MATROIDS OVER ONE-DIMENSIONAL GROUPS

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Abstract. We develop the theory of matroids over one-dimensional algebraic groups, with special emphasis on positive characteristic. In particular, we compute the Lindström valuations and Frobenius flocks of such matroids. Building on work by Evans and Hrushovski, we show that the class of algebraic matroids, paired with their Lindström valuations, is not closed under duality of valuated matroids.

1. Introduction

Given an algebraically closed field K and a natural number n, any irreducible subvariety $X \subseteq K^n$ gives rise to a matroid $M(X)$ on the ground set $[n] := \{1, \dots, n\}$ by declaring $I \subseteq [n]$ to be independent if the coordinate projection $X \to K^I$ is dominant. The variety X is an *algebraic representation* of $M(X)$ and $M(X)$ is called an *algebraic matroid*.

The best-understood algebraic matroids are the *linear matroids*, which are those coming from linear subvarieties $X \subseteq K^n$. However, the class of algebraic matroids is larger than just linear matroids. For example, in the first paper to study algebraic representability of matroids, Ingleton gave an algebraic representation of the non-Fano matroid, over any field, using a variety $X \subseteq K^7$ parametrized by monomials [Ing71, Ex. 15], which we would now call a toric variety. More generally, such parametrizations can be used to show that any linear matroid over $\mathbb{Q}$ is algebraic over any field (this construction is reviewed in Example 4).

In fact, linear spaces and toric varieties have a common generalization to an arbitrary connected, one-dimensional algebraic group G. The key point is that the linear parametrization of a linear space and the monomial parametrization of a toric variety can be replaced by any homomorphism of algebraic groups $\Psi : G^d \to G^n$ with image X, a closed and connected subgroup of G^n. The homomorphism Ψ is described by an $n \times d$ matrix with elements in the ring $\mathbb{E}$ of endomorphisms of the group G. For all one-dimensional groups G, $\mathbb{E}$ is both a left and right Ore domain, so it is contained in a division ring Q (generated by $\mathbb{E}$), and the algebraic matroid represented by X has a linear representation over Q. Here, we write $M(X)$ for the matroid whose bases are all sets $I \subseteq [n]$ such that the projection of X to G^I is all of G^I. This matroid is equivalent to the algebraic matroid defined in the first paragraph because, if we choose a non-constant rational function $h \colon G \dashrightarrow K$ and let $Y \subset K^n$ be the coordinatewise image $h^n(X)$, then $M(X) = M(Y)$.

Theorem 1. *Let G be a connected, one-dimensional algebraic group over an algebraically closed field K. Then the maps in Figure 1 are bijections and the diagram commutes. Furthermore, for a closed, connected subgroup $X \subseteq G^n$ the set $M(X)$ is a matroid and coincides with the linear matroid on $[n]$ defined by the right vector space $P^{-1}(X)Q \subset Q^n$.*

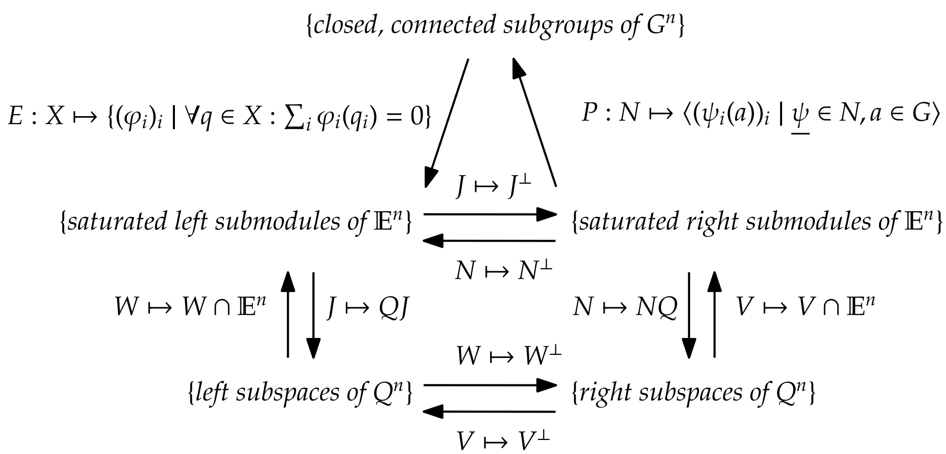

FIGURE 1. The diagram of Theorem 1.

The map P in Figure 1 sends N to the subgroup of G^n generated by the elements of the form $(\psi_i(a))_i$ with $a \in G$ and $\underline{\psi}$ running through N. We think of the right modules N and V as giving *parametrizations* of X and of the left modules J and W as giving *equations* for X, whence the notation E and P. The ring $\mathbb{E}$ is left and right Noetherian, so a right module $N \subseteq \mathbb{E}^n$ has a finite generating set. Use these vectors as the columns of an $n \times d$-matrix Ψ. Then Ψ gives a natural group homomorphism $G^d \to G^n$; $P(N)$ is the image of this homomorphism. A different choice of generators yields a different matrix Ψ with the same column space and a different homomorphism $G^{d'} \to G^n$ with the same image.

We note that our use of the column space of an $n \times m$ matrix Ψ, in order to define a matroid on the ground set $[n]$, differs from the conventional use of row spaces in matroid theory, which define matroids on the set of columns. However, in our construction of Ψ, it is the rows which are labeled by $[n]$, because Ψ is a matrix defining a group homomorphism to G^n. Since $\mathbb{E}$ is possibly non-commutative, the column space of Ψ is not equivalent to the row space of its transpose, and so we use the column space to define our matroids consistently.

The possible one-dimensional groups G are the additive group $\mathbb{G}_a = (K, +)$, the multiplicative group $\mathbb{G}_m = (K^*, \cdot)$, and any elliptic curve over K. In characteristic 0, their endomorphism rings are K, $\mathbb{Z}$, and either $\mathbb{Z}$ or an order in an imaginary quadratic number field, respectively. In characteristic $p > 0$, there are additional possibilities for the endomorphism ring, both of which are non-commutative. The endomorphism ring of $\mathbb{G}_a$ is the ring $K[F]$ of p-polynomials, and the endomorphism ring of an elliptic curve may be a subring of a quaternion ring. These non-commutative rings give examples of non-linear matroids which are algebraic over all fields of positive characteristic [Lin86a]. All of our results are formulated uniformly over the three different types of algebraic groups, and only in the proofs do we sometimes distinguish between them.

Because of Theorem 1, we call the matroids isomorphic to $M(X)$ for some closed, connected $X \subseteq G^n$ $\mathbb{E}$-*linear*. An $\mathbb{E}$-linear matroid M admits an algebraic representation in the sense of the first paragraph of this paper: choosing a non-constant rational function $h : G \dashrightarrow K$ defined near $0 \in G$ we obtain a rational map $h^n : G^n \dashrightarrow K^n$, and the variety $Y := \overline{h^n(X)} \subseteq K^n$ has $M(Y) = M$.

Theorem 2. *The class of* $\mathbb{E}$*-linear matroids is closed under contraction, deletion, and duality.*

Example 3. Let $G = \mathbb{G}_a$ be the additive group over $K = \overline{\mathbb{F}_2}$. Then $\mathbb{E}$ is isomorphic to the skew polynomial ring $K[F]$ in which multiplication is governed by the rule $Fa = a^2F$. The (right) column space N of the matrix

$$\Psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & F \end{pmatrix},$$

is saturated in $K[F]^4$, and $P(N) = \{(a, b, a + b, a + b^2) \mid (a, b) \in \mathbb{G}_a^2\}$. The matroid $M(P(N))$ is the uniform matroid $U_{2,4}$. ♣

Example 4. Let $G = \mathbb{G}_m$ be the multiplicative group over K. Then $\mathbb{E} \cong \mathbb{Z}$ via the map $\mathbb{Z} \to \mathbb{E}, a \mapsto (t \mapsto t^a)$. Any matrix $\Psi \in \mathbb{Z}^{n\times d}$ of full rank d gives rise to an algebraic group homomorphism $\Psi : \mathbb{G}_m^d \to \mathbb{G}_m^n, t = (t_1, \ldots, t_d) \mapsto (t_1^{\psi_{i1}} \cdots t_d^{\psi_{id}})_{i=1}^n$ whose image X is a d-dimensional subtorus of the n-dimensional torus $\mathbb{G}_m^n$. The matroid $M(X)$ is equal to the linear matroid over $\mathbb{Q}$ in which $I \subseteq [n]$ is an independent set if and only if the corresponding rows of Ψ are linearly independent.

This classical argument shows that matroids linear over $\mathbb{Q}$ are algebraic over any field. The group homomorphism $\Psi : \mathbb{G}_m^d \to \mathbb{G}_m^n$ is a closed embedding if and only if the $\mathbb{Z}$-column space of Ψ is a saturated submodule of $\mathbb{Z}^n$. Our framework is a common generalization of linear matroids and these toric matroids. ♣

Positive characteristic. In characteristic zero, by Ingleton's theorem [Ing71], the class of matroids that admit an $\mathbb{E}$-linear representation is the same as the class of matroids that admit a K-linear representation. Therefore, we next specialize to characteristic $p > 0$, and we study $\mathbb{E}$-linear matroids from the perspective of the theory developed in [BDP18, Car18]. There, for any irreducible variety $Y \subseteq K^n$, a canonical matroid valuation on $M(Y)$ is constructed, called the *Lindström valuation*. Furthermore, in [BDP18], a so-called *Frobenius flock* is associated to the pair consisting of Y and a sufficiently general point of Y. Here, in § 3.7, we will define the Frobenius flock of a closed, connected subgroup $X \subseteq G^n$, use it to define the Lindström valuation on $M(X)$, and relate these definitions to the constructions of [BDP18] via a rational function $G \dashrightarrow K$ as above. The goal is, then, to express these invariants of X in the data of Theorem 1.

To this end, we proceed as follows. In § 3.2 we construct a canonical valuation $v \colon \mathbb{E} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that $v(\alpha) > 0$ if and only if α is an inseparable morphism $G \to G$. A routine check shows that v extends to Q. Then, in § 3.3, we describe a general construction of a linear flock over K from a right vector subspace $V \subset Q^n$, and of a compatible valuation on the linear matroid on $[n]$ determined by V. These are related to the Frobenius flock and the Lindström valuation of X and to [BDP18] as follows.

Theorem 5. *Let X be a closed, connected subgroup of G^n and let $N = P^{-1}(X)$ be the saturated right submodule of $\mathbb{E}^n$ representing X. Then the Frobenius flock of X equals the linear flock of NQ, and the Lindström valuation of X equals the matroid valuation corresponding to NQ. Furthermore, let $h : G \dashrightarrow K$ be a rational function defined near 0 and set $Y := \overline{h^n(X)}$. Under mild conditions on h, the point $0 \in Y$ satisfies the genericity condition (*) from* [BDP18]*, and $d_0 h^n$ restricts to a linear bijection between the Frobenius flock of X and the Frobenius flock of* $(Y, 0)$.

By Theorem 2, the algebraic matroids arising from groups are closed under duality. However, the method used to prove this duality does not dualize the Lindström valuation in the sense of [DW92, Proposition 1.4]. On the contrary, using results from [EH91], we prove:

Theorem 6. *The class of algebraic matroids equipped with their Lindström valuations is not closed under duality of valuated matroids. In particular, if K has positive characteristic, then there exists a closed, connected subgroup $X \subset \mathbb{G}_a^n$, with algebraic matroid M and Lindström valuation w, such that the dual valuated matroid (M^*, w^*) is not the algebraic matroid and Lindström valuation of any variety $Y \subset K^n$.*

Relation to existing literature. As noted, the use of endomorphisms of $\mathbb{G}_m$ appears as an example in [Ing71]. The systematic use of endomorphisms of $\mathbb{G}_a$ (dubbed *p-polynomials*) in matroid theory first appeared in [Lin88], though particular cases were studied in [Lin86a] and [Lin86b], which used p-polynomials to show that the non-Pappus matroid is algebraic over any field of positive characteristic. The first uniform treatment of endomorphism rings of all possible one-dimensional algebraic groups is in [EH91], which used model theory to show that each algebraic representation of certain matroids would have to be algebraically equivalent to an $\mathbb{E}$-linear representation for some one-dimensional algebraic group G with endomorphism ring $\mathbb{E}$.

Organization. The remainder is organized as follows. In Section 2 we recall basic facts about one-dimensional algebraic groups G, their endomorphisms rings $\mathbb{E}$, and submodules of $\mathbb{E}^n$; and we prove Theorems 1 and 2. In Section 3, we zoom in on characteristic p and establish Theorem 5. Finally, in Section 5 we give several examples of $\mathbb{E}$-linear matroids with G equal to $\mathbb{G}_m$, $\mathbb{G}_a$, or an elliptic curve and prove Theorem 6.

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2. Arbitrary characteristic

2.1. **One-dimensional algebraic groups.** Let G be a one-dimensional algebraic group over the algebraically closed field K. Thus, G is either the multiplicative group $\mathbb{G}_m = (K^*, \cdot)$, the additive group $\mathbb{G}_a = (K, +)$, or an elliptic curve (for a classification of the affine one-dimensional groups see [Bor91, Theorem III.10.9];

Hurwitz' automorphism theorem rules out that the genus of G is strictly greater than 1). In particular, G is Abelian. We write the operation in G additively in the uniform treatment of these cases.

2.2. **The endomorphism ring.** The set $\mathbb{E} := \mathrm{End}(G)$ of endomorphisms of G as an algebraic group is a ring with operations $\cdot = \circ$ (composition) as multiplication and $(\varphi + \psi)(g) := \varphi(g) + \psi(g)$ as addition; for $\varphi + \psi$ to be an endomorphism, the fact that G is Abelian is crucial.

If $G = \mathbb{G}_{\mathrm{m}}$, then $\mathbb{E} \cong \mathbb{Z}$ via the map $\mathbb{Z} \to \mathbb{E}$, $a \mapsto (t \mapsto t^a)$. If $G = \mathbb{G}_{\mathrm{a}}$ and $\mathrm{char}\, K = 0$, then $\mathbb{E} \cong K$ via the map $K \to \mathbb{E}, c \mapsto (d \mapsto cd)$. For $G = \mathbb{G}_{\mathrm{a}}$ in positive characteristic, $\mathbb{E}$ is the skew polynomial ring $K[F]$ whose elements are polynomials in F and multiplication is governed by the rule $Fa = a^pF$ for all $a \in K$, because $a \in K$ acts by scaling $\mathbb{G}_{\mathrm{a}}$ and F acts as the Frobenius operator. If G is an elliptic curve, then $\mathbb{E}$ is either the ring of integers, an order in an imaginary quadratic number field, or an order in a quaternion algebra (only in positive characteristic) [Sil09, Theorem V.3.1]. For any one-dimensional group G, its endomorphism ring $\mathbb{E}$ embeds into a division ring Q (generated by $\mathbb{E}$). This is clear from our description of the endomorphism rings, except for the case of p-polynomials, which is proved in [Lin88].

2.3. **Submodules of $\mathbb{E}^n$.** Since $\mathbb{E}$ is, in general, a noncommutative ring, we distinguish between left submodules J and right submodules N of $\mathbb{E}^n$. We define QJ (respectively, NQ) to be the left (respectively, right) Q-subspace of Q^n generated by J. The dimension of this vector space is called the *rank* $\mathrm{rk}\, J$ and $\mathrm{rk}\, N$ of J and N, respectively. We call $J^{\mathrm{sat}} := QJ \cap \mathbb{E}^n$ and $N^{\mathrm{sat}} := NQ \cap \mathbb{E}^n$ the *saturations* of J and N, and we call J and N *saturated* if they are equal to their saturations. Hence J is saturated if and only if the following holds: if $\alpha\underline{\varphi} \in J$ for some $\underline{\varphi} \in \mathbb{E}^n$ and $\alpha \in \mathbb{E} \setminus \{0\}$, then $\underline{\varphi} \in J$; and similarly for N. Furthermore, $J \mapsto QJ$ and $N \mapsto NQ$ are bijections between saturated submodules of $\mathbb{E}^n$ and Q-subspaces in Q^n. In particular, if $J \subseteq J'$ are both saturated and $\mathrm{rk}\, J = \mathrm{rk}\, J'$, then $J = J'$; and similarly for N.

2.4. **Orthogonal complements.** We have the natural pairing $\mathbb{E}^n \times \mathbb{E}^n \to \mathbb{E}$:

$$\langle \underline{\varphi}, \underline{\psi} \rangle := \sum_i \varphi_i \psi_i.$$

This pairing is left-$\mathbb{E}$ linear in the first argument and right-$\mathbb{E}$-linear in the second argument. The orthogonal complement $J^\perp$ of a left $\mathbb{E}$-submodule $J \subseteq \mathbb{E}^n$ is:

$$J^\perp = \left\{ \underline{\psi} \mid \sum_i \forall \underline{\varphi} \in J : \langle \underline{\varphi}, \underline{\psi} \rangle = 0 \right\};$$

it is a saturated right $\mathbb{E}$-module whose rank equals $n - \mathrm{rk}\, J$, and similarly for the orthogonal complement of a right submodule N. The operation $\perp$ also extends to left or right subspaces of Q^n.

2.5. **Connected subgroups.**

Lemma 7. *Let X be a closed, connected subgroup of G^n of dimension d. Then there exist surjective algebraic group homomorphisms $\alpha\colon X \to G^d$ and $\beta\colon G^d \to X$ and a natural number e such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are the multiplication with e maps on G^d and X, respectively. Moreover, if $G = \mathbb{G}_{\mathrm{a}}$ or $G = \mathbb{G}_{\mathrm{m}}$, then e can be taken equal to* 1.

Proof. For $G = \mathbb{G}_m$, this follows from [Bor91, Proposition III.8.5]. For $G = \mathbb{G}_a$ in characteristic zero, a closed, connected subgroup of $\mathbb{G}_a^n$ is just a linear subspace of K^n, and the result is basic linear algebra. For $G = \mathbb{G}_a$ in positive characteristic, the result follows from [CGP10, Lemma B.1.10].

Now assume that G is an elliptic curve. Then the lemma follows from the fact that isogeny of Abelian varieties is an equivalence relation. We recall some details. Take any maximal subset $I \subseteq [n]$ such that the projection $\alpha : X \to G^I$ is dominant. Since algebraic group homomorphisms have a closed image, α is surjective. Maximality implies that $|I| = d$, so that $N := \ker \alpha$ is finite. Let $e > 0$ be the exponent of N. Then the multiplication map $\gamma : X \to X, x \mapsto ex$ has $\ker \gamma \supseteq N$. Therefore γ factors as $\beta \circ \alpha$ for some algebraic group homomorphism $\beta : G^I \cong X/N \to X$. So $\beta \circ \alpha$ is multiplication by e. Conversely, for $a \in G^I$ there exists an $x \in X$ such that $\alpha(x) = a$, and we find

$$\alpha(\beta(a)) = \alpha(\beta(\alpha(x))) = \alpha(ex) = e\alpha(x) = ea,$$

so also $\alpha \circ \beta$ is multiplication by e. Since multiplication by e is surjective—here we use that X is connected—both α and β are surjective. □

Remark 8. The proof in the latter paragraph also works in the cases $G = \mathbb{G}_m$, except that it does not yield $e = 1$. It does not work for $G = \mathbb{G}_a$, since the exponent e of N in the proof might be p, so that multiplication with e is not surjective.

2.6. **Proof of Theorems 1 and 2.** The following lemmas establish Theorem 1.

Lemma 9. *The maps P and E in the diagram are well-defined.*

Proof. We begin with E. Let X be a closed, connected subgroup of G^n and define

$$J := E(X) = \{\underline{\varphi} \in \mathbb{E}^n \mid \forall q \in X : \sum_i \varphi_i(q_i) = 0\}.$$

A straightforward computation shows that J is a left $\mathbb{E}$-submodule of $\mathbb{E}^n$. To show that J is saturated, suppose that $\alpha\underline{\varphi} \in J$ with $\alpha \in \mathbb{E} \setminus \{0\}$ and $\underline{\varphi} \in \mathbb{E}^n$. Then for each $q \in X$, $\underline{\varphi}(q) := \sum_i \varphi_i(q_i)$ is in the kernel of the endomorphism α. This kernel is a closed subset of the one-dimensional variety G, and since $\alpha \neq 0$, we find that $\ker \alpha$ is a finite set of points. Since X is irreducible and since $\underline{\varphi}$, regarded as a map $G^n \to G$, is a morphism, $\underline{\varphi}(X) \subseteq \ker \alpha$ is a single point. This point is 0 since $\varphi(0) = 0$. Hence $\underline{\varphi} \in J$, and J is saturated as desired.

Next we show that P is well-defined. Let N be any right submodule of $\mathbb{E}^n$ (not necessarily saturated) and let $\underline{\psi}^{(1)}, \ldots, \underline{\psi}^{(m)}$ be a generating set of N. Write $\Psi = (\underline{\psi}^{(1)}, \ldots, \underline{\psi}^{(m)})$. We think of Ψ as an $n \times m$-matrix over $\mathbb{E}$. It gives rize to the group homomorphism

$$\Psi : G^m \to G^n, \quad (a_1, \ldots, a_m) \mapsto \sum_{j=1}^{m} \underline{\psi}^{(j)}(a_j), \tag{1}$$

whose image X is closed, connected subgroup of G^n. A straightforward computation shows that $X = P(N)$.

The two maps at the bottom in the diagram are clearly well-defined if J is any left $\mathbb{E}$-submodule of $\mathbb{E}^n$, then $J^\perp$ is a saturated right-submodule of $\mathbb{E}^n$, and vice versa. □

Lemma 10. *The maps $J \mapsto J^\perp$ and $N \mapsto N^\perp$ (when restricted to saturated modules) are inverse to each other.*

Proof. The module $(J^\perp)^\perp$ is a saturated left $\mathbb{E}$-submodule of $\mathbb{E}^n$ which on the one hand contains J and on the other hand has the same rank as J. Hence they are equal. The same argument applies to N. □

Lemma 11. *For any right $\mathbb{E}$-submodule N of $\mathbb{E}^n$, $P(N) = P(N^{\mathrm{sat}})$, and $\dim P(N) = \mathrm{rk}\, N$.*

Proof. From $N^{\mathrm{sat}} \supseteq N$ we immediately find $P(N^{\mathrm{sat}}) \supseteq P(N)$. Conversely, for $\psi \in N^{\mathrm{sat}}$ there exists an $\alpha \in \mathbb{E}$ such that $\psi\alpha \in N$, and since the map $\alpha \colon G \to G$ is surjective, $\mathrm{im}(\psi\alpha \colon G \to G^n) = \mathrm{im}(\psi \colon G \to G^n)$. This proves the first statement.

This shows that $P(N) = P(N')$ where $N' \subseteq N$ is any submodule of N of the same rank generated by vectors that are (right-)linearly independent over Q. The parametrization (1) shows that $\dim X \leq \mathrm{rk}\, N' = \mathrm{rk}\, N$. For the converse write $d := \mathrm{rk}\, N$. Then exists a subset $I \subseteq [n]$ with $|I| = d$ such that the projection of N in $\mathbb{E}^I$ has rank d, and one finds vectors $v_i \in N, i \in I$ such that v_i has nonzero entry α_i in position i and zero entries in positions $j \in I \setminus \{i\}$. Then the image of X in G^I contains the elements $(\delta_{ij}\alpha_i(G))_{j\in I}$ for every $i \in I$, and these generate G^I. So $\dim X \geq d$. □

Lemma 12. *The diagram in Theorem 1 commutes.*

Proof. We concentrate on the upper triangle; the lower square was discussed in § 2.4. Let $X \subseteq G^n$ be a closed and connected subgroup. By Lemma 7, X is the image of some homomorphism $\beta : G^d \to G^n$ where $d = \dim X$. Then $\beta = (\beta_1, \ldots, \beta_n)$ where $\beta_i : G^d \to G$ is a homomorphism. Write β_{ij} for the composition of the embedding $G \to G^d, a \mapsto (0, \ldots, 0, a, 0, \ldots, 0)$ (with a in the j-th position) and β_i. Then $\beta = (\beta_{ij})_{ij} \in \mathbb{E}^{n\times d}$; let N' be the right submodule of $\mathbb{E}^n$ generated by the columns of β. Then $X = P(N')$ by (1), and if we write $N = (N')^{\mathrm{sat}}$, then $X = P(N') = P(N)$ by Lemma 11.

Also by Lemma 11, $\mathrm{rk}\, N = d$. Set $J := N^\perp$, a left module of rank $n - d$. The orthogonality to N implies that J is contained in $E(X)$. Let $m \geq n - d$ be the rank of $E(X)$. Then there is a subset $I \subseteq [n]$ of size m such that the projection of $E(X)$ in $\mathbb{E}^I$ has rank m. Then $E(X)$ contains, for every $i \in I$, an element of the form $\alpha_i e_i + \sum_{j\in[n]\setminus I} \alpha_{ij} e_j$, where e_i is the i-th standard basis vector of $\mathbb{E}^n$. This implies that the projection $X \to G^{[n]\setminus I}$ is finite-to-one. In particular, $d = \dim X \leq n - m \leq d$, so equality must hold everywhere and $m = n - d$ and $E(X) = J$. Thus

$$X = P(N) = P((N^\perp)^\perp) = P(J^\perp) = P(E(X)^\perp).$$

To see that the triangle also commutes when we start at some saturated left $\mathbb{E}$-module $J \subseteq \mathbb{E}^n$, set $X := P(J^\perp)$ and note that $\dim X = n - \mathrm{rk}\, J$ (by Lemma 11) and $J \subseteq E(X)$. If $E(X)$ were strictly larger than J, then, since they are both saturated, $\mathrm{rk}\, E(X) > \mathrm{rk}(J)$, but then $\mathrm{rk}(E(X)^\perp) < \mathrm{rk}(J^\perp)$ and

$$n - \mathrm{rk}\, J = \dim X = \dim(P(E(X)^\perp)) = \mathrm{rk}(E(X)^\perp) < n - \mathrm{rk}\, J,$$

a contradiction (in the second equality we use that the triangle commutes when starting at X). The same reasoning applies when we start at some saturated right $\mathbb{E}$-module $N \subseteq \mathbb{E}^n$. □

Proof of Theorem 1. By Lemma 12, the diagram of Figure 1 commutes. Now, let $X \subseteq G^n$ be a closed, connected subgroup, and set $N := P^{-1}(X) \subseteq \mathbb{E}^n$. Let $I \subseteq [n]$, let X_I be the image of X in G^I (X_I is a closed, connected subgroup), and N_I be the saturation of the image of N in $\mathbb{E}^I$. Now we have $X_I = P(N_I)$ and hence $\dim X_I = \mathrm{rk}(N_I) = \dim_Q N_I Q$ by Lemma 11. This proves that $M(X)$ is (a matroid and) equal to the linear matroid on $[n]$ defined by NQ. □

Proof of Theorem 2. Let $A = M(X)$ for some closed, connected subgroup $X \subseteq G^n$. Let $N := E(X)^\perp \subseteq \mathbb{E}^n$. Then the right Q-vector space $V := NQ$ of Q^n determines the same matroid A by Theorem 1. Now the results on deletion and contraction follow from linear algebra over Q.

For duality, we argue that Q has an anti-automorphism τ. When Q is commutative, we take $\tau = 1_Q$. When $G = \mathbb{G}_a$ in characteristic $p > 0$, there is a unique anti-isomorphism $\tau : \mathbb{E} = K[F] \to K[F^{-1}] \subseteq Q$ that sends F to F^{-1} and that is the identity on K. This extends uniquely to an anti-automorphism of Q ($\mathbb{E}$ itself does not have a natural anti-automorphism in this case!). When G is a supersingular elliptic curve in characteristic $p > 0$, we use [Sil09, III.9, item (ii)].

Now $V^\perp$ is a left vector space over Q determining the matroid dual to A, and $\tau(V^\perp) \subseteq Q^n$ a right vector space also determining the matroid dual to A. Let $N' := \tau(V^\perp) \cap \mathbb{E}^n$. This is a saturated right module, and the group $X := P(N')$ determines the matroid dual to E. □

Example 13. Take M and Ψ as in Example 3. The orthogonal complement $M^\perp$ is the left submodule of $K[F]^4$ given as the row space of the matrix

$$\Psi^\perp = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & F & 0 & 1 \end{pmatrix}.$$

The recipe in the proof of Theorem 2 involves taking the right Q-vector space spanned by the columns of

$$\begin{pmatrix} 1 & 1 \\ 1 & F^{-1} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and intersect with $K[F]^4$. This yields the right $K[F]$-module generated by the columns of

$$\begin{pmatrix} 1 & F \\ 1 & 1 \\ 1 & 0 \\ 0 & F \end{pmatrix}.$$

The corresponding subgroup is $\{(a + b^2, a + b, a, b^2) \mid (a, b) \in K^2\}$, a representation of the dual of $U_{2,4}$, which of course is also $U_{2,4}$. ♣

3. Positive characteristic

3.1. **One-dimensional algebraic groups over $\mathbb{F}_p$.** In this section we let p be a fixed prime number, K an algebraically closed field of characteristic p, and G a one-dimensional algebraic group over K. As before, $\mathbb{E}$ denotes the endomorphism ring of G.

3.2. **Valuation on $\mathbb{E}$.** We begin by constructing a valuation on $\mathbb{E}$. Valuations are well-known in commutative algebra, but they can also be generalized to non-commutative rings [Sch45]. We will only use the case of value group $\mathbb{Z}$.

Definition 14. A *valuation* on a ring R is a function $v\colon R \to \mathbb{Z} \cup \{\infty\}$ such that:

- For all $a, b \in R$, $v(ab) = v(a) + v(b)$.
- For all $a, b \in R$, $v(a + b) \geq \min\{v(a), v(b)\}$.

In order to define $v\colon \mathbb{E} \to \mathbb{Z} \cup \{\infty\}$, we let $\alpha \in \mathbb{E}$ be a nonzero endomorphism of G and then obtain an injective homomorphism of function fields $\alpha^*\colon K(G) \to K(G)$ of function fields. If we let L be the set of elements of $K(G)$ which are purely inseparable over $\alpha^* K(G)$, then $L/\alpha^* K(G)$ is a purely inseparable extension of fields, and $K(G)/L$ is a separable extension. The degree $[L : \alpha^* K(G)]$ is a power of p, and is called the inseparable degree of the extension $K(G)/\alpha^* K(G)$, and denoted $[K(G) : \alpha^* K(G)]_i$ [Lan02, §V.6]. We define our valuation by $v(\alpha) = \log_p [K(G)\colon \alpha^* K(G)]_i$, and $v(0) = \infty$.

The main technical point to proving that v is a valuation is the following:

Lemma 15. *If K' is a subfield of $K(G)$ such that $K(G)/K'$ is purely inseparable, then $K' = F^n K(G)$ for some integer n, where F is the Frobenius endomorphism of $K(G)$.*

Proof. Let x and y be two elements in $K(G) \setminus K$. Since $K(G)/K$ has transcendence rank 1, x and y are algebraically dependent, meaning that they satisfy a polynomial relation $f(x, y)$, where $f \in K[X, Y]$. If every exponent of f is divisible by p, then $f = g^p$, using the fact that K is algebraically closed. We can assume that f is irreducible, which means that at least one exponent is not divisible by p. Therefore, either x is separable over $K(y)$ or y is separable over $K(x)$. Thus, the purely inseparable subfields of $K(G)$ are totally ordered by inclusion.

The fields $K(G) \supset FK(G) \supset F^2 K(G) \supset \ldots$ also form a chain of purely inseparable subfields of $K(G)$. Each containment has index p, so there can't be any intermediate fields. Since $K(G)/K'$ is purely inseparable, then $K' \neq K$, and so $K'(G)/K$ has finite index. Therefore, the purely inseparable subfield K' must be $F^n K(G)$ for some n. □

Proposition 16. *The function $v\colon \mathbb{E} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ defines a valuation on $\mathbb{E}$ that extends uniquely to Q.*

Proof. In order to show that v is a valuation on $\mathbb{E}$, we first note that $v(\alpha\beta) = v(\alpha) + v(\beta)$ because of the multiplicativity of inseparable degree [Lan02, Cor. V.6.4].

Second, we want to show that if α and β are elements of $\mathbb{E}$, then $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$. By Lemma 15, $\alpha^* K(G)$ and $\beta^* K(G)$ are contained in $F^{v(\alpha)} K(G)$ and $F^{v(\beta)} K(G)$, respectively, and thus both are contained in $F^{\min\{v(\alpha), v(\beta)\}} K(G)$. Since $(\alpha + \beta)^* K(G)$ is contained in the compositum of $\alpha^* K(G)$ and $\beta^* K(G)$, then it is contained in $F^{\min\{v(\alpha, v(\beta)\}} K(G)$, and thus $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$.

Next we want to show that v extends to a unique valuation on Q, which is an exercise in working with rings of fractions and the Ore condition. We refer to [Coh95, Chapter 1] for an introduction to Ore domains. Every element of Q can be written as a fraction $\varphi^{-1}\psi$ for some $\varphi, \psi \in \mathbb{E}$, and we define $v(\varphi^{-1}\psi) = -v(\varphi) + v(\psi)$. If we write $\varphi^{-1}\psi$ instead as $(\tau\varphi)^{-1}\tau\psi$ for some $\tau \in \mathbb{E}$, then

$$v((\tau\varphi)^{-1}\tau\psi) = -v(\tau\varphi) + v(\tau\psi) = -v(\tau) - v(\varphi) + v(\tau) + v(\psi) = -v(\varphi) + v(\psi) = v(\varphi^{-1}\psi),$$

which shows that this valuation is well-defined. Since the valuation is defined using elements of $\mathbb{E}$ and their reciprocals, this is the unique valuation which extends the one on $\mathbb{E}$.

Now we want to show that v defines a valuation on Q. Again, we write $\varphi^{-1}\psi$ and $\sigma^{-1}\tau$ for elements of Q, where φ, ψ, σ, and τ are in $\mathbb{E}$. Then we take their product by finding ψ' and σ' in $\mathbb{E}$ such that $\sigma'\psi = \psi'\sigma$ (the existence of such elements is the left Ore condition), and then $(\varphi^{-1}\psi)(\sigma^{-1}\tau) = (\sigma'\varphi)^{-1}\psi'\tau$. The valuation of this product is:

$$\begin{aligned} v((\sigma'\varphi)^{-1}\psi'\tau) &= -v(\sigma'\varphi) + v(\psi'\tau) = -v(\sigma') - v(\varphi) + v(\psi') + v(\tau) \\ &= -v(\varphi) - v(\sigma) + v(\psi) + v(\tau) = v(\varphi^{-1}\psi) + v(\sigma^{-1}\tau), \end{aligned}$$

where the first equality on the second line is by our assumption that $\sigma'\psi = \psi'\sigma$.

Second, to compute the sum of $\varphi^{-1}\psi$ and $\sigma^{-1}\tau$, we find φ' and σ' in $\mathbb{E}$ such that $\sigma'\varphi = \varphi'\sigma$ and then:

$$\varphi^{-1}\psi + \sigma^{-1}\tau = (\sigma'\varphi)^{-1}(\sigma'\psi + \varphi'\tau).$$

If we take the valuation of this sum, we get:

$$\begin{aligned} v((\sigma'\varphi)^{-1}(\sigma'\psi + \varphi'\tau)) &= -v(\sigma') - v(\varphi) + v(\sigma'\psi + \varphi'\tau) \\ &\geq -v(\sigma') - v(\varphi) + \min\{v(\sigma') + v(\psi), v(\varphi') + v(\tau)\} \\ &= \min\{-v(\varphi) + v(\psi), -v(\sigma') - v(\varphi) + v(\varphi') + v(\tau)\} \\ &= \min\{-v(\varphi) + v(\psi), -v(\sigma) + v(\tau)\} \\ &= \min\{v(\varphi^{-1}\psi), v(\sigma^{-1}\tau)\}, \end{aligned}$$

which completes the proof the v is a valuation on $\mathbb{E}$. □

The set of all elements with positive valuation is a two-sided ideal I in $\mathbb{E}$. If G is defined by equations with coefficients in $\mathbb{F}_p$, such as $\mathbb{G}_a$ or $\mathbb{G}_m$, then the Frobenius homomorphism itself is an endomorphism of G. Moreover, this element F generates I as either a left, right or two-sided ideal. When G is an elliptic curve not defined over $\mathbb{F}_p$, then I need not be principal.

3.3. **Vector space flocks from right vector spaces.** *Frobenius flocks* are collections of vector spaces which are an enrichment of a valuated matroid, introduced in [BDP18]. Here, we describe a slight generalization called *linear flocks*, where the Frobenius automorphism is replaced by an arbitrary automorphism [Bol18, Ch. 4]. Throughout what follows, we write e_i for the i-th standard basis vector in $\mathbb{Z}^n$.

Let L be a field and let $\varphi\colon L \to L$ an automorphism. Then a (L, φ)-linear flock is a collection of vector spaces $(V_\alpha)_{\alpha\in\mathbb{Z}} \in L^n$, of the same dimensions, such that

(1) For any $\alpha \in \mathbb{Z}^n$ and $1 \leq i \leq n$, $V_\alpha/i = V_{\alpha+e_i} \setminus i$ (here $/i$ stands for intersecting with the i-th coordinate hyperplane and $\setminus i$ stands for setting the i-th coordinate zero); and
(2) For any $\alpha \in \mathbb{Z}^n$, $V_{\alpha+(1,\dots,1)} = \varphi(V_\alpha)$, where φ refers to the coordinate-wise application of the automorphism to L^n.

A Frobenius flock is a linear flock over (L, F^{-1}), where F is the Frobenius automorphism of a perfect field L of positive characteristic. As shown in [BDP18, Thm. 34], an algebraic variety over L defines a Frobenius flock.

We now construct a linear flock from a right vector space, as follows. Let Q be a division ring with a surjective, discrete valuation $v\colon Q \to \mathbb{Z} \cup \{\infty\}$, and let $R \subset Q$ be the valuation ring, which is the set of elements with non-negative valuation. The residue division ring L is the quotient of R by the maximal ideal consisting of all elements of positive valuation. For simplicity, we assume that L is commutative, and hence a field. We define the automorphism $\varphi\colon L \to L$ by sending the residue class of $x \in R$ to the residue class of $\pi x \pi^{-1}$, where π is any *uniformizer*, i.e., element of valuation 1. Thus, if R is commutative, then φ is the identity.

Lemma 17. *If Q is a division ring with a valuation, whose residue division ring L is commutative, then the automorphism $\varphi\colon L \to L$ is well-defined.*

Proof. Let π' be another element of R such that $v(\pi') = 1$ and let x' be another element of R with the same residue class as x. Thus, $\pi' = u\pi$, where u is a unit in R, and $x' = x + t$, where $v(t) > 0$. Then, using $\overline{y}$ to denote the residue class in L of $y \in R$,

$$\overline{\pi' x' (\pi')^{-1}} = \overline{u\pi(x+t)\pi^{-1}u^{-1}} = \overline{u} \cdot \overline{\pi x \pi^{-1}} \cdot \overline{u^{-1}} + \overline{u\pi t \pi^{-1} u^{-1}} = \overline{\pi x \pi^{-1}},$$

because L is commutative and $v(u\pi t\pi^{-1}u^{-1}) = v(t) > 0$. □

We recall the following lemma, and include a proof for completeness.

Lemma 18. *Let $V \subseteq Q^n$ be a right vector space, and let $v_1, \dots, v_r \in V \cap R^n$ map to a basis of the L-vector space $\overline{V \cap R^n} \subseteq L^n$. Then $v_1, \dots, v_r$ are a Q-basis of V. In particular, $\dim_Q V = \dim_L \overline{V \cap R^n}$.*

Proof. Choose $v_{r+1}, \dots, v_n \in R^n$ such that their reductions, $\overline{v_1}, \dots, \overline{v_n}$ form a basis of L^n. By Nakayama's Lemma, $v_1, \dots, v_n$ generate R^n, and therefore they generate Q^n as well, so $v_1, \dots, v_n$ is a basis both for Q^n as a right vector space, and for R^n as a right R-module.

Now let $w \in V$, so that $w = v_1 a_1 + \cdots + v_n a_n$ for some $a_1, \dots, a_n \in Q$. Set

$$u := v_{r+1} a_{r+1} + \cdots + v_n a_n = w - v_1 a_1 - \cdots - v_r a_r \in V.$$

Assume that u is non-zero. Then, by scaling by an appropriate power of π, we can assume that the minimum valuation of the coordinates of u is 0, so $u \in R^n$ and $\overline{u}$ is non-zero. Thus, u can be written as an R-linear combination of $v_1, \dots, v_n$, but we already know that u is uniquely written as $v_{r+1}a_{r+1} + \cdots + v_n a_n$, which means that $a_{r+1}, \dots, a_n$ must be in R. Therefore, $\overline{u}$ is a nonzero linear combination $\overline{v_{r+1}}, \dots, \overline{v_n}$, which implies that $\overline{u}$ is not in $\overline{V \cap R^n}$, a contradiction. Therefore, u is 0, so w is in the span of $v_1, \dots, v_r$. This shows that $\dim_Q V = r = \dim_L \overline{V \cap R^n}$, as desired. □

Now fix a right vector space $V \subset Q^n$. For any $\alpha \in \mathbb{Z}^n$, we define

$$V_\alpha := \overline{(\pi^{-\alpha} V) \cap R^n} \subset L^n,$$

where $\pi^{-\alpha}$ denotes the diagonal matrix with entries $\pi^{-\alpha_1}, \dots, \pi^{-\alpha_n}$.

Lemma 19. *The vector spaces $(V_\alpha)_{\alpha \in \mathbb{Z}^n}$ defined above form a (L, φ^{-1})-linear flock.*

Proof. First, for each α we have

$$\dim_L V_\alpha = \dim_Q \pi^{-\alpha} V = \dim_Q V =: r,$$

where the first equality follows from Lemma 18. Second, for $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n$ we have

$$(\pi^{-\alpha}V) \cap R^n \supseteq \pi^{e_i}((\pi^{-\alpha-e_i}V) \cap R^n).$$

Applying π^{e_i} to a vector in R^n and then reducing is the same thing as first reducing and then setting the i-th coordinate to zero, so we find that $V_\alpha/i \supseteq V_{\alpha+e_i} \setminus i$. If the latter vector space has dimension r, then, since the former vector space has dimension at most r, the two spaces are equal. Otherwise, $V_{\alpha+e_i} \setminus i$ has dimension $r-1$. Then $V_{\alpha+e_i}$ contains the i-th standard basis vector of L^n and therefore $\pi^{-\alpha-e_i}V$ contains an element of the form $(a_1, \ldots, 1, \ldots, a_n)$ with $v(a_j) > 0$ for $j \neq i$. Then $\pi^{-\alpha}V$ contains the vector $(\pi^{-1}a_1, \ldots, 1, \ldots, \pi^{-1}a_n)$ whose reduction has a nonzero i-th coordinate, hence V_α/i has dimension $r-1$, as well, so that again we have $V_\alpha/i = V_{\alpha+e_i} \setminus i$.

Third, every coordinate of every element of $\pi((\pi^{-\alpha}V) \cap R^n)$ has positive valuation, and so $\pi((\pi^{-\alpha V}) \cap R^n)\pi^{-1}$ is contained in R^n, and thus equal to $(\pi^{-\alpha+(1,\cdots,1)}V) \cap R^n$. Thus, $V_{\alpha-(1,\ldots,1)}$ is equal to $\varphi(V_\alpha)$. □

By considering the matroid associated to each V_α, a (L, φ^{-1})-linear flock defines a matroid flock, a notion cryptomorphic to that of a valuated matroid (up to adding scalar multiples of the all-one vector) by [BDP18, Theorem 7].

We will now show that for the flock from Lemma 19, this valuated matroid is the natural noncommutative generalization of the well-known construction of a matroid valuation from a vector space of a field with a non-Archimedean valuation. For this we recall that the *Dieudonné determinant* is the unique group homomorphism $\det : \mathrm{GL}_r(Q) \to Q^*/[Q^*, Q^*]$ that sends a diagonal matrix $\mathrm{diag}(c, 1, \ldots, 1)$ to the class of c and matrices that differ from the identity matrix only in one off-diagonal entry to 1. We define the Dieudonné determinant of a non-invertible square matrix to be the symbol O, and we make $Q^*/[Q^*, Q^*] \cup \{O\}$ into a commutative monoid by $a \cdot O := O$. Since commutators have valuation 0, the valuation v induces a group homomorphism $v : Q^*/[Q^*, Q^*] \to \mathbb{Z}$, and we set $v(O) = \infty$. The Smith normal form algorithm shows that $r \times r$-matrix $A \in R^{r \times r}$ has $v(\det(A)) \geq 0$ with equality if and only if $\overline{A} \in L^{r \times r}$ is invertible.

Proposition 20. *Let $V \subset Q^n$ be a right vector space, and let μ denote the matroid valuation corresponding to the matroid flock associated to the linear flock $(V_\alpha)_\alpha$ (defined using any uniformizer $\pi \in R$). Then the rank in μ of a subset $I \subseteq [n]$ is the dimension of the projection of V into Q^I; and the valuated circuits are:*

$$\{(v(c_1), \ldots, v(c_n)) \mid c_i \in Q, [c_1 \cdots c_n]V = 0, \textit{and the support of } c \textit{ is minimal}\}.$$

Moreover, if V is given as the right column span of an $n \times r$ matrix A, then the valuation $\mu(B)$ of $B \in \binom{[n]}{r}$ is equal to the valuation $v(\det A[B])$ of the Dieudonné determinant of the submatrix of A consisting of the rows labeled by B.

Proof. By [BDP18], the valuation μ can be characterized as follows (again, modulo scalar multiples of the all-one vector): for all $\alpha \in \mathbb{Z}^n$ the expression $\mu(B) - e_B^T\alpha$, where $e_B := \sum_{i \in B} e_i$ is the characteristic vector of B, is minimized by $B_0 \in \binom{[n]}{r}$ if and only if B_0 is a basis for the matroid $M(V_\alpha)$ on $[n]$ defined by V_α. It suffices to prove that the numbers $v(\det A[B])$ have this property.

Suppose that B_0 is a basis for $M(V_\alpha)$. Choose $v_1, \ldots, v_r \in (\pi^{-\alpha}V) \cap R^n$ such that $\overline{v_1}, \ldots, \overline{v_r}$ are a basis of V_α. Let $g \in \mathrm{GL}_r(Q)$ be the unique matrix such that

$A' := \pi^{-\alpha}Ag$ has columns $v_1, \ldots, v_r$. Then for any $B \in \binom{[n]}{r}$ we have

$$\begin{aligned}
v(\det(A[B])) - e_B^T\alpha &= v(\det(\pi^\alpha[B])\det(A'[B])\det(g^{-1})) - e_B^T\alpha \\
&= v(\det(A'[B])) + v(\det(g^{-1})) + (e_B^T\alpha - e_B^T\alpha) \\
&\geq v(\det(A'[B_0])) + v(\det(g^{-1})) \\
&= v(\det(A[B_0])) - e_{B_0}^T\alpha,
\end{aligned}$$

where the inequality follows because A' has entries in R, and $v(\det(A'[B_0])) = 0$ since $\overline{A'[B_0]}$ is invertible. Moreover, equality holds if and only if also B is a basis in the matroid $M(V_\alpha)$. This proves the last statement in the lemma, and also that B is a basis in the underlying matroid of μ if and only if $A[B]$ is invertible; this, in turn, implies the statement about the rank function.

Finally, recall that the valuated circuits corresponding to the basis valuation μ are the vectors $\gamma \in (\mathbb{Z} \cup \{\infty\})^n$ that are supported precisely on some circuit $C \subseteq [n]$ of the matroid underlying μ and have the property that for each $i \in C$, and for each $j \in C - i := C \setminus \{i\}$ we have

$$\mu(C - i) + \gamma_j = \mu(C - j) + \gamma_i.$$

Without loss of generality, we may assume that $C - i = [r]$ and that $i = r + 1$. After performing column operations on A over Q, we may assume that $A[C - i]$ is the identity matrix, so that $\mu(B) = 0$.

Then the unique linear relation among the rows of $A[[r + 1]]$ is the vector $a = (a_{r+1,1}, \ldots, a_{r+1,r}, -1, 0, \ldots, 0)$. We have, for each $j = 1, \ldots, r$,

$$\mu(C - i) + v(a_{r+1,j}) = v(a_{r+1,j}) = v(\det A[C - j]) = \mu(C - j) + v(-1),$$

which means that $v(a)$ satisfies the condition on γ above. This proves the statement about valuated circuits. □

3.4. **The derivative homomorphism.** An element $\alpha \in \mathbb{E}$ is, by definition, an algebraic group homomorphism $G \to G$. In particular, it maps 0 to 0, and the derivative $d_0\alpha$ is a linear map from the tangent space T_0G into itself. Since T_0G is a one-dimensional vector space over K, we can identify $d_0\alpha$ with a scalar in K. Concluding, we have a map

$$\ell : \mathbb{E} \to K, \quad \alpha \mapsto d_0\alpha.$$

In the case of an elliptic curve, this is the same map as constructed in [Sil09, Corollary III.5.6].

Lemma 21. *The map ℓ is a (unitary) ring homomorphism,* $\operatorname{im}\ell$ *is a subfield of K, and* $\ker\ell$ *is the ideal* $\{\varphi \in \mathbb{E} \mid v(\varphi) > 0\}$.

Proof. First, the multiplicative neutral element of $\mathbb{E}$ is the identity $G \to G$, whose derivative is the scalar multiplication $T_0G \to T_0G$ by $1 \in K$. Next, multiplicativity of ℓ follows from the chain rule:

$$\ell(\alpha\beta) = d_0(\alpha \circ \beta) = (d_0\alpha) \circ (d_0\beta) = \ell(\alpha)\ell(\beta).$$

For additivity, we recall that for any algebraic group G with neutral element e the derivative of the group operation $m\colon G \times G \to G$ at (e, e) is the addition map $T_eG \times T_eG \to T_eG$. So in our setting where $e = 0$,

$$\ell(\alpha + \beta) = d_0(\alpha + \beta) = d_0(m \circ (\alpha, \beta)) = d_0\alpha + d_0\beta = \ell(\alpha) + \ell(\beta),$$

where we used the chain rule once more in the third equality.

To show that $\operatorname{im}\ell$ is a field, we use the classification of one-dimensional groups G. If G is $\mathbb{G}_a$, then $\mathbb{E}$ is the ring of p-polynomials $K[F]$, and $\ell(F) = 0$, so $\operatorname{im}\ell = K[F]/K[F]F$ is isomorphic to K. In the other cases, when G is $\mathbb{G}_m$ or an elliptic curve, $\mathbb{E}$ is a finitely generated $\mathbb{Z}$-algebra. Since $\operatorname{im}\ell$ is a subring of a field of characteristic p, it must be a finitely generated $\mathbb{F}_p$-algebra, and also an integral domain, thus it is a field.

For the last statement, if $\alpha \in \mathbb{E}$ has positive valuation, then it is inseparable, so its derivative vanishes. Conversely, let $\alpha \in \mathbb{E}$ with $\ell(\alpha) = d_0\alpha = 0$. For $h \in G$ let $a_h : G \to G$ be the morphism $g \mapsto g + h$. Then $\alpha \circ a_h = a_{\alpha(h)} \circ \alpha$ and therefore

$$(d_h\alpha) \circ (d_0a_h) = d_0(\alpha \circ a_h) = d_0(a_{\alpha(h)} \circ \alpha) = d_0(a_{\alpha(h)}) \circ 0 = 0$$

and using that d_0a_h is invertible with inverse d_ha_{-h} we find that $d_h\alpha = 0$. So α is inseparable, so it has positive valuation. □

Remark 22. The first statement in the lemma also holds when $\operatorname{char} K = 0$. Then the non-existence of inseparable morphisms implies immediately that $\mathbb{E}$ embeds into K and is, in particular, a commutative ring.

Let $R \subseteq Q$ be the valuation ring. For what follows, we need to extend ℓ from $\mathbb{E}$ to R. Let $\alpha, \beta \in \mathbb{E}$ such that $\alpha\beta^{-1} \in R$. By [Sil09, Corollary II.2.12], we can write $\beta = \beta' \circ F^e$ for some $e \in \mathbb{Z}_{\geq 0}$, where $F^e : G \to G^{(p^e)}$ is the e-th power of Frobenius and $\beta' : G^{(p^e)} \to G$ is separable. It follows that $v(\beta) = e$. Similarly, write $\alpha = \alpha'' \circ F^d$ with $\alpha'' : G^{(p^d)} \to G$ separable and hence $v(\alpha) = d$. Since $\alpha\beta^{-1} \in R$, we have $d \geq e$, and we set $\alpha' := \alpha'' \circ F^{d-e}$. Then α', β' are both morphisms $G^{(p^e)} \to G$ and β' is separable. This implies that $d_0\beta' : T_0G^{(p^e)} \to T_0G$ is multiplication by a nonzero scalar, and so an isomorphism. Then $(d_0\alpha')(d_0\beta')^{-1}$ is a linear map $T_0G \to T_0G$, hence multiplication by a scalar, which we denote by $\ell(\alpha\beta^{-1})$.

Lemma 23. *The above is a well-defined extension of $\ell : \mathbb{E} \to K$ to a ring homomorphism $\ell : R \to K$ whose image is a field and whose kernel is the set of elements of positive valuation.*

Proof. First, taking β equal to the identity, the above reduces to the earlier definition of $\ell : \mathbb{E} \to K$. Second, let $\gamma \in \mathbb{E} \setminus \{0\}$ and set $\alpha_1 := \alpha\gamma$ and $\beta_1 := \beta\gamma$, so that $\alpha_1\beta_1^{-1} = \alpha\beta^{-1}$. Write $\gamma = \gamma' F^c$ with $c \in \mathbb{Z}_{\geq 0}$ and $\gamma' : G^{(p^c)} \to G$ separable. Then, with notation as above,

$$\beta_1 = (\beta' F^e)(\gamma' F^c) = \beta'(F^e\gamma' F^{-e})F^{e+c} = \beta'\gamma'' F^{e+c}$$

where $\gamma'' = F^e\gamma' F^{-e}$ is a separable morphism $G^{(p^{e+c})} \to G^{(p^e)}$. Similarly, we have $\alpha_1 = \alpha'\gamma'' F^{e+c}$. The definition of $\ell(\alpha_1\beta_1^{-1})$ reads

$$(d_0\alpha'\gamma'')(d_0(\beta'\gamma''))^{-1} = (d_0\alpha')(d_0\gamma'')(d_0\gamma'')^{-1}(d_0\beta')^{-1} = (d_0\alpha')(d_0\beta')^{-1}.$$

This shows that the definitions of ℓ on $\alpha_1\beta_1^{-1}$ and on $\alpha\beta^{-1}$ agree. More generally, $\alpha\beta^{-1} = \alpha_1\beta_1^{-1}$ holds if and only if there exist nonzero $\gamma, \delta \in \mathbb{E}$ such that $\alpha\gamma = \alpha_1\delta$ and $\beta\gamma = \beta_1\delta$, and applying the above twice we find that $\ell : R \to K$ is well-defined.

Second, to show that ℓ is multiplicative, let $r, r_1 \in R$. If $v(r) > 0$ or $v(r_1) > 0$, then $v(rr_1) > 0$ and $\ell(rr_1) = 0 = \ell(r)\ell(r_1)$. So we may assume that $v(r) = v(r_1) = 0$. We may also write r and r_1 with a common denominator: $r = \alpha\beta^{-1}, r_1 = \alpha_1\beta^{-1}$ where $v(\alpha) = v(\alpha_1) = v(\beta) =: e$. Now find $\gamma, \gamma_1 \in \mathbb{E} \setminus \{0\}$ such that $\beta\gamma = \alpha_1\gamma_1$, so that

$$s := (\alpha\beta^{-1})(\alpha_1\beta^{-1}) = (\alpha\gamma)(\beta\gamma_1)^{-1}$$

and also $v(\gamma) = v(\gamma_1) =: c$. Write $\gamma = \gamma' F^c$ and $\gamma_1 = \gamma'_1 F^c$ and $\alpha = \alpha' F^e$ and $\alpha_1 = \alpha'_1 F^e$ and $\beta = \beta' F^e$ with $\gamma', \gamma'_1 : G^{(p^c)} \to G$ and $\alpha', \alpha'_1, \beta' : G^{(p^e)} \to G$ separable. Then we have, for the denominator of s,

$$\beta\gamma_1 = \beta' F^e \gamma'_1 F^c = \beta' \gamma''_1 F^{e+c}$$

where $\gamma''_1 = F^e \gamma'_1 F^{-e} : G^{(p^{e+c})} \to G^{(p^e)}$ is separable. Similarly, for the numerator of s,

$$\alpha\gamma = \alpha' \gamma'' F^{e+c}$$

where $\gamma'' = F^e \gamma' F^{-e} : G^{(p^{e+c})} \to G^{(p^e)}$ is separable. Now, by definition,

$$\ell(s) = (d_0\alpha'\gamma'')(d_0\beta'\gamma''_1)^{-1} = (d_0\alpha')(d_0\gamma'')(d_0\gamma''_1)^{-1}(d_0\beta')^{-1} \tag{2}$$

On the other hand, by a similar computation, the relation $\beta\gamma = \alpha_1\gamma_1$ implies $\beta'\gamma'' = \alpha_1\gamma''_1$, so that

$$(d_0\beta')(d_0\gamma'') = (d_0\alpha'_1)(d_0\gamma''_1).$$

Writing this as $d_0\gamma'' = (d_0\beta')^{-1}(d_0\alpha'_1)(d_0\gamma''_1)$ and substituting in (2) yields

$$\ell(s) = (d_0\alpha')(d_0\beta')^{-1}(d_0\alpha'_1)(d_0\beta')^{-1} = \ell(r)\ell(r_1),$$

as desired.

Third, for additivity of ℓ we compute, still assuming $r = \alpha\beta^{-1}, r_1 = \alpha_1\beta^{-1} \in R$ and notation as above, but no longer requiring $v(r) = v(r_1) = 0$,

$$\ell(r + r_1) = \ell((\alpha + \alpha_1)\beta^{-1}) = d_0(\alpha' + \alpha'_1)(d_0\beta)^{-1} = (d_0\alpha' + d_0\alpha'_1)(d_0\beta)^{-1} = \ell(r) + \ell(r_1).$$

Finally, $\ker \ell = \{r \in R \mid v(r) > 0\}$ follows directly from the definition. Since every element of R not in this ideal is invertible in R, $\operatorname{im} \ell$ is a field. □

3.5. **The Lie algebra of a subgroup.** We return to the division ring Q generated by the endomorphism ring $\mathbb{E}$ of a connected, one-dimensional algebraic group over K, equipped with the valuation from §3.2. Note that the residue field L here is commutative, since by Lemma 23, L embeds into the ground field K of our algebraic group G. We write ℓ for the map $R^n \to K^n$ defined by applying ℓ component-wise. To prove Theorem 5 we need a description of the Lie algebra of a closed, connected subgroup $X \subseteq G^n$ in terms of the right vector space representing it.

Lemma 24. *Let $X \subseteq G^n$ be a closed, connected subgroup, let $N = P^{-1}(X) \subseteq \mathbb{E}^n$ be the saturated right module representing it, and NQ the right subspace of Q^n generated by Q. Let v be any vector spanning the one-dimensional space T_0G. Then we have*

$$T_0X = \langle \ell(NQ \cap R^n)v \rangle_K.$$

Proof. First, $\dim_K T_0X = \dim X = \dim_Q NQ$ by Lemma 11. On the other hand, by Lemma 18 and the fact that ℓ is just the reduction map followed by an embedding $L \to K$, $\dim_Q NQ = \dim_K \ell(NQ \cap R^n)$. So the two spaces in the lemma have the same dimension. It therefore suffices to prove that the right-hand side is contained in the left-hand side. Since any finite set of elements in Q can be given common denominators, a general element of $NQ \cap R^n$ is of the form $\underline{\psi}\beta^{-1}$ with $\beta \in \mathbb{E} \setminus \{0\}$ and $\underline{\psi} \in N$. Write $\beta = \beta' F^e$ with $e \in \mathbb{Z}_{\geq 0}$ and $\beta' : G^{(p^e)} \to G$ separable, and write $\psi_i = \psi'_i F^e$ where $\psi'_i : G^{(p^e)} \to G$ is a not necessarily separable morphism. From $\underline{\psi} \in N$ and $\underline{\psi} = \underline{\psi}' \circ F^e$ and the fact that $F^e : G \to G^{(p^e)}$ is surjective, it follows that

$\underline{\psi}'$ maps $G^{(p^e)}$ into X. Hence $d_0\underline{\psi}'$ maps $T_0G^{(p^e)}$ into T_0X. On the other hand, by definition of ℓ we have

$$\ell(\underline{\psi}\beta^{-1}) = (d_0\underline{\psi}')(d_0\beta')^{-1},$$

which, therefore, is a linear map $T_0G \to T_0X$, as desired. □

3.6. **Classification of valuation.** Recall than an elliptic curve in positive characteristic is called *supersingular* if its ring of endomorphisms is non-commutative, and thus an order in a quaternion algebra [Sil09, §V.3].

Proposition 25. *For each connected one-dimensional algebraic group G, the valuation on $\mathbb{E}$ is as follows:*

(1) *If $G \cong \mathbb{G}_a$, so that $\mathbb{E}$ is the ring of p-polynomials $K[F]$, then v is the F-adic valuation.*
(2) *If $G \cong \mathbb{G}_m$, so that $\mathbb{E} \cong \mathbb{Z}$, with F corresponding to p, then v is the p-adic valuation.*
(3) *If $G \cong E$, an elliptic curve with j-invariant not in $\overline{\mathbb{F}_p}$, then $\mathbb{E} \cong \mathbb{Z}$ and v is the p-adic valuation.*
(4) *If $G \cong E$, a non-supersingular elliptic curve with j-invariant in $\overline{\mathbb{F}_p}$, then $\mathbb{E}$ is an order in a quadratic number field $\mathbb{Q}(\sqrt{-D})$. Let $\mathcal{O} \supset \mathbb{E}$ denote the ring of integers in $\mathbb{Q}(\sqrt{-D})$. Then there exits a maximal ideal $m \subset \mathcal{O}$ such that $m\overline{m} = (p)$, where $\overline{m}$ denotes complex conjugation, and v is the restriction of the m-adic valuation.*
(5) *If $G \cong E$, a supersingular elliptic curve with j-invariant in $\overline{\mathbb{F}_p}$ then $\mathbb{E}$ is an order in a quaternion algebra, and $v(\alpha)$ is the p-adic valuation of $\alpha\overline{\alpha}$.*

Proof. The first two cases follow from the fact that in each case Frobenius is an endomorphism of G, corresponding to p and F, respectively.

In case (3), we know that multiplication by p, a morphism of degree p^2 by [Sil09, Thm III.6.2(d)], is inseparable but not purely inseparable, and hence $v(p) = 1$. Thus, v must be the p-adic valuation.

In case (4), we know that any valuation on $\mathbb{Q}(\sqrt{-D})$ corresponds to a maximal ideal m of $\mathcal{O}$, and the multiplication by p endomorphism is inseparable, but not purely inseparable so $v(p) = 1$. Thus, it remains to show that the rational prime p splits in $\mathcal{O}$. This is a standard fact in the theory of elliptic curves, but we include a proof for convenience.

As before, we define $G^{(p^i)}$ to be the ith power of Frobenius applied to the defining equations of the elliptic curve G. Since G is defined over $\overline{\mathbb{F}}_p$, $G^{(p^k)} \cong G$ for some positive integer k, thus composition with the kth power of Frobenius followed by this isomorphism defines an endomorphism of G, which we will denote $\alpha_k \colon G \to G$. In particular, α_k is a degree p^k purely inseparable endomorphism and thus $v(\alpha_k) = k$. The dual homomorphism of F is separable by equivalence (ii) of [Sil09, Thm. V.3.1(a)], and so the dual endomorphism $\overline{\alpha}_k$ is separable, meaning that $v(\overline{\alpha}_k) = 0$. Since $\alpha_k\overline{\alpha}_k = p^k$ by [Sil09, Thm. III.6.2(a)], then (p) is not a prime ideal, and $\overline{\alpha}_k \notin m$.

In case (5), the endomorphism p is purely inseparable by the equivalence (iii) of [Sil09, Thm. V.3.1(a)], and has degree p^2 by [Sil09, Thm. III.6.2(d)]. Therefore, $v(p) = 2$. Let α be any endomorphism in $\mathbb{E}$, and $\overline{\alpha}$ its dual. Then $\alpha\overline{\alpha} = d$, where d is the degree of α and of $\overline{\alpha}$, by [Sil09, Thm. III.6.2(a)]. If we write $d = p^k e$, where p does not divide e, so that k is the p-adic valuation of d, then $v(d) = kv(p) + v(e) \geq 2k$. On the other hand, the inseparable degree of α divides d and is a power of p, so

$v(\alpha) \leq k$, and similarly $v(\overline{\alpha}) \leq k$. By the multiplicativity of valuation, $v(\alpha) = k$, which is what we wanted to show. □

We next want to show that the valuation on $\mathbb{E}$ is surjective. Since we already know that the residue division ring of Q is commutative, this means we can use the results of § 3.3.

Lemma 26. *There exists an element $\pi \in \mathbb{E}$ with $v(\pi) = 1$.*

Proof. We consider each of the cases of Proposition 25. If $G \cong \mathbb{G}_\mathrm{a}$, then we can take $\pi = F$. In cases (2) and (3), $\mathbb{E} \cong \mathbb{Z}$, with the p-adic valuation, and in case (4), the valuation restricts to the p-adic valuation on the subring of integers, and so in these case, we can take $\pi = p$.

In case (5), where G is a supersingular elliptic curve, $v(p) = 2$, and we have to find $\pi \in \mathbb{E} \setminus \mathbb{Z}$. By [Voi18, Thm. 42.1.9], Q is ramified at p, which means that $Q \otimes_\mathbb{Q} \mathbb{Q}_p$ is a division algebra over $\mathbb{Q}_p$, the field of p-adics. Therefore, by [Voi18, Thm. 13.3.10(c)], there is an element $\varphi \in Q \otimes_\mathbb{Q} \mathbb{Q}_p$ such that $N(\varphi)$ has p-adic valuation 1. Since $Q \otimes_\mathbb{Q} \mathbb{Q}_p$ is the p-adic completion of Q, then φ can be approximated by an element $\varphi' \in Q$ with the same valuation. We can write $\varphi' = a^{-1}\psi$, where $a \in \mathbb{Z}$ and $\psi \in \mathbb{E}$. Therefore, $N(\psi) = a^2 N(\varphi')$ has odd p-adic valuation, say $2k+1$, so we can write $\psi = \psi' \circ F^{2k+1}$, where $\psi' \colon G^{(p^{2k+1})} \to G$ is separable. Since G is defined over $\mathbb{F}_{p^2}$ [Sil09, Thm. V.3.1(a)], $G^{(p^{2k+1})} \cong G^{(p)}$, and so we take $\pi = \psi' \circ F$. □

3.7. **Proof of Theorem 5.** Before proving the theorem, we need to define the Frobenius flock of a d-dimensional, closed, connected subgroup X of G^n. By Lemma 26 we can choose a uniformizer π of R which is in fact an element of $\mathbb{E}$. For $\alpha \in \mathbb{Z}^n_{\geq 0}$ and $q \in G^n$ we write $\pi^\alpha q := (\pi^{\alpha_1}(q), \ldots, \pi^{\alpha_n}(q)) \in G^n$. This determines an action of the additive monoid $\mathbb{Z}^n_{\geq 0}$ on G^n, and $\pi^\alpha X$ is a closed, connected subgroup of G^n for all $\alpha \in \mathbb{Z}^n_{\geq 0}$ of dimension d.

We extend the definition of $\pi^\alpha X$ to $\alpha \in \mathbb{Z}^n$ as follows. Write $\alpha = \beta - k(1, \ldots, 1)$ where $\beta \in \mathbb{Z}^n_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. Then we let $\pi^\alpha X$ be the connected component of 0 of the preimage of $\pi^\beta X$ under the homomorphism $\pi^{k(1,\ldots,1)} : G^n \to G^n$. This preimage is clearly contained in the preimage of $\pi^{\beta+(1,\ldots,1)} X$ under $\pi^{(k+1)(1,\ldots,1)}$ and hence, since both preimages have dimension d, the connected components of 0 coincide, so $\pi^\alpha X$ is independent of the choice of β as above. One readily checks that we have thus defined an action of $\mathbb{Z}^n$ on the set of d-dimensional closed, connected subgroups of G^n.

Definition 27. The *Frobenius flock* of $X \subseteq G^n$, relative to π, is the collection of vector spaces $(U_\alpha)_{\alpha \in \mathbb{Z}^n}$ defined by $U_\alpha := T_0(\pi^{-\alpha} X) \subseteq (T_0 G)^n$.

In the case where $G = \mathbb{G}_\mathrm{a}$ and $\pi = F$, this notion coincides with the Frobenius flocks of [BDP18]. There, X was allowed to be an arbitrary irreducible closed subset of K^n, and consequently the base point at we took the tangent spaces had to be chosen somewhat carefully. In our current setting, where X is a subgroup, all points lead to equivalent Frobenius flocks, which is why we chose 0 as the base point. Also, in order to work with elliptic curves for which Frobenius is not an endomorphism, we allow π to be any endomorphism of valuation 1.

The next proposition says that the Frobenius flock of X equals the linear flock of the corresponding right Q-subspace of Q^n, up to a natural identification. Recall that the residue field L of Q is a subfield of K via the homomorphism ℓ from §3.4.

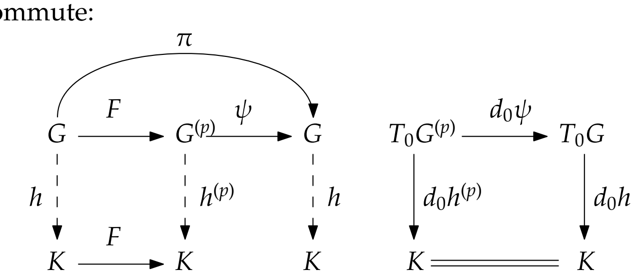
Proposition 28. *Let $X \subseteq G^n$ be a closed, connected subgroup and $N := P^{-1}(X) \subseteq \mathbb{E}^n$ the corresponding right module. Let $v \in T_0G \setminus \{0\}$. Let $(V_\alpha)_\alpha$ be the L-linear flock of NQ and let $(U_\alpha)_\alpha$ be the Frobenius flock of X. Then the map $L^n \to (T_0G)^n, c = (c_1, \dots, c_n) \mapsto (\ell(c_1)v, \dots, \ell(c_n)v) = \ell(c)v$ induces a linear bijection $(K \otimes_L V_\alpha) \to U_\alpha$ for each $\alpha \in \mathbb{Z}^n$.*

Proof. Let $\alpha \in \mathbb{Z}^n$ and set $Y := \pi^{-\alpha}X$. A straightforward computation shows that $P^{-1}(Y)Q = \pi^{-\alpha}NQ$. By Lemma 24 applied to Y, we therefore have

$$U_\alpha = T_0Y = \langle \ell((\pi^{-\alpha}NQ) \cap R^n)v\rangle_K = (K \otimes_L \overline{(\pi^{-\alpha}NQ) \cap R^n})v = (K \otimes_L V_\alpha)v,$$

as desired. □

In what follows, we decompose $\pi = \psi \circ F$ for some separable homomorphism $\psi\colon G^{(p)} \to G$ and F the Frobenius map. Let $h\colon G \dashrightarrow K$ be a rational function defined near 0 with $h(0) = 0$ and such that $d_0h\colon T_0G \to T_0K = K$ is an isomorphism. Let $h^{(p)}\colon G^{(p)} \dashrightarrow K$ be the Frobenius twist of h, i.e., the rational map making the diagram on the left commute:



We want the diagram on the right, at the level of tangent spaces, to commute as well. *A priori*, $d_0h \circ d_0\psi = cd_0h^{(p)}$ for some constant $c \in K^*$. Multiplying h by a scalar $a \in K$, the left-hand side is multiplied by a and the right-hand side is multiplied by a^p. Hence if we choose a such that $a^{p-1} = 1/c$, then we have the desired equality:

$$d_0h \circ d_0\psi = d_0h^{(p)}.$$

Applying the Frobenius twist to both sides, we obtain

$$d_0h^{(p)} \circ d_0\psi^{(p)} = d_0h^{(p^2)}$$

where $\psi^{(p)} : G^{(p^2)} \to G^{(p)}$. Combining the two formulas yields

$$d_0h \circ d_0\psi \circ d_0\psi^{(p)} = d_0h^{(p^2)}.$$

We abbreviate $\psi \circ \psi^{(p)}$ to ψ^2. Then the above reads

$$d_0h \circ d_0\psi^2 = d_0h^{(p^2)}.$$

More generally, for each nonnegative integer k, writing $\psi^k := \psi \circ \psi^{(p)} \circ \cdots \circ \psi^{(p^{k-1})}$, we have

$$d_0h \circ d_0\psi^k = d_0h^{(p^k)}.$$

Extending this componentwise to tuples we have

$$d_0h^n \circ d_0\psi^\alpha = d_0h^{(p^\alpha)} \text{ for all } \alpha \in \mathbb{Z}^n_{\geq 0}.$$

Proposition 29. *Set $Y := \overline{h^n(X)}$ and assume that $Y^{(p^\alpha)}$ is smooth at 0 for all $\alpha \in \mathbb{Z}^n$. Let $(U_\alpha)_\alpha$ be the Frobenius flock of X, and for $\alpha \in \mathbb{Z}^n$ set $W_\alpha := T_0Y^{(p^\alpha)}$. Then the matroid on $[n]$ defined by W_α equals the matroid defined by $T_{F^\alpha y}Y^{(p^\alpha)}$ for y a general point in Y, and*

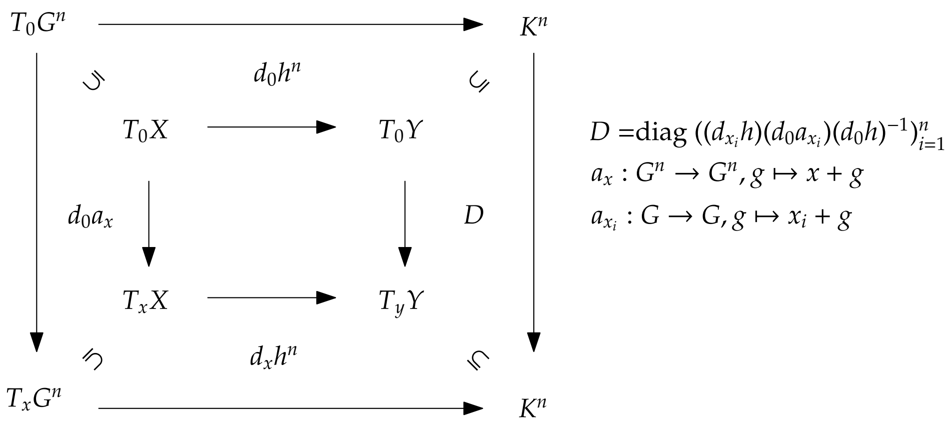
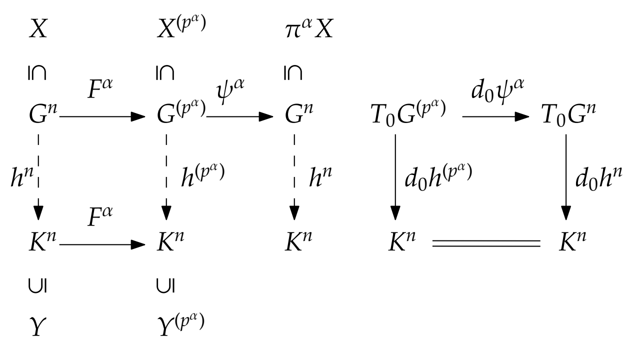


Figure 2. The last commuting diagram in the proof of Proposition 29.

the Frobenius flock $(W_\alpha)_\alpha$ *of* Y *at* 0 *is the image of* $(U_\alpha)_\alpha$ *under the linear isomorphism* $(d_0h)^n : T_0G^n \to T_0K^n$*.*

Proof. Since Frobenius flocks are uniquely determined by their restriction to the positive orthant, it suffices to prove both statements for $\alpha \in \mathbb{Z}^n_{\geq 0}$. For the last statement, consider the following two diagrams; here we have left out the obvious solid arrows between $X, X^{(p^\alpha)}, \pi^\alpha X$ and between $Y, Y^{(p^\alpha)}$ as well as the dashed arrows $h^n : X \dashrightarrow Y$ and $h^{(p^\alpha)} : X^{(p^\alpha)} \dashrightarrow Y^{(p^\alpha)}$.



The left-most diagram commutes by definition, and the right-most diagram commutes by the discussion above. So, by the left-most diagram, $A := d_0h^{(p^\alpha)} \circ (d_0\psi^\alpha)^{-1}$ maps $U_\alpha = T_0\pi^\alpha X$ into $W_\alpha = T_0Y^{(p^\alpha)}$, and by the rightmost diagram A equals d_0h^n. This proves the last statement.

For the first statement, we will use the homogeneity of X. A general point y in Y has the same properties as 0: y is of the form $h^n(x)$ for some $x = (x_1, \ldots, x_n) \in X$, $d_{x_i}h$ is nonzero for each i, and d_xh^n is an isomorphism between T_xX and T_yY. In that case, we have the commuting diagram of Figure 2, where all linear maps are isomorphisms and the right-most vertical map has a diagonal matrix relative to the standard basis. Consequently, the matroids represented by T_0Y and T_yY are equal. □

4. Equivalence of algebraic representations

In our examples below we will need to characterize the algebraic representations of certain matroids, up to equivalence. Here we discuss briefly what this means. For this we go back to the original definition of algebraic matroids: let $x_1, \dots, x_n$ be elements of an extension field L of our algebraically closed ground field K. In the matroid M on $[n]$ determined by this data, I is independent if and only if the $x_i, i \in I$ are algebraically independent over K. In the set-up of the introduction, L is the function field of $X \subseteq K^n$ and the x_i are the coordinate functions. Clearly, if we enlarge L, the matroid remains the same, and if we add $x'_1, \dots, x'_n \in L$ such that the algebraic closure of $K(x_i)$ in L equals that of $K(x'_i)$ for each i, then the corresponding elements i and i' are parallel. Thus, the matroid of the restriction to $x'_1, \dots, x'_n$ is again M, and we call this algebraic realization *equivalent* to the original one.

In this section we will show that the Lindström valuations of equivalent algebraic representations only differ by a trivial valuation. This follows from a more general statement about valuations of matroids with parallel elements. We use the following lemma, which is a straightforward consequence of submodularity of matroid valuations.

Lemma 30. *Let $\nu : \binom{E}{r} \to \mathbb{R} \cup \{\infty\}$ be a matroid valuation. Let $S \in \binom{E}{r-2}$ and $\{a, b, c, d\} \in \binom{E \setminus S}{4}$ be given. Then the minimum of*

$$\begin{aligned} \nu(S \cup \{a, b\}) &+ \nu(S \cup \{c, d\}), \\ \nu(S \cup \{a, c\}) &+ \nu(S \cup \{b, d\}), \text{ and} \\ \nu(S \cup \{a, d\}) &+ \nu(S \cup \{b, c\}) \end{aligned}$$

is attained at least twice. □

Proposition 31. *Let M be a matroid on E, and suppose $i, j \in E$ are parallel in M. Let ν be a valuation of M. Then there exists $c_{i,j} \in \mathbb{R}$ such that*

$$\nu(S \cup \{i\}) - \nu(S \cup \{j\}) = c_{i,j}$$

for all $S \subseteq E \setminus \{i, j\}$ such that $S \cup \{i\}$ (and thus $S \cup \{j\}$) are bases of M.

Proof. Suppose S, S' are such that

$$\nu(S \cup \{i\}) - \nu(S \cup \{j\}) \neq \nu(S' \cup \{i\}) - \nu(S' \cup \{j\}),$$

and $|S \setminus S'|$ is minimal. Clearly $S \neq S'$. Using the basis exchange axiom of matroids, let $a \in S \setminus S'$ and $b \in S' \setminus S$ be given such that $S \setminus \{a\} \cup \{b, i\}$ is a basis. Then

$$\nu(S \setminus \{a\} \cup \{b, i\}) - \nu(S \setminus \{a\} \cup \{b, j\}) = \nu(S' \cup \{i\}) - \nu(S' \cup \{j\})$$

by minimality of $|S \setminus S'|$. Since i and j are parallel, $\nu(S \setminus \{a\} \cup \{i, j\}) = \infty$. By Lemma 30, we have

$$\nu(S \cup \{i\}) - \nu(S \cup \{j\}) = \nu(S \setminus \{a\} \cup \{b, i\}) - \nu(S \setminus \{a\} \cup \{b, j\}),$$

which is a contradiction. □

Proposition 32. *Let $L \supseteq K$ be a field extension and let $x_1, \dots, x_n, x'_1, \dots, x'_n \in L$ be elements. Suppose that, for each i, the algebraic closure of $K(x_i)$ in L is the same as that of $K(x'_i)$, i.e., that the elements $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ determine equivalent algebraic representations of the same matroid M. Let μ, μ' be the Lindström valuations of these representations. Then there exists an $\alpha \in \mathbb{R}^n$ such that*

$$\mu'(B) = \mu(B) + e_B^T \alpha \text{ for all bases } B \text{ of } M.$$

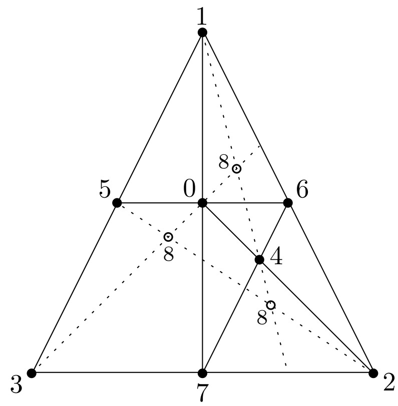


Figure 3. A matroid of rank 3 on 9 elements. A triple is collinear if and only if it is dependent in the matroid. The point 8 is the common intersection of the lines through $\{0, 3\}$, $\{1, 4\}$ and $\{2, 5\}$.

Proof. The tuple $(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$ is an algebraic representation of the matroid obtained from M by adding a parallel copy i' to each element i. Denote the Lindström valuation of this algebraic representation by ν. From [BDP18, Car18] it follows that ν restricts to μ on the copy of $[n]$ corresponding to $x_1, \ldots, x_n$ and to μ' on the copy of $[n]$ corresponding to $x'_1, \ldots, x'_n$. Furthermore, by Proposition 31, for any basis $\{i_1, \ldots, i_r\}$ of M, we have

$$\nu(\{i_1, \ldots, i_r\}) = \nu(\{i'_1, \ldots, i'_r\}) + c_{i_1, i'_1} + \ldots + c_{i_r, i'_r}.$$

Hence μ and μ' differ by the trivial valuation $B \mapsto e_B^T \alpha$ with $\alpha_i = c_{i,i'}$ for each i. □

5. Examples

Example 33. Consider the matroid M from Figure 3.

We construct a general matrix over a division ring S such that each dependent set of M is dependent in the matrix:

$$\Psi = \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & a \\ 1 & 0 & -1 \\ 1 & a & a \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & a & -1 \end{pmatrix},$$

where $a \in K$ satisfies $a^2 = -1$. After choosing a basis and fixing row and column scalars, the remaining entries were chosen as freely as possible given the dependent triples of M. As it turns out, the only freedom that is left is the choice of a.

No assumptions on the characteristic or commutativity of S have been made at this point. If S has characteristic 2, then (among others) the rows corresponding to the basis $\{3,4,5\}$ become dependent, so that Ψ cannot be a representation of M. So S must have characteristic $\neq 2$. Indeed, if $S = \mathbb{Q}(i)$ and $a = i$, then a subset of rows of Ψ is dependent if and only if it is dependent in M. Hence the column space of Ψ is a representation of M over $\mathbb{Q}(i)$.

Now suppose that K is a field of characteristic 2, and G is a connected one-dimensional algebraic group over K. Let X be a closed, connected subgroup of G^n, representing M algebraically. Then by Theorem 1, there is a linear representation of M over the endomorphism ring $\mathbb{E}$ of G. Due to the above, M is not representable over $\mathbb{Q}$, nor over $K(F)$. Hence G cannot be either the additive or multiplicative group. So G must be an elliptic curve.

Consider the supersingular elliptic curve E over $\mathbb{F}_2$ given by the equation $y^2+y = x^3+x^2+1$. Then $\mathbb{E}$ is the rational Hamilton quaternion order. Due to Proposition 25(5), $\mathbb{E}$ is equipped with a valuation v where $\nu(\alpha)$ is the 2-adic valuation of $\alpha\bar{\alpha}$. Then $B \mapsto v(\det \Psi_B)$ is the corresponding Lindström valuation of M. ♣

Example 34. Let M be the non-Fano matroid. As is well-known, M is realizable over a division ring if and only if the characteristic is not 2, in which case there is a projectively unique realization, given by the rows of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Nonetheless, M has algebraic realizations over a field of characteristic 2 using either the group $\mathbb{G}_m$ or an elliptic curve, both of whose endomorphism rings $\mathbb{E}$ have characteristic 0.

If w denotes the Lindström valuation of one of these algebraic realizations, then, using Proposition 20, we can compute the following invariant:

$$\begin{aligned} \gamma(w) &= w(\{4,5,6\}) - w(\{1,2,5\}) - w(\{1,3,6\}) - w(\{2,3,4\}) + 2w(\{1,2,3\}) \\ &= v(-2) - v(1) - v(-1) - v(1) + 2v(1) = v(2), \end{aligned}$$

where v is the valuation on $\mathbb{E}$. For each element of M, the sum of the coefficients of the valuations in which it appears in the definition of $\gamma(w)$ is 0, and so γ is invariant under adding trivial valuations. By the classification of the valuations in Proposition 25, $\gamma(w)$ is either 2 if the algebraic group G is a supersingular elliptic curve or 1 otherwise.

Moreover, by results in [EH91], every algebraic realization of M is equivalent to a $\mathbb{E}$-linear realization, and so by Proposition 32, $\gamma(w)$ is either 1 or 2 for the Lindström valuation on any algebraic realization of M. In particular, [EH91, Thm. 2.1.2] states that any algebraic realization of the matroid $M(K_4)$ of the complete graph on 4 vertices is equivalent to an $\mathbb{E}$-linear realization. The restriction of M to the first 6 rows is isomorphic to $M(K_4)$, and therefore in any realization of M, the first 6 elements are equivalent to a $\mathbb{E}$-linear realization. It suffices to show that the last element is the realization of M is equivalent to the element corresponding to

$(1, 1, 1)$ in the $\mathbb{E}$-linear representation. If not, then these two elements would define a rank-2 flat in a 1-element extension of M, which is contained in the rank-2 flat spanned by the first and sixth element and the distinct rank 2 flat spanned by the second and fifth element. Such an arrangement of flats contradicts the grading of the lattice of flats of a matroid.

In conclusion, scaling the valuation M coming from the $\mathbb{G}_m$-realization by a positive integer yields infinitely many valuations, of which exactly two are realizable as the Lindström valuation of an algebraic realization over a field of characteristic 2. In contrast, all of these scaled valuations are realizable by Frobenius flocks over $\mathbb{F}_2$ by scaling the original Frobenius flock. ♣

We can now prove the last of the results from the introduction. We refer to [DW92] for contractions, restrictions, and duality of matroid valuations.

Proof of Theorem 6. We use the universality construction in [EH91, Lem. 3.4.1] to construct the dual matroid M^*, from the following system of equations in $p^3 + 2$ noncommuting variables $x_0, x_1, y_1, \ldots, y_{p^3-2}, z, w$:

$$\begin{aligned} x_0 &= 0 & y_2 &= y_1 y_1 & x_1 &= y_1 y_{p^3-2} \\ x_1 &= 1 & y_3 &= y_1 y_2 & w &= z y_1 \\ & & &\vdots & w &= y_p z \\ & & y_{p^3-2} &= y_1 y_{p^3-3}. & & \end{aligned} \tag{3}$$

Then, by [EH91, Lem. 3.4.1], there exists a matroid M^* such that any algebraic realization of M^* is equivalent to a realization by a closed, connected subgroup $Y \subset G^n$ for some one-dimensional algebraic group G. Moreover, setting $\mathbb{E} := \mathrm{End}(G)$ and letting Q be the division ring generated by $\mathbb{E}$, that subgroup realization corresponds to an assignment of distinct values from Q to the variables $x_0, \ldots, z, w$ such that the equations (3) are satisfied; and any such choice yields a subgroup realization. We now assume that we have such a realization and study the solutions in an endomorphism ring $\mathbb{E}$ of some one-dimensional group.

In particular, we have $y_i = y_1^i$ for all i, and $1 = x_1 = y_1 y_{p^3-2} = y_1^{p^3-1}$, which means that y_1 is a $(p^3 - 1)$-root of unity, and since the y_i are distinct, y_1 it is a primitive $(p^3 - 1)$-root of unity. Over $\mathbb{Q}$, primitive $(p^3 - 1)$-roots of unity have degree at least 6, but all elements in the endomorphism rings of $\mathbb{G}_m$ and of elliptic curves have degree at most 2 over $\mathbb{Q}$. Therefore, any $\mathbb{E}$-linear realization of M^* comes from $G = \mathbb{G}_a$ and $\mathbb{E} = K[F]$, and y_1 is an element of $\mathbb{F}_{p^3} \setminus \mathbb{F}_p$.

We now look at the last two equations, which give us the relation $z y_1 = y_p z = y_1^p z$. Consider the semidirect product $K^* \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on K^* by the Frobenius automorphism, and let μ denote the homomorphism from the monoid $(K[F] \setminus \{0\}, \cdot)$ to $K^* \rtimes \mathbb{Z}$ defined by

$$\mu\left(\sum_{i=0}^{d} a_i F^i\right) = (a_d, d),$$

where d is chosen so that a_d is non-zero. Note that the valuation on $K[F]$ is the projection of μ onto the second coordinate. Since $K^* \rtimes \mathbb{Z}$ is a group, μ extends uniquely to a group homomorphism $\mu\colon Q^* \to K^* \rtimes \mathbb{Z}$, where Q is the division ring generated by $K[F]$. Applying μ to the equation $z y_1 = y_1^p z$, and using (a, d) to denote

$\mu(z)$, we have:

$$(a, d) \cdot (y_1, 0) = (y_1^p, 0) \cdot (a, d)$$
$$(a y_1^{p^d}, d) = (y_1^p a, d)$$

Since K^* is commutative, $y_1^{p^d} = y_1^p$. Also, y_1 is in $\mathbb{F}_{p^3} \setminus \mathbb{F}_p$, so the action of Frobenius on y_1 has order 3, and so this means that $d = v(z) \equiv 1 \bmod 3$.

The construction of M^* in [EH91, Lem. 3.4.1] represents the variable z as a cross-ratio, meaning that there are 4 elements in M^* (denoted x_0, x_∞, x_1, and z in the proof there but denoted $1, 2, 3, 4$ here) such that any Q-realization of M^* is equivalent to one whose restriction to these 4 elements is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & z \end{pmatrix},$$

whose matroid is the uniform matroid $U_{2,4}$. By Proposition 20, the restriction of the Lindström valuation to this $U_{2,4}$, denoted w^*, satisfies:

$$\begin{aligned} w^*(\{1,4\}) + w^*(\{2,3\}) - w^*(\{1,3\}) - w^*(\{2,4\}) &= \\ v(z) + v(-1) - v(1) - v(-1) = v(z), \end{aligned} \tag{4}$$

as does any valuation that differs from w^* by a trivial valuation.

Now we construct a realization of the matroid M dual to M^*. First, we can construct a $K[F]$-realization of M^* by choosing y_1 to be an element of $\mathbb{F}_{p^3} \setminus \mathbb{F}_p$, $y_i = y_1^i$, $z = F$, and $w = y_1^p F$. Then, by Theorem 2, M also has a $K[F]$-realization, which is constructed by taking the orthogonal complement of the realization of M^*—this orthogonal complement is a left Q-vector space—and applying the anti-isomorphism τ which sends F to F^{-1}. In particular, the contraction of all of M except the elements $1, \ldots, 4$ from the previous paragraph has the realization:

$$\begin{bmatrix} -1 & -1 \\ -1 & -F^{-1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Again, using Proposition 20 to compute the valuation w on this realization, we have:

$$w(\{1,4\}) + w(\{2,3\}) - w(\{1,3\}) - w(\{2,4\}) = v(-1) + v(F^{-1}) - v(1) - v(-1) = -1$$

Therefore, if w^* is the dual valuation of w, defined by $w^*(B) = w(\{1,2,3,4\} \setminus B)$, then it satisfies:

$$w^*(\{2,3\}) + w^*(\{1,4\}) - w^*(\{2,4\}) - w^*(\{1,3\}) = -1.$$

This valuation is not the Lindström valuation of an algebraic realization by (4) and the requirement that $v(z) \equiv 1 \bmod 3$. □

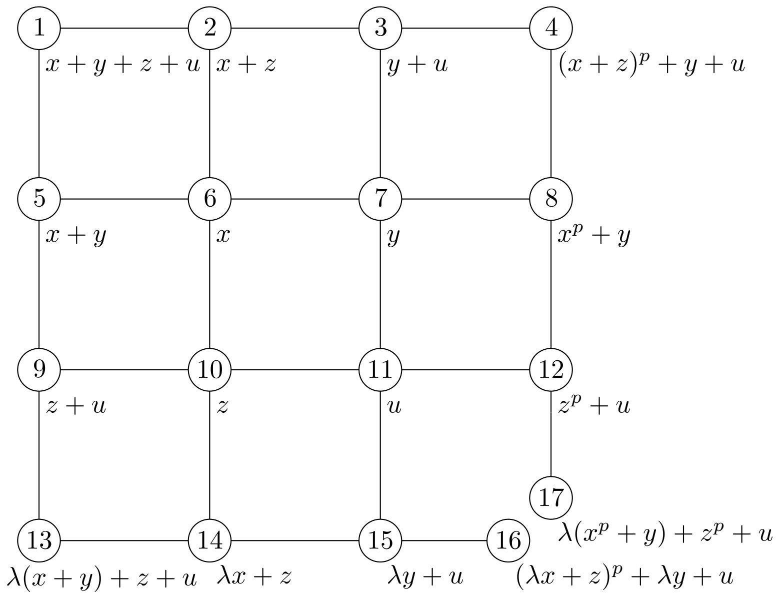


FIGURE 4. An algebraic matroid in $K[x, y, z, u]$ [Lin86a].

Example 35. We take $G = \mathbb{G}_a$ over any field of positive characteristic p, and let λ be any element of $K \setminus \mathbb{F}_p$. Then, N will be given by the matrix

	x	y	z	u
1	1	1	1	1
2	1	0	1	0
3	0	1	0	1
4	F	1	F	1
5	1	1	0	0
6	1	0	0	0
7	0	1	0	0
8	F	1	0	0
9	0	0	1	1
10	0	0	1	0
11	0	0	0	1
12	0	0	F	1
13	λ	λ	1	1
14	λ	0	1	0
15	0	λ	0	1
16	λF	λ	F	1
17	$F\lambda$	λ	F	1

Then the matroid M for N is depicted in Figure 4, where the lines denote the rank 2 flats. The matroid M is not linear over any field, but is linear over any non-commutative division ring, and, as we have shown, algebraic over any field of positive characteristic. Many restrictions of M have the same property, such as the restriction to $\{1, 4, 7, 8, 10, 11, 13, 14\}$, which appears in [Ing71] and [Lin86a]. The restriction to $\{3, 4, 7, 8, 9, 10, 13, 14\}$ appears in [Lin85], where an algebraic realization over $\mathbb{F}_2$ is given. ♣

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