C$ such that $\gamma \circ \theta = \Delta_C \circ f$, where $\Delta_C\colon C \to \prod_{\mathcal{F}}^G C$ is the diagonal *-homomorphism. Various equivalent formulations of this notion were considered in Subsection 2.2.2.

We now present the natural generalization of positive weak $\mathcal{L}_G^{\mathrm{C}^*}$-containment where instead of a single *-homomorphism one considers a tuple of completely positive contractive order zero maps. Whenever $f\colon A \to B$ is a $G$-equivariant *-homomorphism, we will regard $B$ as a $G$-equivariant $A$-bimodule, as defined in Subsection 3.5.

**Definition 5.1.** Let $A$, $B$, and $C$ be $G$-C*-algebras, and let $\theta\colon A \to B$ and $f\colon A \to C$ be $G$-equivariant *-homomorphisms. We say that $\theta$ is *$G$-equivariantly $d$-contained* in $f$ if for any separable $G$-C*-subalgebras $A_0 \subset A$ and $B_0 \subset B$ such that $\theta(A_0) \subset B_0$, and for some (equivalently, any) countably incomplete filter $\mathcal{F}$, there exist $G$-equivariant completely positive contractive order zero $A$-bimodule maps $\psi_0, \ldots, \psi_d\colon B \to \prod_{\mathcal{F}}^G C$ whose sum $\psi = \psi_0 + \cdots + \psi_d$ is contractive and such that $\psi \circ \theta = \Delta_C \circ f$.

The notion of $G$-equivariant $d$-containment from Definition 5.1 admits a natural syntactic reformulation: $\theta\colon A \to B$ is $G$-equivariantly $d$-contained in $f\colon A \to C$ if and only if for any tuples $\overline{a}$ in $A$, $\overline{b}$ in $B$, and for any tuple $\overline{w}$ of finite dimensional C*-algebras, for any positive quantifier-free $\mathcal{L}_G^{\mathrm{oz},A\text{-}A}$-formulas $\varphi(\overline{z}, \overline{y})$, for any positive quantifier-free $\mathcal{L}_G^{\mathrm{osos},A\text{-}A}$-formulas $\psi(\overline{x}, \overline{z}, \overline{y})$, where the variables $\overline{z}$ have finite-dimensional C*-algebras as sorts, and for any $\varepsilon > 0$, there exist tuples $\overline{c}_0, \ldots, \overline{c}_d$ in $C$ such that

$$\psi(f(\overline{a}), \overline{w}, \overline{c}_0 + \ldots + \overline{c}_d) \leq \psi(\theta(\overline{a}), \overline{w}, \overline{b}) + \varepsilon \quad \text{and} \quad \varphi(\overline{w}, \overline{c}_j) \leq \varphi(\overline{w}, \overline{c}) + \varepsilon \quad \text{for } j = 0, \ldots, d.$$

**Remark 5.2.** When $B$ is nuclear, in the syntactic characterization of $d$-containment, one can replace $\mathcal{L}_G^{\mathrm{osos},A\text{-}A}$-formulas with $\mathcal{L}_G^{\text{osos-nuc},A\text{-}A}$-formulas, and $\mathcal{L}_G^{\mathrm{oz},A\text{-}A}$-formulas with $\mathcal{L}_G^{\text{oz-nuc},A\text{-}A}$-formulas. This follows from the characterization of $\mathcal{L}_G^{\text{osos-nuc},A\text{-}A}$-morphisms from Lemma 3.4.

**Definition 5.3.** The *$G$-equivariant order zero dimension* $\dim_{\mathrm{oz}}^G(\theta)$ of a $G$-equivariant *-homomorphism $\theta\colon A \to B$ is the smallest integer $d \geq 0$ such that $\theta$ is $G$-equivariantly $d$-contained in the identity map $\mathrm{id}_A\colon A \to A$. If no such $d$ exists, we set $\dim_{\mathrm{oz}}^G(\theta) = \infty$.

The proof of the following proposition is an easy consequence of the syntactic characterization of $G$-equivariantly $d$-containment.

**Proposition 5.4.** *Let $\Lambda$ be a directed set.*

(1) *Let $\theta_0\colon A \to B$ and $\theta_1\colon B \to C$ be $G$-equivariant *-homomorphisms between $G$-C*-algebras. Then*
$$\dim_{\mathrm{oz}}^G(\theta_1 \circ \theta_0) + 1 \leq (\dim_{\mathrm{oz}}^G(\theta_1) + 1)(\dim_{\mathrm{oz}}^G(\theta_0) + 1);$$

(2) *Let $\theta\colon A \to B$ be a $G$-equivariant *-homomorphism, and let $C$ be a $G$-C*-algebra. Then*
$$\dim_{\mathrm{oz}}^G(\theta \otimes \mathrm{id}_C) \leq \dim_{\mathrm{oz}}^G(\theta);$$

(3) *Let $(\{A_\lambda\}_{\lambda\in\Lambda}, \{\theta_{\lambda,\mu}\}_{\lambda,\mu\in\Lambda,\lambda<\mu})$ be a direct system of $G$-C*-algebras (with $G$-equivariant *-homomorphisms). For $\lambda \in \Lambda$, denote by $\theta_{\lambda,\infty}\colon A_\lambda \to \varinjlim A_\mu$ denote the canonical equivariant *-homomorphism. Then*
$$\dim_{\mathrm{oz}}^G(\theta_{\lambda,\infty}) \leq \limsup_{\mu\in\Lambda} \dim_{\mathrm{oz}}^G(\theta_{\lambda,\mu}).$$

(4) *For $j = 0, 1$, let $(\{A_\lambda^{(j)}\}_{\lambda\in\Lambda}, \{\theta_{\lambda\mu}^{(j)}\}_{\lambda,\mu\in\Lambda,\lambda<\mu})$ be a direct system of $G$-C*-algebras (with $G$-equivariant *-homomorphisms). Let $\left\{\eta_\lambda\colon A_\lambda^{(0)} \to A_\lambda^{(1)}\right\}_{\lambda\in\Lambda}$ be a family of intertwining $G$-equivariant *-homomorphisms. Then*
$$\dim_{\mathrm{oz}}^G(\varinjlim_{\lambda\in\Lambda} \eta_\lambda) \leq \limsup_{\mu\in\Lambda} \dim_{\mathrm{oz}}^G(\eta_\mu).$$

Let $\theta\colon A \to B$ is a $G$-equivariant completely positive contractive order zero map. We recall that, by Proposition 2.3 in [40], there is a naturally induced completely positive contractive order zero map $A \rtimes G \to B \rtimes G$ between the crossed products, which we will denote in the following by $\widehat{\theta}$. Explicitly, one can define $\widehat{\theta}$ on the dense subalgebra $L^1(G, A)$ of $A$ by letting $\widehat{\theta}(f)$ be the element $g \mapsto \varphi(f(g))$ of $L^1(G, B) \subset B \rtimes G$. If $\theta$ is a *-homomorphism, then $\widehat{\theta}$ is a *-homomorphism as well. If $\theta$ is an $A$-$A$-bimodule, then $\widehat{\theta}$ is an $A$-$A$-bimodule as well.

**Lemma 5.5.** Let $A$ and $B$ be $G$-C*-algebras and let $\theta\colon A \to B$ be a $G$-equivariant *-homomorphism. Then $\dim_{\mathrm{oz}}(\widehat{\theta}) \leq \dim_{\mathrm{oz}}^G(\theta)$. If moreover $G$ is compact, then also $\dim_{\mathrm{oz}}(\theta|_{A^G}) \leq \dim_{\mathrm{oz}}^G(\theta)$, where $A^G$ denotes the fixed point algebra.

*Proof.* Observe that if $A$ is a $G$-C*-algebra and $\mathcal{F}$ is a countably incomplete filter, then there exists a canonical *-homomorphism $\prod_{\mathcal{F}}^G A \rtimes G \to \prod_{\mathcal{F}}^G (A \rtimes G)$ in view of the universal property of the full crossed product; see [44]. This and remarks above prove the first assertion. When $G$ is compact, the second assertion can be proved similarly, by averaging with respect to the Haar measure on $G$. □

5.2. **Commutant $d$-containment.** The notion of commutant positive existential $\mathcal{L}_G^{\mathrm{C}^*}$-theory of a $G$-C*-algebra has been introduced in Subsection 4.4. In this section, we consider $d$-dimensional generalizations of such a notion. We will regard (not necessarily unital) C*-algebras as structures in the Kirchberg language introduced in Subsection 2.4. This will allow us to formulate a definition applicable in both the unital and the nonunital settings.

**Definition 5.6.** Let $d \in \mathbb{N}$, and let $A$ and $B$ be $G$-C*-algebras. Fix a cardinal $\kappa$ larger than the density character of $A$ and $B$. We say that $A$ is *$G$-equivariantly commutant $d$-contained* in $B$, and write $A \precsim_d B$, if for some (equivalently, any) countably incomplete $\kappa$-good filter $\mathcal{F}$, and for any separable unital $G$-C*-subalgebra $C$ of $F_{\mathcal{F}}^G(A)$, there exist $G$-equivariant completely positive contractive order zero maps $\eta_0, \ldots, \eta_d\colon C \to F_{\mathcal{F}}^G(B)$ with unital sum.

We say that $A$ is *$G$-equivariantly commutant $d$-contained* in $B$ *with commuting towers*, and write $A \precsim_d^{\mathrm{c}} B$, if one choose the maps $\eta_0, \ldots, \eta_d\colon C \to F_{\mathcal{F}}^G(B)$ as above to also have pairwise commuting ranges.

Using Proposition 3.7 one can give a syntactic reformulation of Definition 5.6, which in particular shows that the choice of the countably incomplete $\kappa$-good filter $\mathcal{F}$ is irrelevant. When $A, B$ are separable, one can take any countably incomplete filter. It is not difficult to see that if $A \precsim_{d-1} B$ and $B \precsim_{k-1} C$ then $A \precsim_{dk-1} C$.

**Remark 5.7.** Suppose that $A$ is a separable unital $G$-C*-algebra that admits a $G$-equivariant unital *-homomorphism into $A' \cap \prod_{\mathcal{F}}^G A$ for some (equivalently, any) countably incomplete filter. Then $A$ is *$G$-equivariantly commutant $d$-contained* (with commuting towers) in $B$ if and only if there exist $G$-equivariant completely positive contractive order zero maps $\eta_0, \ldots, \eta_d\colon A \to F_{\mathcal{F}}^G(B)$ (with commuting ranges) such that $\eta_0 + \cdots + \eta_d$ is unital. In particular, this applies when $A$ is commutative, or when $A$ is a semi-strongly self-absorbing $G$-C*-algebra; see [75, Theorem 4.6].

**Remark 5.8.** Suppose that $D$ is a semi-strongly self-absorbing $G$-C*-algebra. Theorem 4.7 from [75] shows that a separable $G$-C*-algebra $B$ is $G$-equivariantly $D$-absorbing if and only if $D$ is commutant 0-contained in $B$. We will prove in Theorem 5.33 that this is in turn equivalent to $D$ being commutant $d$-contained with commuting towers in $B$ for any $d \in \mathbb{N}$.

**Remark 5.9.** Let $A$ and $B$ be separable $G$-C*-algebras with $A$ $G$-equivariantly commutant $d$-contained (with commuting towers) in $B$. Let $C$ be a separable subalgebra of $F_{\mathcal{F}}^G(B)$. It is a consequence of Proposition 3.7 that there exist completely positive contractive order zero maps $\eta_0, \ldots, \eta_d\colon A \to C' \cap F_{\mathcal{F}}^G(B)$ (with commuting ranges) such that $\eta_0 + \cdots + \eta_d$ is unital.

5.3. **Relationship between order zero dimension and $d$-containment.** Suppose that $G$ is a *compact* group. The notion of *Rokhlin dimension* (with commuting towers) for a $G$-C*-algebra—see [57, Definition 1.1], [41, Definition 3.2]—can be naturally presented in terms of $d$-containment. Precisely, a $G$-C*-algebra $A$ has Rokhlin dimension (with commuting towers) at most $d$ if and only if the $G$-C*-algebra $C(G)$ endowed with the canonical left translation action $\mathtt{Lt}$ is $G$-equivariantly commutant $d$-contained (with commuting towers) in $A$. We will denote by $\dim_{\mathrm{Rok}}(A)$ the Rokhlin dimension of a $G$-C*-algebra $A$, and by $\dim_{\mathrm{Rok}}^{\mathrm{c}}(A)$ the Rokhlin dimension with commuting towers of $A$. We point out that Rokhlin dimension has recently been generalized to actions of residually finite groups in [76] and to $\mathbb{R}$-actions (flows) in [55].

In this subsection, we will observe that there exists a relationship between the notion of $G$-equivariant order zero dimension of a $G$-equivariant *-homomorphism introduced in Subsection 5.1, and the notion of $G$-equivariant commutant $d$-containment introduced in Subsection 5.2. Precisely, if $\theta\colon A \to B$ has $G$-equivariant order zero dimension at most $d$, then $B$ is commutant $d$-contained in $A$; see Proposition 5.12.

**Lemma 5.10.** Let $C$ be a unital $G$-C*-algebra, let $A$ and $B$ be $G$-C*-algebras, let $\kappa$ be a cardinal larger than the density character of $A$ and $C$, and let $\mathcal{F}$ be a countably incomplete $\kappa$-good filter. Suppose that $\theta\colon A \to B$ is a

$G$-equivariant *-homomorphism, and let $1_C \otimes \theta\colon A \to C \otimes_{\max} B$ be the map $a \mapsto 1_C \otimes \theta(a)$. If $\dim_{\mathrm{oz}}^G(\theta) \leq d < +\infty$, then there exist $G$-equivariant completely positive contractive order zero maps $\eta_0, \ldots, \eta_d\colon C \to F_{\mathcal{F}}^G(A)$ such that $\eta_0 + \cdots + \eta_d$ is unital. The converse holds if $A = B$ and $\theta\colon A \to A$ is the identity map.

*Proof.* Observe that $\theta$ is necessarily injective. We can therefore identify $A$ with its image under $\theta$ inside $B$. Let $\psi_0, \ldots, \psi_d\colon C \otimes_{\max} B \to \prod_{\mathcal{F}}^G A$ be $G$-equivariant completely positive contractive order zero $A$-bimodule maps witnessing the fact that $\dim_{\mathrm{oz}}^G(1_C \otimes \theta) \leq d$. Fix $c_0 = 1, c_1, \ldots, c_n \in C$. Let $t(\overline{x})$ be a positive quantifier-free $\mathcal{L}_G^{\mathrm{oz}}$-type that is realized by $(c_0, \ldots, c_n)$ in $C$. Consider the corresponding multiplier $\mathcal{L}_G^K(A)$-type $t_A^{\mathrm{m}}$ defined as in Subsection 3.6. Let $t_A^{\mathrm{c}}(\overline{x})$ be the commutant type associated with $A$, and consider the $\mathcal{L}_G^K(A)$-type $q_A(\overline{y}_0, \ldots, \overline{y}_d)$ consisting of conditions $\varphi(\overline{y}_j) \leq r$ for any condition $\varphi(\overline{x}) \leq r$ in $t_A^{\mathrm{m}}(\overline{x}) \cup t_A^{\mathrm{c}}(\overline{x})$ and $j = 0, 1, \ldots, d$, and $\|a(y_{0,0} + \cdots + y_{0,d}) - a\| = 0$ for every $a \in A$. Fix an approximate unit $(a_\lambda)_{\lambda \in \Lambda}$ for $A$. Considering the tuple $\overline{b} := (c_0 \otimes a_\lambda, \ldots, c_n \otimes a_\lambda)$ in $C \otimes_{\max} B$, for large enough $\lambda$, we conclude that the type $t_A^{\mathrm{m}}(\overline{x}) \cup t_A^{\mathrm{c}}(\overline{x})$ is approximately realized in $C \otimes_{\max} B$. Recall that, by definition of $G$-equivariant order zero dimension, $\psi_0, \ldots, \psi_d$ are completely contractive order zero $A$-bimodule maps with contractive sums such that $(\psi_0 + \cdots + \psi_d)\,|_A$ is the canonical diagonal $G$-equivariant $*$-homomorphism $A \to \prod_{\mathcal{F}}^G A$. Therefore by Theorem 2.5 considering the elements $\psi_0(\overline{b}), \ldots, \psi_d(\overline{b})$ shows that the type $q_A(\overline{x})$ is approximately realized in $A$. The conclusion follows from quantifier-free positive $\mathcal{L}_G^K(A)$-saturation of $F_{\mathcal{F}}^G(A)$; see Proposition 3.7.

Conversely, suppose that $A = B$, that $\theta\colon A \to A$ is the identity map $\mathrm{id}_A$ of $A$, and that there exist $G$-equivariant completely positive contractive order zero maps $\eta_0, \ldots, \eta_d\colon C \to F_{\mathcal{F}}^G(A)$ such that $\eta_0 + \cdots + \eta_d$ is unital. Then the function $F_{\mathcal{F}}^G(A) \times A \to \prod_{\mathcal{F}}^G A$, given by $([a_i]_{i \in I}, b) \mapsto [a_i b]_{i \in I}$, induces a *-homomorphism $\Psi\colon F_{\mathcal{F}}^G(A) \otimes_{\max} A \to \prod_{\mathcal{F}}^G A$, by the universal property of the maximal tensor product. One can then define $\psi_j = \Psi \circ (\eta_j \otimes \mathrm{id}_A)\colon C \otimes_{\max} A \to \prod_{\mathcal{F}}^G A$, for $j = 0, \ldots, d$. These are well defined $G$-equivariant completely positive contractive order zero $A$-bimodule maps, which witness that $\dim_{\mathrm{oz}}^G(\theta) \leq d$. ☐

The particular instance of Lemma 5.10 is the case when $G$ is a compact group and $C = C(G)$ gives the following:

**Lemma 5.11.** Let $G$ be a compact group, and let $A$ be a $G$-C*-algebra. Denote by $\theta\colon A \to C(G) \otimes A$ the second factor embedding. Then $\dim_{\mathrm{Rok}}(A) = \dim_{\mathrm{oz}}^G(\theta)$.

**Proposition 5.12.** *Suppose that $\theta\colon A \to B$ is a $G$-equivariant *-homomorphism. If $\dim_{\mathrm{oz}}^G(\theta) \leq d$, then $B \precsim_d A$.*

*Proof.* Fix a countably incomplete filter $\mathcal{F}$. Suppose that $C$ is a separable unital $G$-C*-subalgebra of $F_{\mathcal{F}}(B)$. Then by Lemma 5.10 the second factor embedding $1_C \otimes \mathrm{id}_B\colon B \to C \otimes_{\max} B$ has order zero dimension equal to zero. By Proposition 5.4(2), the $G$-equivariant *-homomorphism $(1_C \otimes \mathrm{id}_B) \circ \theta\colon A \to C \otimes_{\max} B$ has order zero dimension at most $d$. Therefore by Lemma 5.10 again there exist $G$-equivariant completely positive contractive order zero maps $\eta_0, \ldots, \eta_d\colon C \to F_{\mathcal{F}}^G(A)$ such that $\eta_0 + \cdots + \eta_d$ is unital. By Lemma 5.10, this concludes the proof. ☐

5.4. **Dimension functions.** By a *dimension function* for (nuclear) $G$-C*-algebras we mean a function from the class of (nuclear) C*-algebras to $\{0, 1, 2, \ldots, \infty\}$.

**Definition 5.13.** A dimension function dim for $G$-C*-algebras is said to be *positively* $\forall\exists$*-axiomatizable* if there exists a collection $\mathcal{F}$ of formulas $\xi(\overline{x}, \overline{z}_0, \ldots, \overline{z}_d, \overline{y}_0, \ldots, \overline{y}_d)$ of the form

$$\max\left\{\eta(\overline{x}, \overline{z}_0, \ldots, \overline{z}_d), \varphi_0(\overline{z}_0, \overline{y}_0), \ldots, \varphi_d(\overline{z}_d, \overline{y}_d), \psi(\overline{x}, \overline{z}_0, \ldots, \overline{z}_d, \overline{y}_0 + \cdots + \overline{y}_d)\right\},$$

where

(1) $\overline{z}_0, \ldots, \overline{z}_d$ have finite-dimensional C*-algebras as sorts,
(2) $\eta$ is a quantifier-free positive primitive $\mathcal{L}_G^{\mathrm{C}^*}$-formula,
(3) $\varphi$ is a quantifier-free positive primitive $\mathcal{L}_G^{\mathrm{oz}}$-formula,
(4) $\psi$ is a quantifier-free positive primitive $\mathcal{L}_G^{\mathrm{osos}}$-formula,

such that the following holds: for a $G$-C*-algebra $A$, $\dim(A) \leq d$ if and only if

$$A \models \sup_{\overline{x}} \inf_{\overline{z}_0} \cdots \inf_{\overline{z}_d} \inf_{\overline{y}_0} \cdots \inf_{\overline{y}_d} \xi(\overline{x}, \overline{z}_0, \ldots, \overline{z}_d, \overline{y}_0, \ldots, \overline{y}_d) = 0.$$

**Definition 5.14.** A dimension function for *nuclear* $G$-C*-algebras is said to be *nuclearly* positively ∀∃-axiomatizable if in Definition 5.13 we can simultaneously choose $\varphi$ and $\psi$ to be positive quantifier-free formulas in $\mathcal{L}_G^{\text{oz-nuc}}$ and $\mathcal{L}_G^{\text{C*-nuc}}$, respectively.

**Example 5.15.** The following are positively ∀∃-axiomatizable dimension functions for nuclear C*-algebras:

(1) Nuclear dimension. Indeed, one can consider variables $(\overline{z}_0, \ldots, \overline{z}_d)$ with sorts finite-dimensional C*-algebras $F_0, \ldots, F_d$ and then the formulas
- $\eta(\overline{x}, \overline{z}_0, \ldots, \overline{z}_d) \equiv \max\limits_{j=0,\ldots,d} \inf\limits_{s\in \mathrm{CPC}(A,F_j)} \max\limits_{k} \|s(x_k) - z_{j,k}\|$;
- $\varphi_j(\overline{y}_j, \overline{z}_j) \equiv \inf\limits_{t\in \mathrm{CPC}(F_j,A)} \max\limits_{k} \|t(z_{j,k}) - y_{j,k}\|$, for a fixed $j = 0, \ldots, d$;
- $\psi(\overline{x}, \overline{z}_0, \ldots, \overline{z}_d, \overline{y}) \equiv \max_k \|x_k - y_k\|$.

(2) Decomposition rank. In fact, one may just consider the same formulas $\eta$ and $\psi$ as in (1), together with
$$\varphi_j(\overline{z}_j, \overline{y}_j) \equiv \inf_{t\in \mathrm{CPC}(F_j,A)} \max_k \|t(z_{j,k}) - y_{j,k}\| , \quad \text{for } j = 0, \ldots, d.$$

If $\overline{x}, \overline{y}$ are $n$-tuples of variables, we write $\delta(\overline{x}, \overline{y})$ for the formula $\max\limits_{1\leq j,k\leq n} \|x_j y_k - y_k x_j\|$.

**Definition 5.16.** Suppose that dim is a dimension function for (separable) $G$-C*-algebras. We say that dim is *commutant positively existentially axiomatizable* if there exists a collection $\mathcal{F}$ of formulas of the form
$$\xi(\overline{x}, \overline{y}_0, \ldots, \overline{y}_d) = \max\{\delta(\overline{x}, \overline{y}_j), \varphi(\overline{y}_j), \|x_\ell(y_{0,j} + \cdots + y_{d,j}) - x_\ell\| : 0 \leq j < k \leq d, 1 \leq \ell \leq n\},$$
where $\varphi$ is a quantifier-free positive primitive $\mathcal{L}_G^{\text{oz}}$-formula with parameters from finite-dimensional C*-algebras, such that the following hods: for a (separable) $G$-C*-algebra $A$, one has $\dim(A) \leq d$ if and only if
$$A \models \sup_{\overline{x}} \inf_{\overline{y}_0} \cdots \inf_{\overline{y}_d} \xi(\overline{x}, \overline{y}_0, \ldots, \overline{y}_d) = 0.$$

**Definition 5.17.** Suppose that dim is a dimension function for (separable) $G$-C*-algebras. We say that dim is *commutant positively existentially axiomatizable with commuting towers* if there exists a collection $\mathcal{F}$ of formulas of the form
$$\xi(\overline{x}, \overline{y}_0, \ldots, \overline{y}_d) = \max\{\delta(\overline{x}, \overline{y}_j), \delta(\overline{y}_j, \overline{y}_k), \varphi(\overline{z}_j, \overline{y}_j), \|x_\ell(y_{0,j} + \cdots + y_{d,j}) - x_\ell\| : 0 \leq j < k \leq d, 1 \leq \ell \leq n\}$$
where $\varphi$ is a quantifier-free positive primitive $\mathcal{L}_G^{\text{oz}}$-formula with parameters from finite-dimensional C*-algebras, such that the following hods: for a (separable) $G$-C*-algebra $A$, one has $\dim(A) \leq d$ if and only if
$$A \models \sup_{\overline{x}} \inf_{\overline{y}_0} \cdots \inf_{\overline{y}_d} \xi(\overline{x}, \overline{z}_0, \ldots, \overline{z}_d, \overline{y}_0, \ldots, \overline{y}_d) = 0.$$

**Example 5.18.** Suppose that $C$ is a fixed $G$-C*-algebra. Set $\dim(A) \leq d$ if and only if $C$ is commutant $d$-contained in $C$ (with commuting towers). Then dim is a dimension function for $G$-C*-algebras that is commutant positively existentially axiomatizable (with commuting towers).

In the particular case when $G$ is a compact group and $C$ is the $G$-C*-algebra $C(G)$ endowed with the canonical shift action of $G$, this says that Rokhlin dimension (with commuting towers) is a commutant positively existentially axiomatizable (with commuting towers) dimension function.

The following is a consequence of Definition 5.17 and the syntactic characterization of commutant $d$-containment.

**Proposition 5.19.** *Let* dim *be a dimension function for separable $G$-C*-algebras that is positively existentially axiomatizable (with commuting towers). Let $A$ and $B$ be separable $G$-C*-algebras such that $A$ is commutant $d$-contained (with commuting towers) in $B$. Then*
$$\dim(B) + 1 \leq (d + 1)(\dim(A) + 1).$$

Similarly, the following fact is a consequence of the syntactic characterization of $G$-equivariant $d$-containment, Remark 5.2, and Proposition 5.12.

**Proposition 5.20.** *Let $A$ and $B$ be $G$-C*-algebras, and let $\theta\colon A \to B$ be a $G$-equivariant *-homomorphism. Suppose that* dim *is a dimension function for C*-algebras. If* dim *is positively ∀∃-axiomatizable dimension or commutant positively existentially axiomatizable, then*
$$\dim(A) + 1 \leq (\dim_{\mathrm{oz}}^G(\theta) + 1)(\dim(B) + 1).$$

*Moreover, if $B$ is nuclear and* dim *is nuclearly* $\forall\exists$*-axiomatizable, then again*

$$\dim(A) + 1 \leq (\dim_{\mathrm{oz}}^G(\theta) + 1)(\dim(B) + 1).$$

In particular, Proposition 5.20 applies when dim is either nuclear dimension $\dim_{\mathrm{nuc}}$, decomposition rank dr, or, when $G$ is compact, Rokhlin dimension $\dim_{\mathrm{Rok}}$; see Example 5.15 and Example 5.18. More generally, one can define the nuclear dimension and decomposition rank of a *-homomorphism $f\colon A \to B$, and then show that if $\theta\colon A \to B$ is $d$-contained in $f$, then

$$\dim_{\mathrm{nuc}}(\theta) + 1 \leq (d+1)(\dim_{\mathrm{nuc}}(f) + 1) \quad \text{and} \quad \mathrm{dr}(\theta) + 1 \leq (d+1)(\mathrm{dr}(f) + 1).$$

For a compact group $G$, the following result relates the order zero dimension of the canonical inclusions $A^G \to A$ and $A \rtimes G \to A \otimes \mathcal{K}(L^2(G))$ to the Rokhlin dimension of a $G$-C*-algebra $A$.

**Proposition 5.21.** *Let $G$ be a compact group, let $A$ be a $G$-C*-algebra $A$, and denote by $\iota\colon A^G \hookrightarrow A$ and $\sigma\colon A \rtimes G \to A \otimes \mathcal{K}(L^2(G))$ the canonical inclusion maps. Then $\dim_{\mathrm{oz}}(\iota) \leq \dim_{\mathrm{Rok}}(A)$ and $\dim_{\mathrm{oz}}(\sigma) \leq \dim_{\mathrm{Rok}}(A)$.*

*Proof.* Denote by $\theta\colon A \to C(G) \otimes A$ the second factor embedding. Let $\mathtt{Lt}$ denote the action of $G$ on $C(G)$ by left translation, and denote by $\alpha$ the given action on $A$. Endow $C(G) \otimes A$ with the tensor product action $\gamma = \mathtt{Lt} \otimes \alpha$. Then $\theta$ is $G$-equivariant, and hence it induces a *-homomorphism $A \rtimes G \to (C(G) \otimes A) \rtimes G$. Observe that $(C(G) \otimes A, \gamma)$ is canonically $G$-equivariantly isomorphic to $(C(G) \otimes A, \mathtt{Lt} \otimes \iota_A)$ by [39, Proposition 2.3]. Then the crossed product $(C(G) \otimes A) \rtimes_\gamma G$ is canonically isomorphic to $A \otimes \mathcal{K}(L^2(G))$, and the fixed point algebra $(C(G) \otimes A)^\gamma$ is canonically isomorphic to $A$. It follows that the map $\widehat{\theta}$—defined right before Lemma 5.5—is canonically conjugate to $\sigma$, and $\theta|_{A^G}$ is canonically conjugate to $\iota$.

Using Lemma 5.11 at the first step, Lemma 5.5 at the second, and the above observations at the third, we deduce that

$$\dim_{\mathrm{Rok}}(A) = \dim_{\mathrm{oz}}^G(\theta) \geq \dim_{\mathrm{oz}}(\widehat{\theta}) = \dim_{\mathrm{oz}}(\widehat{\sigma}).$$

Similarly, we have $\dim_{\mathrm{Rok}}(A) \geq \dim_{\mathrm{oz}}(\iota)$, as desired. ☐

**Corollary 5.22.** Let $G$ be a compact group, let $A$ be a $G$-C*-algebra $A$, and let dim be a positively $\forall\exists$-axiomatizable dimension function for C*-algebras. Then

$$\dim(A^G) + 1 \leq (\dim_{\mathrm{Rok}}(A) + 1)(\dim(A) + 1) \quad \text{and} \quad \dim(A \rtimes G) \leq (\dim_{\mathrm{Rok}}(A) + 1)(\dim(A) + 1).$$

For separable unital $A$, the following first appeared as [40, Theorem 3.3]. The particular case of commuting towers has also been independently obtained in [45], using completely different methods.

**Corollary 5.23.** Let $G$ be a compact group, and let $A$ be a $G$-C*-algebra $A$. Then

$$\dim_{\mathrm{nuc}}(A^G) + 1 \leq \dim_{\mathrm{nuc}}(A \rtimes G) + 1 \leq (\dim_{\mathrm{Rok}}(A) + 1)(\dim_{\mathrm{nuc}}(A) + 1)$$

and

$$\mathrm{dr}(A^G) + 1 \leq \mathrm{dr}(A \rtimes G) + 1 \leq (\dim_{\mathrm{Rok}}(A) + 1)(\mathrm{dr}(A) + 1).$$

*Proof.* The first inequalities in Corollary 5.23 are due to the fact that the fixed point algebra $A^G$ of a $G$-C*-algebra is a corner of the crossed product $A \rtimes G$—see [72, Theorem]—and the fact that decomposition rank and nuclear dimension are nonincreasing when passing to hereditary subalgebras; see [66, Proposition 3.8] and [85, Proposition 2.5]. The second inequalities are an immediate consequence of Corollary 5.22 and Example 5.15. ☐

5.5. **Bundles.** In this subsection, we generalize the main result of [22] to equivariant bundles; see Theorem 5.26. This result will be crucial in our applications to actions with finite Rokhlin dimension in Subsections 6.4 and 6.5.

We will need the following equivariant version of the Choi-Effros lifting theorem for compact groups. In the general case, one can average over compact subsets of $G$, to get lifts that are equivariant over sufficiently large parts of the groups. Since we only consider compact groups in this section, we do not pursue this direction any further.

**Proposition 5.24.** *Let $G$ be a compact group, let $(A, \alpha)$ and $(B, \beta)$ be $G$-C*-algebras, and let $\varphi\colon A \to B$ be a surjective, $G$-equivariant, nuclear *-homomorphism. Then there exists a $G$-equivariant completely positive contractive lift $\sigma\colon B \to A$. If $\varphi$ is unital, then we can also choose $\sigma$ to be unital.*

*Proof.* Use Choi-Effros to find a completely positive contractive lift $\rho\colon B \to A$ (which may be chosen to be unital if $\varphi$ is). If $\mu$ denotes the normalized Haar measure on $G$, then it is easy to check that the map $\sigma\colon B \to A$ given by $\sigma(b) = \int_G \alpha_g(\rho(\beta_{g^{-1}}(b)))\, \mathrm{d}\mu$, for all $b \in B$, is as in the statement. ☐

Suppose that $X$ is a compact metrizable space. The definition of $C(X)$-algebra can be found in [22, Definition 2.1]. We consider here the natural equivariant analog of a $C(X)$-algebra:

**Definition 5.25.** Let $G$ be a locally compact group, and let $A$ be a $C(X)$-algebra. For $x \in X$, denote by $U_x$ the open subset $X \setminus \{x\}$ of $X$, and denote by $A(U_x)$ the corresponding ideal of $A$.

We say that $A$ is a *$G$-$C(X)$-algebra*, if $A$ is endowed with an action $\alpha\colon G \to \mathrm{Aut}(A)$ satisfying $\alpha_g(A(U_x)) \subset A(U_x)$ for all $x \in X$ and all $g \in G$.

In the context of the above definition, given $x \in X$, denote by $A_x$ the quotient $A/A(U_x)$ and by $\pi_x\colon A \to A_x$ the canonical quotient map. Then $\alpha$ induces actions $\alpha^{(x)}\colon G \to \mathrm{Aut}(A_x)$, that make each $\pi_x$ equivariant.

The definition of *unitarily regular action* is given in [78, Definition 1.18]. Observe that the trivial action on a strongly self-adsorbing C*-algebra is unitarily regular. More generally, this applies to any strongly self-absorbing $G$-C*-algebra that $G$-equivariantly absorbs the trivial action on the Jiang-Su algebra; see [78, Proposition 1.20]. The main theorem of this subsection is the following:

**Theorem 5.26.** *Let $G$ be a compact group, and let $X$ be a compact metrizable space of finite covering dimension. Let $(D, \delta)$ be a strongly self-absorbing, unitarily regular $G$-C*-algebra, and let $(A, \alpha)$ be a separable, unital $G$-$C(X)$-algebra such that $A_x$ is $G$-equivariantly isomorphic to $D$, for all $x \in X$. Then there is a $G$-equivariant $C(X)$-linear isomorphism $(A, \alpha) \cong (C(X) \otimes D, \iota_{C(X)} \otimes \delta)$.*

Our proof follows the lines of Dadarlat-Winter's proof of the nonequivariant version of Theorem 5.26 from [22, Section 4]. In fact, for the sake of succinctness, we only mention what changes are needed in said proof, and leave the smaller details to the reader. We are confident that a similar result holds for arbitrary locally compact groups, and suspect that a proof can be obtained by appropriately modifying the one here provided (for example, unitaries will in general not be fixed, but rather approximately fixed over large portions of the group). Since we do not have applications in mind for the general statement, we restrict ourselves to the compact case.

Throughout the rest of the subsection, we fix a compact metrizable space $X$, a strongly self-absorbing $G$-C*-algebra $(D, \delta)$, and a separable unital $G$-$C(X)$-algebra $A$.

**Definition 5.27.** Let $(B, \beta)$ and $(C, \gamma)$ be $G$-C*-algebras, let $\varepsilon > 0$ and let $F \subset B$ be a compact set. We say that a linear map $\varphi\colon B \to C$ is *$\varepsilon$-multiplicative* (respectively, *$\varepsilon$-equivariant*) on $F$, if $\|\varphi(b_1 b_2) - \varphi(b_1)\varphi(b_2)\| < \varepsilon$ for all $b_1, b_2 \in F$ (respectively, $\|\gamma_g(\varphi(b)) - \varphi(\beta_g(b))\| < \varepsilon$ for all $g \in G$ and for all $b \in F$).

The following is the analog of Proposition 4.1 of [22].

**Proposition 5.28.** *Denote by $\mu\colon C(X) \to A$ the structure map. Suppose that for any $\varepsilon > 0$ and for any compact subsets $F \subset A$, $H_1 \subset C(X)$ and $H \subset D$, there are completely positive contractive maps $\psi\colon A \to C(X) \otimes D$ and $\varphi\colon C(X) \otimes D \to A$ satisfying*

1. *$\|(\varphi \circ \psi)(a) - a\| < \varepsilon$ for all $a \in F$;*
2. *$\|\varphi(f \otimes 1_D) - \mu(f)\| < \varepsilon$ for all $f \in H_1$;*
3. *$\|(\psi \circ \mu)(f) - f \otimes 1_D\| < \varepsilon$ for all $f \in H_1$;*
4. *$\varphi$ is $\varepsilon$-multiplicative and $\varepsilon$-equivariant on $(1_{C(X)} \otimes \mathrm{id}_D)(H_2)$;*
5. *$\psi$ is $\varepsilon$-multiplicative and $\varepsilon$-equivariant on $F$.*

*Then there is a $G$-equivariant $C(X)$-linear isomorphism $(A, \alpha) \cong (C(X) \otimes D, \iota_{C(X)} \otimes \delta)$.*

*Proof.* The only thing that needs to be checked is that the isomorphisms $\overline{\varphi}$ and $\overline{\psi}$ constructed in [22], are equivariant, which is a routine computation. ☐

We need an equivariant version of [22, Proposition 3.5], in order to prove the analog of [22, Lemma 4.5]. We note here that when $G$ is compact and $\varepsilon > 0$ is small enough, then any unitary in $A^G_\varepsilon$ can be perturbed to a nearby unitary in $A^G$. Moreover, if the original unitary is in the connected component of the unit, then so is its perturbation (and the path can be chosen to be in $A^G$).

**Proposition 5.29.** *Suppose that $D$ is unitarily regular. Then for any finite set $F \subset D$ and every $\varepsilon > 0$, there exist a finite set $H \subset D$ and $\delta > 0$ with the following property: for any unital $D$-absorbing $G$-C*-algebra $A$, and any unital completely positive maps $\varphi, \psi \colon D \to A$ that are $\delta$-multiplicative and $\delta$-equivariant on $H$, there is a unitary $u \in \mathcal{U}_0(A^G)$ such that $\|\varphi(d) - u\psi(d)u^*\| < \varepsilon$ for all $d \in F$.*

*Proof.* The proof in [22] applies almost verbatim, with the following changes: the maps $\Phi$ and $\Psi$ are also equivariant. Instead of [79, Corollary 1.12], use [75, Proposition 3.4(iii)]; the obtained unitary $V$ can be chosen to belongs to $(B \otimes D)^G$, and similarly with $V_n$. The equivariant analog of [79, Proposition 1.9] is straightforward to show for compact groups (choosing unitaries in the fixed point algebra). The unital homomorphisms $\theta_n$ can then be chosen to be equivariant, and the maps $\gamma_n$ are also equivariant (observe that $\gamma_n$ denotes two different things in [22]). Finally, one must use [78, Theorem 2.15] instead of [22, Theorem 3.1] (this is where unitary regularity of the action on $D$ is used). ☐

Lemma 4.2 in [22] goes through with only minor changes:

**Lemma 5.30.** Adopt the notation from [22, Lemma 4.2]. Assume furthermore that $D$ is unitarily regular and that the maps $\sigma_1$ and $\sigma_2$ are $\delta(F, \gamma)$-equivariant on $\mathcal{E}(F, \gamma)$. Then there is a continuous path $(u_t)_{t \in [0,1]}$ of unitaries in $(C(K) \otimes D)^G$ satisfying $u_0 = 1_{C(K)} \otimes 1_D$ and $\|u_1\sigma_1(d)u_1^* - \sigma_2(d)\| < \gamma$ for all $d \in F \cdot F$.

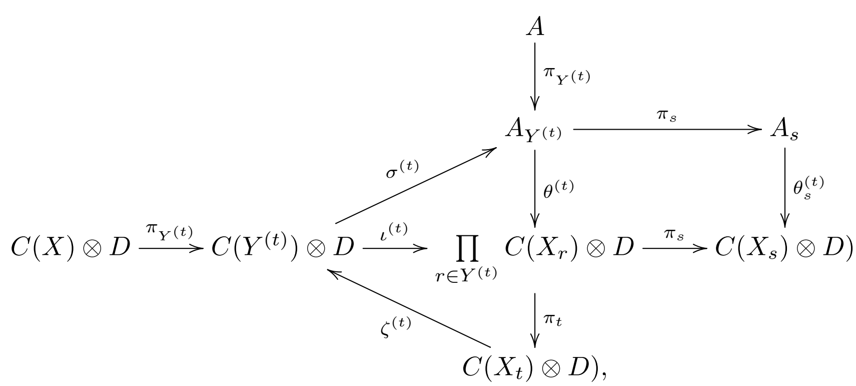
*Proof.* Replace every application of [22, Proposition 3.5] with an application of Proposition 5.29. ☐

We need an equivariant analog of a local approximate trivialization; see [22, Definition 4.3]. Since our notation differs slightly from the one used in said paper, we reproduce the definition entirely.

**Definition 5.31.** For $n \in \mathbb{N}$, we write $p \colon [0,1]^n \to [0,1]$ for the first coordinate projection. Given a compact subset $X \subset [0,1]^n$, set $Y = p(X)$. If $C \subset Y$ is a closed subset, we write $X_C = p^{-1}(C)$. Let $A$ be a unital $G$-$C(X)$-algebra $A$. We abbreviate $A_{X_C}$ to $A_C$, and $A_{X_{\{s\}}}$ to $A_s$, for $s \in Y$, while the fiber maps are denoted $\pi_C$ and $\pi_s$, respectively. (We will not distinguish, as far as the notation is concerned, between fiber maps of different $C(X)$-algebras associated to the same closed subset of $X$.)

Suppose that $D$ is a strongly self-absorbing $G$-C*-algebra, that each fiber of $A$ is $G$-equivariantly isomorphic to $D$, and that for each $s \in Y$, there is a $G$-$C(X_s)$-algebra isomorphism $A_s \cong C(X_s) \otimes D$. Let $\eta > 0$, let $t \in Y$, and let $\theta \colon A_t \to C(X_t) \otimes D$ be a $G$-$C(X_t)$-algebra isomorphism. Fix compact subsets $F \subset A$ containing $1_A$, $H \subset C(X) \otimes D$, and $\widehat{H} \subset C(X_t) \otimes D$.

Let $Y^{(t)}$ be a closed neighborhood of $t$ in $Y$. A *$G$-equivariant $(\theta, F, H, \widehat{H}, \eta)$-trivialization of $A$ over $Y^{(t)}$* is a family of diagrams, indexed over $s \in Y^{(t)}$, as follows:




where all C*-algebras are $G$-$C(X)$-algebras in the obvious way; each map is $G$-equivariant, unital and completely positive; and conditions (i) through (xii) in [22, Definition 4.3] are satisfied.

Existence of equivariant local approximate trivializations, in the sense of the definition above, is established similarly as in the nonequivariant case:

**Lemma 5.32.** Adopt the notation and assumptions of the first two paragraphs of Definition 5.31, and assume moreover that $D$ is unitarily regular. Then there exist a closed neighborhood $Y^{(t)} \subset Y$ of $t$ and a $G$-equivariant $(\theta, F, H, \widehat{H}, \eta)$-trivialization of $A$ over $Y^{(t)}$.

*Proof.* Again, the proof given in [22] requires only minor changes: the isomorphisms $\widetilde{\theta}_s^{(t)}\colon A_s \to C(X_s)\otimes D$ are chosen to be $G$-equivariant. Also, the $G$-equivariant, unital completely positive lifts $\overline{\zeta}^{(t)}\colon C(X_t) \to C(X_{\widetilde{Y}^{(t)}})$ and $\overline{\sigma}^{(t)}\colon D \to A_{\widetilde{Y}^{(t)}}$ are obtained using Proposition 5.24. The applications of [22, Lemma 4.2] are replaced by applications of Lemma 5.30. In particular, $\widetilde{u}^{(s)}$ can be chosen to be $G$-invariant. It follows that $\theta_s^{(t)}$ is equivariant, since so are $\widetilde{\pi}^{(s)}$ and $\widetilde{\theta}_s^{(t)}$. The verification of (xi) and (xii) in Definition 5.31 is routine, and we omit it. ☐

Finally, we come to the proof of the main result of this section:

*Proof of Theorem 5.26.* Use Proposition 5.28 instead of [22, Proposition 4.1]. The basis of induction must also assume that $\theta_t\colon A_t \to C(X_t)\otimes D$ is $G$-equivariant. Apply Lemma 5.32 in place of [22, Lemma 4.5]. The unital completely positive maps $\lambda^{(i)}, \varrho^{(i)}\colon C(X_{y_i})\otimes D \to C(X_{t_i})\otimes D$ are $G$-equivariant because so are $\zeta^{(y_i)}$, $\sigma^{(y_i)}$, $\pi_{t_i}$, $\theta_{t_i}^{(y_i)}$, and $\theta_{t_i}^{(y_{i+1})}$. The unitaries $u_t^{(i)}$, for $t\in[0,1]$ and $i\in I$, can be chosen to be $G$-invariant by Lemma 5.30; in other words, the path $t\mapsto u_t^{(i)}$ determines a $G$-invariant unitary in $C([0,1])\otimes C(X_{t_1})\otimes D$, where $C([0,1])$ carries the trivial $G$-action. The unitaries defined in (31) and (32) are automatically $G$-invariant. Finally, the maps $\psi\colon A\to C(X)\otimes D$ and $\varphi\colon C(X)\otimes D\to A$ are readily checked to be equivariant (observe that the structure map of a $G$-$C(X)$-algebra is equivariant when $C(X)$ is endowed with the trivial $G$-action). This finishes the proof. ☐

5.6. **$G$-equivariant $D$-absorption.** We start by providing a new characterization of $G$-equivariant $D$-absorption for a compact group $G$. The nonequivariant case (when the group is trivial), has recently been observed in [55].

**Theorem 5.33.** *Let $G$ be a compact group, let $A$ be a separable $G$-C*-algebra, let $\mathcal{F}$ be a countably incomplete filter, and let $D$ be a strongly self-absorbing, unitarily regular $G$-C*-algebra. Then $A$ is $G$-equivariantly $D$-absorbing if and only if there exist $d\in\mathbb{N}$ and $G$-equivariant completely positive contractive order zero maps $\psi_0,\ldots,\psi_d\colon D\to F_{\mathcal{F}}^G(A)$ with commuting ranges such that $\psi_0+\cdots+\psi_d$ is unital.*

*Proof.* By [75, Theorem 3.7], being $D$-absorbing is equivalent to the condition in Theorem 5.33 with $d=0$. We now prove the converse implication. We let $C(D)=C_0((0,1])\otimes D$ denote the cone of $D$, and $C(D)^\dagger$ denote its minimal unitization, endowed with the canonical $G$-action. The tensor product $C(D)^\dagger\otimes\cdots\otimes C(D)^\dagger$ of $d+1$ copies of $C(D)^\dagger$ has a canonical $G$-equivariant character. We let $E$ be its quotient by the kernel of such a character, which is a $G$-invariant ideal. Observe that if $B$ is a unital C*-algebra, then $(d+1)$-tuples of $G$-equivariant completely positive contractive order zero maps $D\to B$ with commuting ranges and unital sum, are into one-to-one correspondence with unital $G$-equivariant *-homomorphisms $E\to B$. This follows form the structure theorem for completely positive contractive order zero maps [84, Corollary 4.1]—or, more precisely, its equivariant counterpart [41, Corollary 2.10]—and the universal properties of unitization and tensor products. Therefore, in order to conclude the proof, it is enough to show that $E$ is $G$-equivariantly $D$-absorbing.

Denote by $X$ the spectrum of the center of $E$, which is a quotient of the $(d+1)$-dimensional cube $[0,1]^{d+1}$. Thus, $X$ is a compact metrizable space. Moreover, the $G$-C*-algebra $E$ is easily seen to be a $G$-$C(X)$-algebra with fibers isomorphic to $D$. By Theorem 5.26, we conclude that $E$ is $G$-equivariantly isomorphic to $C(X)\otimes D$, and in particular is $G$-equivariantly $D$-absorbing. This finishes the proof. ☐

In view of Remark 5.7, one can reformulate Theorem 5.33 by asserting that $A$ is $G$-equivariantly $D$-absorbing if and only if it is commutant $d$-contained in $D$ with commuting towers for some $d\in\mathbb{N}$.

**Corollary 5.34.** Let $G$ be a compact group, let $A$ be a separable $G$-C*-algebra, let $\mathcal{F}$ be a countably incomplete filter, and let $D$ be a strongly self-absorbing, unitarily regular $G$-C*-algebra. Then $A$ is $D$-absorbing if and only if there exist $d\in\mathbb{N}$ and completely positive contractive order zero maps $\psi_0,\ldots,\psi_d\colon D\to F_{\mathcal{F}}^G(A)$ with commuting ranges such that $\psi_0+\cdots+\psi_d$ is unital.

Suppose that $D$ is a strongly self-absorbing $G$-C*-algebra. Consider the $\{0,\infty\}$-valued dimension function for separable $G$-C*-algebras obtained by setting $\dim_D(A)=0$ if and only if $A$ is $G$-equivariantly $D$-absorbing. The following proposition is an immediate consequence of Theorem 5.33; see also Example 5.18.

**Proposition 5.35.** *Let $G$ be a compact group, and let $D$ be a strongly self-absorbing, unitarily regular $G$-C*-algebra. Then $\dim_D$, as defined above, is commutant positively existentially axiomatizable with commuting towers.*

The following is the main result of this subsection. The conclusion is new even in the nonequivariant setting.

**Corollary 5.36.** Let $G$ be a compact group, let $A$ and $B$ be separable $G$-C*-algebras, and let $D$ be a strongly self-absorbing $G$-C*-algebra. If $A$ is $G$-equivariantly $D$-absorbing and $A \precsim^{\mathrm{c}}_{d} B$ for some $d \in \mathbb{N}$, then $B$ is $G$-equivariantly $D$-absorbing.

*Proof.* If $A$ is $G$-equivariantly $D$-absorbing, then $D \precsim^{\mathrm{c}}_{0} A$. If furthermore $A \precsim^{\mathrm{c}}_{d} B$, then we have $D \precsim^{\mathrm{c}}_{d} B$. Therefore $B$ is $G$-equivariantly $D$-absorbing by Theorem 5.33 and Remark 5.7. ☐

5.7. **Examples and applications to dimensional inequalities.** In this section, we exhibit some examples of embeddings with finite order zero dimension, and use them to deduce some dimensional inequalities, particularly for nuclear dimension and decomposition rank. We need to extract a technical fact from Section 5 of [68]. If $a, b$ are elements of a C*-algebra $A$, we write $a \approx_{\varepsilon} b$ to denote that $\|a - b\| < \varepsilon$.

**Lemma 5.37.** Let $n \in \mathbb{N}$, and let $\varepsilon > 0$. Then there exist completely positive contractive maps $\lambda_0, \lambda_1 \colon M_n \to \mathcal{Z}$ such that $\lambda_0(1_{M_n}) + \lambda_1(1_{M_n}) \approx_{\varepsilon} 1_{\mathcal{Z}}$.

*Proof.* See the first part of proof of Theorem 1.1 in Section 5 of [68]. ☐

**Theorem 5.38.** *Let $U$ be a UHF-algebra of infinite type, and let $\theta \colon \mathcal{Z} \to U$ be any unital embedding. Then $\dim_{\mathrm{oz}}(\theta) = 1$.*

*Proof.* Since any two unital embeddings of $\mathcal{Z}$ into $U$ are approximately unitarily equivalent, and $\mathcal{Z} \otimes U$ is isomorphic to $U$, we may assume, without loss of generality, that $\theta$ is the first tensor factor embedding $\mathcal{Z} \to \mathcal{Z} \otimes U$. Let $\mathcal{F}$ be the filter of cofinite subsets of $\mathbb{N}$. Write $U$ as an increasing union $U = \overline{\bigcup_{n \in \mathbb{N}} M_{k_n}}$ of matrix algebras $M_{k_n}$. Using injectivity of $M_{k_n}$, choose a conditional expectation $E_n \colon U \to M_{k_n}$. For $n, m \in \mathbb{N}$, let $\lambda_0^{(n,m)}, \lambda_1^{(n,m)} \colon M_{k_n} \to \mathcal{Z}$ denote the order zero maps obtained from Lemma 5.37 for $\varepsilon = 1/m$. For $j = 0, 1$, set

$$\lambda_j^{(n)} = (\lambda_j^{(n,m)})_{m \in \mathbb{N}} \colon M_{k_n} \to \prod\nolimits_{\mathcal{F}} \mathcal{Z},$$

which is an order zero map. Note that $\lambda_0^{(n)}(1_{M_{k_n}}) + \lambda_1^{(n)}(1_{M_{k_n}})$ is equal to the identity of $\prod_{\mathcal{F}} \mathcal{Z}$. For $j = 0, 1$, let $\psi_j \colon U \to \prod_{\mathcal{F}}(\prod_{\mathcal{F}} \mathcal{Z})$ be given by $\psi_j(x) = (\lambda_j(E_n(x)))_{n \in \mathbb{N}}$ for all $x \in U$. Then $\psi_j$ is order zero, and $\psi_0(1_U) + \psi_1(1_U)$ is equal to the identity of $\prod_{\mathcal{F}}(\prod_{\mathcal{F}} \mathcal{Z})$. We obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & U \\ \big\downarrow & & \big\downarrow{\scriptstyle \psi_j} \\ \prod_{\mathcal{F}} \mathcal{Z} & \longrightarrow & \prod_{\mathcal{F}}(\prod_{\mathcal{F}} \mathcal{Z}), \end{array}$$

where the maps from $\mathbb{C}$ are the canonical unital homomorphisms, and the lower horizontal map is the canonical diagonal *-homomorphism $\Delta_{\prod_{\mathcal{F}} \mathcal{Z}} \colon \prod_{\mathcal{F}} \mathcal{Z} \to \prod_{\mathcal{F}}(\prod_{\mathcal{F}} \mathcal{Z})$. We claim that there are completely positive contractive order zero maps $\varphi_0, \varphi_1 \colon U \to \prod_{\mathcal{F}} \mathcal{Z}$ such that $\psi_j = \Delta_{\prod_{\mathcal{F}} \mathcal{Z}} \circ \varphi_j$ for $j = 0, 1$. This (and in fact, a more general statement) can be proved along the lines of [39, Lemma 4.18], replacing condition (2) in its proof with the following:

$$\left\| (\psi_j)_m^{(n_r)}(b^*) - (\psi_j)_m^{(n_r)}(b)^* \right\| < \frac{1}{r} \quad \text{and} \quad \left\| (\psi_j)_m^{(n_r)}(cc') \right\| < \frac{1}{r}$$

whenever $b, c, c' \in G_r$ satisfy $cc'^*c' = c'^* = c'c = 0$. We omit the details.

The fact that $\dim_{\mathrm{oz}}(\theta) \leq 1$ now follows from Lemma 5.10. It remains to show that $\dim_{\mathrm{oz}}(\theta) > 0$. Since $\dim_{\mathrm{nuc}}(\mathcal{Z}) = 1$ and $\dim_{\mathrm{nuc}}(U) = 0$, the claim follows from Proposition 5.20 for $\dim = \dim_{\mathrm{nuc}}$. ☐

In the proof of the next theorem, given C*-algebras $A$ and $B$, given $\varepsilon > 0$ and given a finite subset $F \subset A$, we say that a linear map $\gamma \colon A \to B$ is *$\varepsilon$-order zero* on $F$, if $\|\varphi(ab)\| < \varepsilon$ for all $a, b \in F$ satisfying $ab = a^*b = ab^* = a^*b^* = 0$.

**Theorem 5.39.** *Let $A$ be a unital Kirchberg algebra, and let $\theta \colon A \to \mathcal{O}_2$ be any unital embedding. Then $\dim_{\mathrm{oz}}(\theta) \leq 1$. Moreover, $\dim_{\mathrm{oz}}(\theta) = 1$ unless $A = \mathcal{O}_2$.*

*Proof.* Assume first that $A = \mathcal{O}_\infty$. As in the proof of Theorem 5.38, we may assume, without loss of generality, that $\theta$ is the first tensor factor embedding $\mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_2$. We will verify the finitary version of order zero dimension. To that effect, let $\varepsilon > 0$, and let $F \subset \mathcal{O}_\infty$ and $H \subset \mathcal{O}_2$ be finite subsets consisting of positive contractions.

Use [41, Lemma 4.17]—see also the first part of the proof of Theorem 3.3 in [3]—to find *-homomorphisms $\varphi_0, \varphi_1 \colon \mathcal{O}_2 \to \mathcal{O}_\infty$ and positive contractions $k_0, k_1 \in \mathcal{O}_2$ such that $\|\varphi_0(k_0) + \varphi_1(k_1) - 1_{\mathcal{O}_\infty}\| < \varepsilon/5$. Since $\mathcal{O}_\infty$ is isomorphic to its infinite tensor product, we may choose $\varphi_0$ and $\varphi_1$ to satisfy $\|\varphi_j(y)a - a\varphi_j(y)\| < \|y\|\varepsilon/5$ for $j = 0, 1$, for all $y \in \mathcal{O}_2$ and for all$a \in F$. (For instance, find $m \in \mathbb{N}$ and a finite subset $F' \subset \otimes_{j=1}^m \mathcal{O}_\infty \subset \otimes_{j=1}^\infty \mathcal{O}_\infty$, such that for every $a \in F$ there exists $a' \in F'$ with $\|a - a'\| < \varepsilon/5$. With $\iota_{m+1} \colon \mathcal{O}_\infty \to \otimes_{j=1}^\infty \mathcal{O}_\infty$ denoting the $(m+1)$-st tensor factor embedding, the maps $\varphi_j \circ \iota_{m+1}$, for $j = 0, 1$, will satisfy the condition above.) Likewise, since $\mathcal{O}_2$ is isomorphic to its infinite tensor product, we may also assume that $\|k_j b - b k_j\| < \varepsilon/5$ for $j = 0, 1$ and for all $b \in H$. Define completely positive contractive maps $\gamma_0, \gamma_1 \colon \mathcal{O}_\infty \otimes \mathcal{O}_2 \to \mathcal{O}_\infty$ on simple tensors as follows: for $x \in \mathcal{O}_\infty$ and for positive $y \in \mathcal{O}_2$, set

$$\gamma_j(x \otimes y) = \varphi_j(k_j)^{1/2} \varphi_j(y)^{1/2} x \varphi_j(y)^{1/2} \varphi_j(k_j)^{1/2}, \quad \text{for } j = 0, 1.$$

We claim that $\gamma_0$ and $\gamma_1$ are $\varepsilon$-order zero on $F \otimes H$, that $((\gamma_0 + \gamma_1) \circ \theta)(a) \approx_\varepsilon a$, and that $\gamma_j(a) \approx_\varepsilon \gamma_j(1)a$ for all $j = 0, 1$ and all $a \in F$. To show the first part of the claim, it is enough to observe that when $x \in F$ and $y \in H$, we have $\gamma_j(x \otimes y) \approx_{4\varepsilon/5} \varphi_j(k_j y)x$ for $j = 0, 1$. For the second one, a similar reasoning applies, since for $a \in F$ we have $\gamma_j(a \otimes 1_{\mathcal{O}_2}) \approx_{2\varepsilon/5} \varphi_j(k_j)a$, and hence

$$(\gamma_0 + \gamma_1)(a \otimes 1_{\mathcal{O}_2}) \approx_{4\varepsilon/5} (\varphi_0(k_0) + \varphi_1(k_1))a \approx_{\varepsilon/5} a,$$

as desired. The third part of the claim also follows, since we have $\gamma_j(a) \approx_{2\varepsilon/5} \varphi_j(k_j)a = \gamma_j(1)a$ for $j = 0, 1$ and for $a \in F$. This proves the result for $A = \mathcal{O}_\infty$.

When $A$ is an arbitrary Kirchberg algebra, the claim follows from the first part of the proof and part (2) of Proposition 5.4, together with Kirchberg's absorption theorems $A \otimes \mathcal{O}_\infty \cong A$ and $A \otimes \mathcal{O}_\infty \cong \mathcal{O}_2$.

When $A = \mathcal{O}_2$, then any inclusion into $\mathcal{O}_2$ is approximately unitarily equivalent to the identity, which clearly has order zero dimension zero. Since having a positively existential embedding into $\mathcal{O}_2$ implies absorbing $\mathcal{O}_2$, it follows that $\dim_{\mathrm{oz}}(\theta) = 1$ whenever $A$ is not $\mathcal{O}_2$. □

In particular, we recover from Theorem 5.39 the following dimensional estimate from [68, Theorem 7.1]. The actual nuclear dimension of Kirchberg algebras has recently been computed in [11, Theorem G]: it is always 1. We nevertheless present this consequence to illustrate the applicability of our techniques.

**Corollary 5.40.** Let $A$ be a Kirchberg algebra. Then $\dim_{\mathrm{nuc}}(A) \leq 3$.

*Proof.* This follows immediately from Theorem 5.39, Proposition 5.20, and the fact that $\dim_{\mathrm{nuc}}(\mathcal{O}_2) = 1$. □

In the next result, we endow $\mathcal{Z}, \mathcal{O}_2, \mathcal{O}_\infty$ and the UHF-algebra with the trivial $G$-action, and we endow all tensor products with the diagonal action.

**Theorem 5.41.** *Let $A$ be a $G$-C*-algebra, and let* dim *be a positively $\forall\exists$-axiomatizable dimension function for $G$-C*-algebra. Let $U$ be a UHF-algebra of infinite type. Then*

$$\dim(A \otimes \mathcal{Z}) \leq 2\dim(A \otimes U) + 1 \quad \textit{and} \quad \dim(A \otimes \mathcal{O}_\infty) \leq 2\dim(A \otimes \mathcal{O}_2) + 1.$$

*Proof.* This is a consequence of Theorem 5.38, Theorem 5.39, part (2) of Proposition 5.4, and Proposition 5.20. □

We want to highlight two important consequences of Theorem 5.41. One of them is obtained by letting dim be the Rokhlin dimension of a compact group action. In this case, and again endowing $\mathcal{Z}, \mathcal{O}_2, \mathcal{O}_\infty$ and the UHF-algebra with the trivial $G$-action, and all tensor products with the diagonal action, we deduce the following dimensional inequalities (compare with Section 4 of [41]).

**Corollary 5.42.** Let $G$ be a compact group, let $A$ be a $G$-C*-algebra, and let $U$ be a UHF-algebra of infinite type. Then

$$\dim_{\mathrm{Rok}}(A \otimes \mathcal{Z}) \leq 2\dim_{\mathrm{Rok}}(A \otimes U) + 1$$

and

$$\dim_{\mathrm{Rok}}(A \otimes \mathcal{O}_\infty) \leq 2\dim_{\mathrm{Rok}}(A \otimes \mathcal{O}_2) + 1.$$

The other consequence is obtained be letting dim be either the nuclear dimension or the decomposition rank. The estimates involving nuclear dimension have previously been observed in [3, Section 3].

**Corollary 5.43.** Let $A$ be a C*-algebra, and let $U$ be any UHF-algebra of infinite type. Then

$$\dim_{\mathrm{nuc}}(A\otimes\mathcal{Z})\leq 2\dim_{\mathrm{nuc}}(A\otimes U)+1 \quad \text{and} \quad \dim_{\mathrm{nuc}}(A\otimes\mathcal{O}_\infty)\leq 2\dim_{\mathrm{nuc}}(A\otimes\mathcal{O}_2)+1.$$

Furthermore,

$$\mathrm{dr}(A\otimes\mathcal{Z})\leq 2\mathrm{dr}(A\otimes U)+1.$$

5.8. **Rokhlin dimension and strongly self-absorbing $G$-C*-algebras.** In this subsection we consider different actions on the same C*-algebra. If $\alpha$ is a continuous action of a compact group $G$ on a C*-algebra $A$, then we let $\dim_{\mathrm{Rok}}^{\mathrm{c}}(A,\alpha)$ be the Rokhlin dimension with commuting towers of the $G$-C*-algebra $(A,\alpha)$. The following is one of our main technical results.

**Theorem 5.44.** *Let $\alpha$ be a continuous action of a compact group $G$ on a C*-algebra $A$. If $\dim_{\mathrm{Rok}}^{\mathrm{c}}(A,\alpha)\leq d$, then $(A,\alpha)\precsim_d^{\mathrm{c}}(A,\iota_A)$.*

*Proof.* Fix a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$. We denote by $F_{\mathcal{U}}(A)$ the Kirchberg invariant of $(A,\iota_A)$ (endowed with the trivial action), and by $F_{\mathcal{U}}^G(A)$ the Kirchberg invariant of $(A,\alpha)$ (endowed with the canonical $G$-action obtained from $\alpha$). Since $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha)\leq d$, it follows from the reformulation of Rokhlin dimension in terms of commutant $d$-containment and Proposition 3.7 that there exist $G$-equivariant completely positive contractive order zero maps $\psi_0,\ldots,\psi_d\colon C(G)\to C'\cap F_{\mathcal{U}}^G(A)$ with pairwise commuting ranges, such that $\psi_0+\cdots+\psi_d$ is unital.

Fix a separable C*-subalgebra $C$ of $F_{\mathcal{U}}(A)$. When $G$ is finite, the maps witnessing that $(A,\alpha)\precsim_d^{\mathrm{c}}(A,\iota_A)$ can be constructed explicitly, so we outline this first. For $g\in G$, let $\delta_g\in C(G)$ be the characteristic function of $\{g\}$. Define maps $\eta_j\colon C\to F_{\mathcal{U}}^G(A)$, for $j=0,\ldots,d$, by $\eta_j(x)=\sum_{g\in G}\psi_j(\delta_g)\alpha_g(x)$. Then these maps witness the fact that $(A,\iota_A)$ is $G$-equivariantly commutant $d$-contained with commuting towers in $(A,\alpha)$.

Suppose now that $G$ is an arbitrary compact second countable group. Below, if $a$ and $b$ are elements of a C*-algebra and $\varepsilon>0$, we write $a\approx_\varepsilon b$ to mean that $\|a-b\|<\varepsilon$. Let $\rho$ be a left invariant metric on $G$. Fix a finite subset $F$ of positive elements of $C$ and $\varepsilon>0$. The argument in [39, Proposition 2.11] shows that there exist $\delta>0$, a finite subset $K$ of $G$, and a partition of unity $(f_g)_{g\in K}$ of $G$ satisfying:

(1) $f_g\in C(G)$ is a positive contraction for all $g\in G$;
(2) $f_g$ and $f_h$ are orthogonal whenever $g,h\in G$ satisfy $\rho(g,h)>\delta$;
(3) for every $a\in F$, we have $\alpha_g(a)\approx_\varepsilon\alpha_h(a)$ whenever $g,h\in G$ satisfy $\rho(g,h)<\delta$; and
(4) $\alpha_h(\sum_{g\in K}\psi_j(f_g)a)\approx_\varepsilon\sum_{g\in K}\psi_j(f_g)a$ for all $h\in G$ and all $a\in F$.

Define now $\eta_j\colon C\to F_{\mathcal{U}}^G(A)$ by $\eta_j(x)=\sum_{g\in K}\psi_j(f_g)\alpha_g(x)$ for $j=0,\ldots,d$. Observe that $\eta_0,\ldots,\eta_d$ are completely positive contractive maps with commuting ranges. Furthermore, for every $0\leq j\leq d$, if $a,b\in F$ satisfy $ab\approx_\varepsilon 0$, then (1) and (2) imply that

$$\eta_j(a)\eta_j(b)=\sum_{g,h\in K}\psi_j(f_g)\psi_j(f_h)\alpha_g(a)\alpha_h(b)\approx_\varepsilon\sum_{g,h\in K}\psi_j(f_g)\psi_j(f_h)\alpha_g(ab)\approx_\varepsilon 0.$$

By (3), we have $\alpha_g(\eta_j(a))\approx_\varepsilon\eta_j(a)$ for every $g\in G$, every $j=0,\ldots,d$, and every $a\in F$. Since $\varepsilon>0$ and $F\subset C_+$ are arbitrary, it follows from Proposition 3.7 that there exist $G$-equivariant completely positive contractive order zero maps $\eta_0,\ldots,\eta_d\colon C\to F_{\mathcal{F}}^G(A)$ with commuting ranges and unital sum. Since this is true for every separable C*-subalgebra $C$ of $F_{\mathcal{F}}(A)$, we conclude that $(A,\iota_A)\precsim_d^{\mathrm{c}}(A,\alpha)$, as desired. ☐

The following corollary is then a consequence of Theorem 5.44 and Proposition 5.19.

**Corollary 5.45.** Let dim be a dimension function for $G$-C*-algebras that is commutant positively existentially axiomatizable with commuting towers. Suppose that $A$ is a C*-algebra, $\alpha$ is a continuous action of a compact group $G$ on $A$, and $\iota_A$ is the trivial $G$-action on $A$. Then

$$\dim(A,\alpha)+1\leq(\dim_{\mathrm{Rok}^{\mathrm{c}}}(A,\alpha)+1)(\dim(A,\iota_A)+1).$$

We now arrive at one of the main results of this section. It asserts that, for a compact group $G$ and a strongly self-absorbing C*-algebra, $G$-actions with finite Rokhlin dimension with commuting towers, on $D$-absorbing C*-algebras, automatically absorb the trivial action on $D$.

**Theorem 5.46.** *Let $D$ be a strongly self-absorbing C*-algebra, let $A$ be a separable $D$-absorbing C*-algebra, and let $\alpha\colon G\to\mathrm{Aut}(A)$ be an action of a compact group $G$ with $\dim_{\mathrm{Rok}}^{\mathrm{c}}(A,\alpha)<\infty$. Then $(A,\alpha)$ is $G$-equivariantly $(D,\iota_D)$-absorbing.*

*Proof.* Let $\dim_D$ be the $\{0, \infty\}$-valued dimension function for $G$-C*-algebras which is finite if and only if the given $G$-C*-algebra is $(D, \iota_D)$-absorbing. The action $(D, \iota_D)$ is unitarily regular since it absorbs $(\mathcal{Z}, \iota_{\mathcal{Z}})$ tensorially; see [78, Proposition 1.20]. It follows from this and Proposition 5.35 that $\dim_D$ is commutant positively existentially axiomatizable with commuting towers. The result now follows from Corollary 5.45. □

**Corollary 5.47.** Let $D$ be a strongly self-absorbing C*-algebra, let $G$ be a compact group, and let $(A, \alpha)$ be a separable $G$-C*-algebra with $\dim^{c}_{\mathrm{Rok}}(A, \alpha) < \infty$. If $A$ is (nonequivariantly) $D$-absorbing, then so are $A^G$ and $A \rtimes G$.

*Proof.* By Theorem 5.46, there is a $G$-equivariant isomorphism between $(A, \alpha)$ and $(A \otimes D, \alpha \otimes \iota_D)$. Upon taking crossed products, we deduce that

$$A \rtimes_\alpha G \cong (A \otimes D) \rtimes_{\alpha\otimes\iota_D} G \cong (A \rtimes_\alpha G) \otimes D,$$

so $A \rtimes_\alpha G$ is $D$-absorbing. The same argument applies to the fixed point algebra, since we have $A^\alpha \cong (A \otimes D)^{\alpha\otimes\iota_D} = A^\alpha \otimes D$. □

Corollary 5.47 is a significant generalization of previously known results concerning Jiang-Su absorption: for finite groups this was shown by Hirshberg-Winter-Zacharias in [57, Theorem 5.9], and in [44, Theorem 5.4.4] by first-named author for compact groups. Similar results have been independently obtained in [45].

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