MODULE STRUCTURE OF THE CENTER OF THE UNIVERSAL CENTRAL EXTENSION OF A GENUS ZERO KRICHEVER-NOVIKOV ALGEBRA.

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Abstract. We describe how the center of the universal central extension of the genus zero Krichever-Novikov current algebra decomposes as a direct sum of irreducible modules for automorphism group of the coordinate ring of this algebra.

1. Introduction

Let $a_1, \dots, a_N$ be distinct complex numbers. In previous work of the author together with R. Lu, X. Guo, and K. Zhao, we described explicitly the automorphism groups of $\mathrm{Der}\,(R)$ for the rings $R = \mathbb{C}[t, (t-a_1)^{-1}, \dots, (t-a_N)^{-1}]$ where the possible groups that can appear are the cyclic groups C_n, the dihedral groups D_n, S_4, A_4 and A_5 (see [CGLZ14]). This famous list of groups discovered by F. Klein appear naturally in the McKay correspondence and in the study of resolution of singularities.

In the present work we describe in a series of propositions the irreducible subrepresentations that appear in the decomposition of Ω^1_R/dR under the action of $\mathrm{Aut}(R)$ for the various groups listed above. The irreducible representations that do appear all have multiplicity one, but not all irreducible representations make their appearance. The proofs use classical results of Schur and Frobenius. These techniques can be found in nearly any book on the representation theory of finite groups such as Serre's book [Ser77] or Fulton and Harris [FH91]. We don't know of any conceptual nor geometric reason why not all irreducible representations appear and when they appear why they occur with multiplicity one.

From the work of S. Block, C. Kassel and J.L. Loday (see [Blo81], [KL82], and [Kas84]) we know if $\mathfrak{g}$ is a simple Lie algebra and R is a commutative algebra, both defined over the complex numbers, then the universal central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g} \otimes R$ is the vector space $\hat{\mathfrak{g}} := (\mathfrak{g} \otimes R) \oplus \Omega^1_R/dR$ where Ω^1_R/dR is the space of Kähler differentials modulo exact forms (see [Kas84]). The vector space $\hat{\mathfrak{g}}$ then becomes a Lie algebra by defining

$$[x \otimes f, y \otimes g] := [xy] \otimes fg + (x, y)\overline{fdg}, \quad [x \otimes f, \omega] = 0$$

for $x, y \in \mathfrak{g}$, $f, g \in R$, $\omega \in \Omega^1_R/dR$ and $(-,-)$ denotes the Killing form on $\mathfrak{g}$. In the above $\overline{a}$ denotes the image of $a \in \Omega^1_R$ in the quotient space Ω^1_R/dR. When R is the ring of meromorphic functions on Riemann surface with a fixed finite number of poles, the algebra $\hat{\mathfrak{g}}$ is called a current Krichever-Novikov algebra and has been extensively studied (see for example the books [Sch14] and [She12] and their references). There are a number of natural questions that arise in studying such Lie algebras and one of them arises as follows.

It is known that l-adic cohomology groups tend to be acted on by Galois groups, and the way in which these cohomology groups decompose can give interesting and important number theoretic

1991 *Mathematics Subject Classification.* Primary 17B67, 81R10.

Key words and phrases. Automorphism groups, Universal Central Extensions.

information (see for example R. Taylor's review of Tate's conjecture [Tay04]). Moreover it is an interesting and very difficult problem to describe the group $\mathrm{Aut}(R)$ where R is the space of meromorphic functions on a compact Riemann surface X and to determine the module structure of its induced action on the module of holomorphic differentials $\mathcal{H}^1(X)$ (see [Bre00]). Now if one realizes the fact that Ω^1_R/dR can be identified with the $H_2(\mathfrak{sl}(R),\mathbb{C})$ (see [Blo81]), it is natural to ask how Ω^1_R/dR decomposes into a direct sum of irreducible modules under the action of the $\mathrm{Aut}(R)$. We answer this question when R is the N-point algebra $R=\mathbb{C}[t,(t-a_1)^{-1},\dots,(t-a_N)^{-1}]$ with $a_1,\dots,a_N$ distinct complex numbers giving rise to the appropriate Kleinian groups $\mathrm{Aut}(R)$.

Let us give a bit of background where these algebras appear in conformal field theory. The ring of functions on the Riemann sphere regular everywhere except at a finite number of points appears naturally in Kazhdan and Luszig's explicit study of the tensor structure of modules for affine Lie algebras (see [KL93] and [KL91]). M. Bremner gave these the name of an *N-point algebra* (see [Bre94]). In the monograph [FBZ01, Ch. 12] algebras of the form $\oplus_{i=1}^N\mathfrak{g}((t-x_i))\oplus\mathbb{C}c$ appear in the description of the conformal blocks. These contain the n-point algebras $\mathfrak{g}\otimes\mathbb{C}[t,(t-a_1)^{-1},\dots,(t-a_N)^{-1}]\oplus\mathbb{C}c$ modulo part of the center Ω_R/dR. M. Bremner explicitly described the universal central extension of such an algebra in [Bre94] where the center has basis $\overline{(t-a_1)^{-1}\,dt},\dots,\overline{(t-a_N)^{-1}\,dt}$. The current algebra $\mathfrak{g}\otimes R\oplus\Omega^1_R/dR$ and the derivation algebra $\mathrm{Der}(R)$ are examples of Krichever-Novikov algebras for the genus zero Riemann sphere minus a finite number of points $R=\mathbb{C}[t,(t-a_1)^{-1},\dots,(t-a_N)^{-1}]$. In Krichever and Novikov's original work they only dealt with a particular one dimensional central extension (see for example [KN87a], [KN87b], [KN89], [Sch14] and [She12]).

2. The Universal Central Extension of the Current Algebra $\mathfrak{g}\otimes\mathcal{R}$.

Let R be a commutative algebra defined over $\mathbb{C}$. Consider the left R-module $F=R\otimes R$ with left action given by $f(g\otimes h)=fg\otimes h$ for $f,g,h\in R$ and let K be the submodule generated by the elements $1\otimes fg-f\otimes g-g\otimes f$. Then $\Omega^1_R=F/K$ is the module of *Kähler differentials*. The element $f\otimes g+K$ is traditionally denoted by fdg. The canonical map $d:R\to\Omega^1_R$ is given by $df=1\otimes f+K$. The *exact differentials* are the elements of the subspace dR. The coset of fdg modulo dR is denoted by $\overline{fdg}$. As C. Kassel showed the universal central extension of the current algebra $\mathfrak{g}\otimes R$ where $\mathfrak{g}$ is a simple finite dimensional Lie algebra defined over $\mathbb{C}$, is the vector space $\hat{\mathfrak{g}}=(\mathfrak{g}\otimes R)\oplus\Omega^1_R/dR$ with Lie bracket given by

$$[x\otimes f,Y\otimes g]=[xy]\otimes fg+(x,y)\overline{fdg},[x\otimes f,\omega]=0,[\omega,\omega']=0,$$

where $x,y\in\mathfrak{g}$, and $\omega,\omega'\in\Omega^1_R/dR$ and (x,y) denotes the Killing form on $\mathfrak{g}$.

Proposition 2.1 ([Bre94])**.** Let $a=(a_1,\dots,a_N)$ and $R_a=\mathbb{C}[t,(t-a_1)^{-1},\dots,(t-a_N)^{-1}]$ be as above. The set

$$\{\omega_1:=\overline{(t-a_1)^{-1}dt},\dots,\omega_N:=\overline{(t-a_N)^{-1}\,dt}\}$$

is a basis of $\Omega^1_{R_a}/dR_a$.

3. Recollections

In [CGLZ14] one of the main results we proved together with Rencai Lu, Xiangqian Guo, and Kaiming Zhao was the following result.

Theorem 3.1. *Suppose $\{a_1,a_2,\cdots,a_n\}$ and $\{a'_1,a'_2,\cdots,a'_m\}$ are two sets of distinct complex numbers and $\phi:R_a\to R_{a'}$ is an isomorphism of commutative algebras. Then $m=n$, ϕ is a fractional linear transformation, and one of the following two cases holds*

(a). There exists some constant $c\in\mathbb{C}$ such that $a_i-a_1=c(a'_i-a'_1)$ and $\phi((t-a_i)^k)=c^k(t-a'_i)^k$ for all $k\in\mathbb{Z}$ and $i=1,2,\cdots,n$ after reordering a'_i if necessary.

(b). There exists some constant $c \in \mathbb{C}$ such that $(a_i - a_1)(a_i' - a_1') = c$ for all $i \neq 1$, and

$$\phi((t-a_1)^k) = \frac{c^k}{(t-a_1')^k},$$

$$\phi((t-a_i)^k) = \frac{(a_1-a_i)^k(t-a_i')^k}{(t-a_1')^k}, \ \forall\, i > 1,$$

for all $k \in \mathbb{Z}$ after reordering a_i and a_i' if necessary.

Moreover, the maps given in (a) and (b) indeed define algebra isomorphisms between R_a and $R_{a'}$ under the corresponding conditions.

We call isomorphisms defined in (a) and (b) **isomorphisms of the first kind** and **isomorphisms of the second kind** respectively.

We also note the following theorems.

Theorem 3.2 ([Shu97] Theorem 2.3.1)**.** *Any finite subgroup of the automorphism group Aut($\hat{\mathbb{C}}$) of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is conjugate to a rotation group.*

Theorem 3.3 ([Shu97] Theorem 2.6.1)**.** *If G is a finite rotation group of the group of automorphisms of the Riemann sphere, then it is isomorphic to one of the following:*
(a). A cyclic group $C_n = \{s | s^n = 1\}$ for $n \geq 1$.
(b). A dihedral group $D_n = \langle s, t \,|\, s^n = 1 = t^2,\ \ tst = s^{-1}\rangle$.
(c). A platonic rotation group A_4, S_4 or A_5.

These are the only groups that appear as automorphisms groups of R_a and all of them appear (see the appendix of [CGLZ14] and the lemmas below).

4. Decompositions of the center under action of Aut(R_a)

Let R be an associative commutative ring defined over a field $\mathbb{F}$. We first observe that Aut(R) acts on Ω_R/dR via the tensor product action on $R \otimes_{\mathbb{F}} R$ since this action preserves K and dR.

4.1. **Decomposition of Ω_{R_a}/dR_a under C_n.** We give in this section examples of various a that give the automorphism groups of Klein listed in Theorem 3.3 as a series of lemmas followed by propositions that describe how the centers decompose under the action of the respective automorphism groups.

Lemma 4.1 ([CGLZ14], Appendix)**.** *Let $n \geq 4$ and $a = (a_1, a_2, \cdots, a_n)$ where $a_k = \zeta^k$ for the primitive n-th root of unity $\zeta = \exp(2\pi\imath/n)$. Then Aut($\mathcal{V}_a$) $\cong C_n$, the cyclic group of order n.*

In the proof of this Lemma it was noted that all automorphisms of R_a where of the first kind

$$\phi_k(t) = \zeta^k t,\ \ k = 0, 1, \cdots, n-1$$

with $\phi_k = \phi_1^k$.

Fix $n \geq 2$. Let $\zeta_i = \exp(2\pi\imath/n)$, $\omega_i = \overline{(t-a_i)^{-1}dt}$, and $R_a = \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_n)^{-1}]$.

Proposition 4.2. Let $a = (\zeta, \zeta^2, \ldots, \zeta^{n-1}, 1)$. The center Ω_{R_a}/dR_a is the direct sum of each irreducible representation of C_n with multiplicity one. More precisely

$$\Omega_{R_a}/dR_a = \oplus_{k=0}^{n-1}\mathbb{C}u_k \tag{4.1}$$

where

$$u_k = \sum_{i=1}^{n} \zeta^{ki}\omega_i$$

are eigenvectors for C_n with eigenvalue ζ^k for the automorphism $\phi(z) = \zeta z$, $0 \leq k \leq n-1$. In particular the subspaces $\mathbb{C}u_n$ are distinct irreducible one dimensional C_n submodules of Ω_{R_a}/dR_a.

Proof. The only automorphisms of R_a are of the first kind and are powers of ϕ where

$$\phi(z) = \zeta z.$$

Hence

$$\phi(\omega_i) = \begin{cases} \omega_{i-1}, & \text{for } 2 \leq i \leq n \\ \omega_n, & \text{for } i = 1. \end{cases}$$

As a consequence

$$\begin{aligned} \phi(u_k) &= \sum_{i=1}^{n} \zeta^{ki}\phi(\omega_i) = \zeta^k \phi(\omega_1) + \sum_{i=2}^{n} \zeta^{ki}\phi(\omega_i) \\ &= \zeta^k \omega_n + \sum_{i=2}^{n} \zeta^{ki}\omega_{i-1} = \zeta^k \omega_n + \sum_{i=1}^{n-1} \zeta^{k(i+1)}\omega_i \\ &= \zeta^k \left(\omega_n + \sum_{i=1}^{n-1} \zeta^{ki}\omega_i \right) = \zeta^k \left(\sum_{i=1}^{n} \zeta^{ki}\omega_i \right) \\ &= \zeta^k u_k. \end{aligned}$$

(See [Ser77] Section 5.1 for more information on the representation theory of C_n.) □

4.2. **Decomposition of Ω_{R_a}/dR_a under S_4 and D_n.**

Lemma 4.3 ([CGLZ14], Appendix)**.** *Let $n \geq 3$, and $a = (0, a_1, a_2, \cdots, a_n)$ where $a_k = \zeta^k$ for a primitive n-th root of unity ζ.*

(a). If $n \neq 4$ then $Aut(\mathcal{V}_a) \cong D_n$, the dihedral group of order $2n$.
(b). If $n = 4$ then $Aut(\mathcal{V}_a) \cong S_4$.

In the proof of this Lemma it was noted that the only automorphisms of R_a are products of the automorphims ϕ and ψ where

$$\phi(t) = \zeta t, \quad \text{and} \quad \psi(t) = \zeta/t. \tag{4.2}$$

Fix $n \geq 2$. Let $\zeta = \exp(2\pi\imath/n)$, $a_i = \zeta^i$, $\omega_0 = \overline{t^{-1}\,dt}$, $\omega_i = \overline{(t-a_i)^{-1}dt}$, and $R_a = \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_n)^{-1}]$.

4.2.1. *The decomposition for D_n.*

Proposition 4.4. Let $a = (0, \zeta, \zeta^2, \ldots, \zeta^{n-1}, 1)$. The center Ω_{R_a}/dR_a is the direct sum of all of the irreducible two dimensional representation of D_n with multiplicity one together with at most three other distinct irreducible representations. More precisely

i). If $n = 2m \neq 4$ is even, then

$$\Omega_{R_a}/dR_a = \mathbb{C}\omega_0 \oplus \left(\oplus_{k=1}^{m-1} U_k\right) \oplus, \mathbb{C}u_m \oplus \mathbb{C}u_n$$

where

$$U_k = \mathbb{C}u_k \oplus \mathbb{C}u_{n-k}, \;\; 1 \leq k \leq m-1, \quad u_k = \sum_{i=1}^{n} \zeta^{ki}\omega_i, \quad 1 \leq k \leq n.$$

The subspaces $\mathbb{C}\omega_0$, $\mathbb{C}u_m$, $\mathbb{C}u_n$ and the U_k are irreducible subrepresentations for $D_n = \langle \phi, \psi \rangle$ for the automorphism $\phi(t) = \zeta t$ and $\psi(t) = \zeta/t$. The irreducible modules U_k, $1 \leq k \leq m-1$ exhaust all of the irreducible two dimensional modules for D_n.

ii). If $n = 2m+1$ is odd, then

$$\Omega_{R_a}/dR_a = \mathbb{C}\omega_0 \oplus (\oplus_{k=1}^m U_k) \oplus \mathbb{C}u_n$$

where

$$U_k = \mathbb{C}u_k \oplus \mathbb{C}u_{n-k}, \quad 1 \le k \le m-1 \quad u_k = \sum_{i=1}^n \zeta^{ki}\omega_i, \quad 1 \le k \le n.$$

The subspace $\mathbb{C}\omega_0$, $\mathbb{C}u_n$ and the U_k are irreducible subrepresentations for D_n. The irreducible modules that appear in the above decomposition exhaust all of the irreducible modules for D_n with multiplicity one.

Proof. Recall only automorphisms of R_a are products of the automorphims ϕ and ψ where

$$\phi(t) = \zeta t, \quad \text{and} \quad \psi(t) = \zeta/t.$$

Now we calculate

$$\phi(\omega_0) = \overline{\phi(t)^{-1}\, d\phi(t)} = \overline{\zeta^{-1}t^{-1}\zeta\, dt} = \omega_0$$
$$\psi(\omega_0) = \overline{\psi(t)^{-1}\, d\psi(t)} = \overline{\zeta^{-1}t\zeta\, dt^{-1}} = -\overline{t^{-1}\, dt} = -\omega_0.$$

As a consequence the subspace $\mathbb{C}\omega_0$ is a one dimensional irreducible representation of D_n occurring as a direct summand of Ω_{R_a}/dR_a.

As above $\phi(u_i) = \zeta^i u_i$. On the other hand

$$\begin{aligned}\psi(\omega_i) &= \overline{\psi(t-\zeta^i)^{-1}\, d\psi(z)} = \overline{((\zeta/t)-\zeta^i)^{-1}\,\zeta dt^{-1}} \\ &= -\zeta^{1-i}\overline{(\zeta^{1-i}-t)^{-1}t^{-1}\, dt} \\ &= \zeta^{1-i}\left(-\zeta^{i-1}\overline{t^{-1}\, dt} + \zeta^{i-1}\overline{(t-\zeta^{1-i})^{-1}\, dt}\right) \\ &= -\overline{t^{-1}\, dt} + \overline{(t-\zeta^{1-i})^{-1}\, dt} \\ &= -\omega_0 + \omega_{n+1-i},\end{aligned}$$

and so

$$\begin{aligned}\psi(u_k) &= \sum_{i=1}^n \zeta^{ki}\psi(\omega_i) = -\left(\sum_{i=1}^n \zeta^{ki}\right)\omega_0 + \sum_{i=1}^n \zeta^{ki}\omega_{n+1-i} \\ &= \sum_{m=1}^n \zeta^{k(n-m+1)}\omega_m = \zeta^k \sum_{m=1}^n \zeta^{(n-k)m}\omega_m \\ &= \zeta^k u_{n-k}.\end{aligned}$$

For $k = n$ we have $\psi(u_n) = u_0 = u_n$. Similarly $\phi(u_n) = u_n$ and as a consequence $\mathbb{C}u_n$ is the trivial irreducible representation of D_n.

Suppose n is even with $n = 2m$, $m \in \mathbb{Z}$. Then $\psi(u_m) = \zeta^m u_m = -u_m$. Similarly $\phi(u_m) = \zeta^m u_m = -u_m$. This means that $\mathbb{C}u_m$ is the one dimensional alternating representation.

Let $n = 2m$ be still even. Now suppose $1 \le k \le m-1$. If we set $w_k = u_k$ for $1 \le k \le m-1$ and $w_{n-k} = \zeta^k u_{n-k}$ for $m \le k \le n-1$, then the matrices for ϕ and ψ for the basis $\{w_k, w_{n-k}\}$, $1 \le k \le m-1$ are

$$\phi = \begin{pmatrix} e^{2\pi\imath k/n} & 0 \\ 0 & e^{-2\pi\imath k/n}\end{pmatrix}, \quad \psi = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}. \tag{4.3}$$

Suppose now $n = 2m+1$ is odd. If we set $w_k = u_k$ for $1 \le k \le m$ and $w_{n-k} = \zeta^k u_{n-k}$ for $m+1 \le k \le n-1$, then the matrices for ϕ and ψ for the basis $\{w_k, w_{n-k}\}$ are as in (4.3).

(See [Ser77] Section 5.3 for more information on the representation theory of D_n.) ∎

For D_n we are left with considering the case of D_4.

Lemma 4.5 ([CGLZ14], Example 3. Case 2)**.** *Let $a = (0, 1, -1)$. Then we have the automorphism of the first kind $\tau(t) = -t$ and an automorphism of the second kind*

$$\sigma(t) = \frac{t+1}{t-1}$$

Then $\mathrm{Aut}(\mathcal{V}_a) = H \cup H\tau \cong D_4 = \langle x, b \,|\, x^4 = b^2 = 1,\ \ bxb = x^{-1}\rangle$ *where for example $x = \sigma\tau$ and $b = \tau$, in this case.*

Proposition 4.6. Again suppose $a = (0, 1, -1)$. Let $\omega_0 = \overline{t^{-1}\,dt}$, $\omega_1 = \overline{(t-1)^{-1}\,dt}$ and $\omega_2 = \overline{(t+1)^{-1}\,dt}$. Then Ω_{R_a}/dR_a, is the direct sum of a one dimensional representation $\mathbb{C}u_1$ where $u_1 = \omega_0 - \omega_1 - \omega_2$ and the unique irreducible representation of D_4 of dimension 2 which we denote by $U_2 = \mathbb{C}u_2 + \mathbb{C}u_3$ where $u_2 = \omega_0$, $u_3 = \omega_1 - \omega_2$.

Proof. There are four irreducible one dimensional representations which we denote by ψ_i, $i = 1, 2, 3, 4$ and one irreducible two dimensional representation which we denote by χ_2. The character table of D_4 is

TABLE 1. Character table of D_4.

	x^k	τx^k
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$
χ_2	$2(\delta_{k,0} - \delta_{k,2})$	0

for $0 \le k \le 3$. (See [Ser77] Section 5.3.) Here $\delta_{k,l}$ is the Kronecker delta function.
The one irreducible representation ρ of D_4 of dimension 2 defined on $\mathbb{C}^2$ is given by

$$\rho(x^k) = \begin{pmatrix} e^{k\imath\pi/2} & 0 \\ 0 & e^{-k\imath\pi/2} \end{pmatrix}, \quad \rho(\tau x^k) = \begin{pmatrix} 0 & e^{-k\imath\pi/2} \\ e^{k\imath\pi/2} & 0 \end{pmatrix}.$$

There are five conjugacy classes of D_4 and they are $\{I\}$, $\{x^2\}$, $\{\tau, x^2\tau\}$, $\{x\tau, x^3\tau\}$, and $\{x, x^3\}$.

On the representation Ω_{R_a}/dR_a we have

$$\begin{aligned}\tau(\omega_0)&=\overline{\tau(t^{-1})\,d\tau(t)}=\overline{t^{-1}\,dt}=\omega_0,\\ \tau(\omega_1)&=\overline{\tau((t-1)^{-1})\,d\tau(t)}=\overline{(t+1)^{-1}\,dt}=\omega_2,\\ \tau(\omega_2)&=\overline{\tau((t+1)^{-1})\,d\tau(t)}=\overline{(t-1)^{-1}\,dt}=\omega_1,\end{aligned}$$

$$\begin{aligned}x\tau(\omega_0)&=\sigma(\omega_0)=\overline{\sigma(t^{-1})\,d\sigma(t)}=\overline{\frac{t-1}{t+1}\,d\left(\frac{t+1}{t-1}\right)}\\ &=-2\overline{\frac{1}{(t-1)(t+1)}}\,dt=\overline{(t+1)^{-1}\,dt}-\overline{(t-1)^{-1}\,dt}=\omega_2-\omega_1,\\ x\tau(\omega_1)&=\sigma(\omega_1)=\overline{\sigma((t-1)^{-1})\,d\sigma(t)}=\overline{\left(\frac{t+1}{t-1}-1\right)^{-1}\,d\left(\frac{t+1}{t-1}\right)}\\ &=-\overline{(t-1)^{-1}\,dt}=-\omega_1\\ x\tau(\omega_2)&=\sigma(\omega_2)=\overline{\sigma((t+1)^{-1})\,d\sigma(t)}=\overline{\left(\frac{t+1}{t-1}+1\right)^{-1}\,d\left(\frac{t+1}{t-1}\right)}\\ &=\overline{t^{-1}\,dt}-\overline{(t-1)^{-1}\,dt}=\omega_0-\omega_1.\end{aligned}$$

Thus

$$\begin{aligned}x(\omega_0)&=\sigma\tau(\omega_0)=\sigma(\omega_0)=\omega_2-\omega_1,\\ x(\omega_1)&=\sigma\tau(\omega_1)=\sigma(\omega_2)=\omega_0-\omega_1,\\ x(\omega_2)&=\sigma\tau(\omega_2)=\sigma(\omega_1)=-\omega_1,\end{aligned}$$

$$\begin{aligned}x^2(\omega_0)&=x(\omega_2-\omega_1)=-\omega_1-(\omega_0-\omega_1)=-\omega_0,\\ x^2(\omega_1)&=x(\omega_0-\omega_1)=\omega_2-\omega_1-(\omega_0-\omega_1)=\omega_2-\omega_0,\\ x^2(\omega_2)&=x(-\omega_1)=\omega_1-\omega_0.\end{aligned}$$

As a consequence we have $\chi_{\Omega_{R_a}/dR_a}(\tau)=1$, $\chi_{\Omega_{R_a}/dR_a}(x\tau)=-1=\chi_{\Omega_{R_a}/dR_a}(x)=\chi_{\Omega_{R_a}/dR_a}(x^2)$. We know that there exists a unique set of nonnegative integers n_1,n_2,n_3,n_4,m such that

$$\chi_{\Omega_{R_a}/dR_a}=n_1\psi_1+n_2\psi_2+n_3\psi_3+n_4\psi_4+m\chi_2. \tag{4.4}$$

Then

$$\begin{aligned}3&=n_1+n_2+n_3+n_4+2m\\ 1&=n_1\psi_1(\tau)+n_2\psi_2(\tau)+n_3\psi_3(\tau)+n_4\psi_4(\tau)+m\chi_2(\tau)\\ &=n_1-n_2+n_3-n_4\\ -1&=n_1\psi_1(x\tau)+n_2\psi_2(x\tau)+n_3\psi_3(x\tau)+n_4\psi_4(x\tau)+m\chi_2(x\tau)\\ &=n_1-n_2-n_3+n_4 \qquad\qquad \text{as } x\tau=\tau x^3\\ -1&=n_1\psi_1(x)+n_2\psi_2(x)+n_3\psi_3(x)+n_4\psi_4(x)+m\chi_2(x)\\ &=n_1+n_2-n_3-n_4\\ -1&=n_1\psi_1(x^2)+n_2\psi_2(x^2)+n_3\psi_3(x^2)+n_4\psi_4(x^2)+m\chi_2(x^2)\\ &=n_1+n_2+n_3+n_4-2m.\end{aligned}$$

If $m = 0$ then the last equation is inconsistent. Moreover as $\dim \Omega_{R_a}/dR_a = 3$, $m \leq 1$ so we must have $m = 1$. Then the only solution to the above equations is $n_1 = n_2 = n_4 = 0$ and $n_3 = 1$. Thus $\chi_{\Omega_{R_a}/dR_a} = \psi_3 + \chi_2$.

To find the irreducible subspaces we need the projection formula with respect to the basis $\omega_0, \omega_1, \omega_2$ for the representation ρ_{D_4} of D_4 on Ω_{R_a}/dR_a:

$$\begin{aligned}\pi_{\chi_2} &= \frac{1}{|D_4|}\sum_{g\in S_4}\chi_2(g)\rho_{D_4}(g)\\ &= \frac{1}{8}\left(2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - 2\begin{pmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{pmatrix}\right)\\ &= \frac{1}{4}\begin{pmatrix}2&1&1\\0&1&-1\\0&-1&1\end{pmatrix}.\end{aligned}$$

Thus a basis for U_2 is $u_2 = \omega_0$, $u_3 = \omega_1 - \omega_2$.

The projection formula for ψ_3 with respect to the basis $\omega_0, \omega_1, \omega_2$ is given by

$$\begin{aligned}\pi_{\psi_3} &= \frac{1}{|D_4|}\sum_{g\in S_4}\psi_3(g)\rho_{D_4}(g)\\ &= \frac{1}{8}\Big(\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + \begin{pmatrix}-1&-1&-1\\0&0&1\\0&1&0\end{pmatrix} + \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + \begin{pmatrix}-1&-1&-1\\0&1&0\\0&0&1\end{pmatrix}\\ &\quad - \begin{pmatrix}0&0&1\\-1&-1&-1\\1&0&0\end{pmatrix} - \begin{pmatrix}0&0&1\\1&0&0\\-1&-1&-1\end{pmatrix} - \begin{pmatrix}0&1&0\\-1&-1&-1\\1&0&0\end{pmatrix} - \begin{pmatrix}1&0&0\\1&0&0\\-1&-1&-1\end{pmatrix}\Big)\\ &= \frac{1}{8}\begin{pmatrix}0&-4&-4\\0&4&4\\0&4&4\end{pmatrix}.\end{aligned}$$

Thus $\omega_0 - \omega_1 - \omega_2$ is a basis of the one dimensional irreducible subrepresentation whose character is ψ_3.

□

4.2.2. *The decomposition of* Ω_{R_a}/dR_a *for* S_4.

Lemma 4.7 ([Kle56], [Shu97] and [CGLZ14] Appendix)**.** *The automorphism group of* $\mathcal{V}_{(0,1,\imath,-1,-\imath)}$ *is* S_4.

Here the automorphism group of $\mathcal{V}_{(0,1,\imath,-1,-\imath)}$ is generated by an automorphism of the first kind

$$\phi(t) = \imath t$$

and also an automorphism of the second kind given by

$$\psi(t) = \frac{t+\imath}{t-\imath}.$$

Proposition 4.8. Set $a = (0, 1, \imath, -1, -\imath)$. For the automorphism group of $\mathcal{V}_a$, S_4, we have

$$\Omega_{R_a}/dR_a = U_\theta \oplus U_{\rho\epsilon} \tag{4.5}$$

where

$$U_\theta = \mathbb{C}(-\omega_0 + \omega_1 + \omega_3) + \mathbb{C}(-\omega_0 + \omega_2 + \omega_4)$$
$$U_{\rho\epsilon} = \mathbb{C}\omega_0 + \mathbb{C}(-\omega_1 + \omega_3) + \mathbb{C}(-\omega_2 + \omega_4)$$

are the irreducible S_4-subrepresentations of Ω_{R_a}/dR_a.

Proof. First observe

$$\begin{aligned}
\phi(\omega_0) &= -\imath^2\overline{t^{-1}\,dt} = \omega_0,\\
\phi(\omega_1) &= \overline{(t+\imath)^{-1}\,dt} = \omega_4,\\
\phi(\omega_2) &= \overline{(t-1)^{-1}\,dt} = \omega_1,\\
\phi(\omega_3) &= \overline{(t-\imath)^{-1}\,dt} = \omega_2,\\
\phi(\omega_4) &= \overline{(t+1)^{-1}\,dt} = \omega_3,
\end{aligned}$$

and

$$\begin{aligned}
\psi(\omega_0) &= -\omega_2 + \omega_4,\\
\psi(\omega_1) &= -\omega_2,\\
\psi(\omega_2) &= -\omega_2 + \omega_1,\\
\psi(\omega_3) &= -\omega_2 + \omega_0,\\
\psi(\omega_4), &= -\omega_2 + \omega_3.
\end{aligned}$$

Thus the linear transformations ϕ and ψ have with respect to this basis the matrices

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0\\ -1 & -1 & -1 & -1 & -1\\ 0 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

We let $\chi_{\Omega_R/dR} = a\chi_0 + b\chi_\epsilon + c\chi_\theta + d\chi_\rho + e\chi_{\epsilon\rho}$ where χ_0 is the character of the trivial representation, χ_ϵ is the character of the alternating representation, χ_θ is the character of the unique two dimensional irreducible representation χ_ρ, χ_ρ is the character of the natural representation on $\mathbb{C}^3$ and $\chi_{\epsilon\rho}$ is the character of the tensor product of the alternating representation and the natural representation.

We then have the following values of the trace of the conjugacy classes

TABLE 2. Character of S_4 on Ω_R/dR

	I	$\psi\phi$	ϕ^2	ψ	ϕ
$\chi_{\Omega_R/dR}$	5	-1	1	-1	1
I	1	1	1	1	1
χ_ϵ	1	-1	1	1	- 1
χ_θ	2	0	2	-1	0
χ_ρ	3	1	-1	0	- 1
$\chi_{\epsilon\rho}$	3	-1	-1	0	1

Note ϕ has order 4 while ψ has order 3. The number of elements in the conjugacy class of $\psi\phi$ is 6: $\psi\phi$, $\phi^3\psi\phi^2$, $\phi^2\psi\phi^3$, $\phi\psi$, $\psi^2\phi\psi^2$, $\phi\psi^2\phi\psi^2\phi^3$. The number of elements in the conjugacy class of ϕ^2 is 3: ϕ^2, $\psi\phi^2\psi^2$, $\psi^2\phi^2\psi$. The number of elements in the conjugacy class of ψ is 8: ψ, $\phi\psi\phi^3$, $\phi^2\psi\phi^2$, $\phi^3\psi\phi$, $\psi\phi^2\psi\phi^2\psi^2$, $\psi^2\phi^2\psi\phi^2\psi$, $\phi\psi^2\phi^2\psi\phi^2\psi\phi^3$, $\phi^3\psi^2\phi^2\psi\phi^2\psi\phi$. The number of elements in the conjugacy class of ϕ is 6: ϕ, $\psi\phi\psi^2$, $\psi^2\phi\psi$, $\phi^2\psi\phi\psi^2\phi^2$, $\phi^2\psi^2\phi\psi\phi^2$, $\psi\phi^2\psi^2\phi\psi\phi^2\psi^2$.

Solving

$$\begin{aligned} 5 &= a + b + 2c + 3d + 3e \\ -1 &= a - b + d - e \\ 1 &= a + b + 2c - d - e \\ -1 &= a + b - c \\ 1 &= a - b - d + e \end{aligned}$$

we get $a = b = d = 0$ and $c = 1 = e$.

Now we need to find a basis for these two irreducible components for χ_θ, $\chi_{\epsilon\rho}$. Let $\rho : S_4 \to \mathrm{GL}(\Omega_{R_a}/dR_a)$ is the given induced representation. To find the irreducible component for χ_θ, we need the projection formula:

$$\begin{aligned} \pi_{\chi_\theta} &= \frac{1}{|S_4|} \sum_{g \in S_4} \chi_\theta(g)\rho(g) \\ &= \frac{1}{24} \sum_{g \in S_4} \chi_\theta(g)\rho(g) \\ &= \frac{1}{24} \begin{pmatrix} 0 & -6 & -6 & -6 & -6 \\ 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 6 \\ 0 & 6 & 0 & 6 & 0 \\ 0 & 0 & 6 & 0 & 6 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus the subspace spanned U_θ by $-\omega_0+\omega_1+\omega_3$ and $-\omega_0+\omega_2+\omega_4$ is an irreducible subrepresentation of Ω_{R_a}/dR_a.

For the irreducible component of type $\chi_{\epsilon\theta}$ we have

$$\begin{aligned} \pi_{\chi_{\epsilon\theta}} &= \frac{1}{|S_4|} \sum_{g \in S_4} \chi_{\epsilon\theta}(g)\rho(g) \\ &= \frac{1}{24} \sum_{g \in S_4} \chi_{\epsilon\theta}(g)\rho(g) \\ &= \frac{1}{6} \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} \end{aligned}$$

Thus a basis for the three dimensional irreducible subrepresentation with character $\chi_{\epsilon\rho}$ is $\omega_0, -\omega_1 + \omega_3, -\omega_2 + \omega_4$. □

(In the Propositions below the author will not be as detailed about the calculations, but they all are of a similar level of difficulty and if the reader is so interested in the details, please feel free to email him - the author will send you the tex file with nearly all calculations written out explicitly.)

4.3. **Decomposition of Ω_{R_a}/dR_a under A_4.**

Lemma 4.9 ([CGLZ14], Example 3. Case 3)**.** *Let $a=(0,1,x)$ where $x=\dfrac{1+\imath\sqrt{3}}{2}$. Then $Aut(\mathcal{V}_a)\cong A_4$, the alternating group on 4 letters. Here*

$$Aut(\mathcal{V}_a)=H\cup H\tau_2\cup H\tau_2^2$$

where $H=\{1,\sigma_1,\sigma_2,\sigma_3\}$ and the automorphisms are defined by

$$\sigma_1(t)=\frac{x}{t},\ \ \sigma_2(t)=\frac{t-x}{t-1},\ \ \sigma_3(t)=\frac{x(t-1)}{t-x},\ \ \tau_2(t)=-x(t-1).$$

Proposition 4.10. Again suppose $a=(0,1,x)$ where $x=\dfrac{1+\imath\sqrt{3}}{2}$. Let $\omega_0=\overline{t^{-1}\,dt}$, $\omega_1=\overline{(t-1)^{-1}\,dt}$ and $\omega_2=\overline{(t-x)^{-1}\,dt}$. Then Ω_{R_a}/dR_a, is the unique irreducible representation of A_4 of dimension 3.

Proof. Set $\zeta=\exp(2\pi\imath/3)$. Let χ_3 be the character of the unique irreducible 3-dimensional representation of A_4 and χ_1 and χ_2 the characters of the two unique non-trivial representations of A_4. The character table of A_4 is

TABLE 3. Character table of A_4.

	I	σ_1	τ_2	τ_2^2
I	1	1	1	1
χ_1	1	1	ζ	ζ^2
χ_2	1	1	ζ^2	ζ
χ_3	3	-1	0	0

Now one can calculate

$$\begin{aligned}\sigma_1(\omega_0)&=\overline{\sigma_1(t^{-1})\,d\sigma_1(t)}=\overline{t/x\,d(x/t)}=-\overline{t^{-1}\,dt}=-\omega_0,\\ \sigma_1(\omega_1)&=-\omega_0+\omega_2,\\ \sigma_1(\omega_2)&=-\omega_0+\omega_1.\end{aligned}$$

Thus $\chi_{\Omega_{R_a}/dR_a}(\sigma_1)=-1$.

Moreover

$$\begin{aligned}\tau_2(\omega_0)&=\overline{\tau_2(t^{-1})\,d\tau_2(t)}=\overline{(-x(t-1))^{-1}\,d(-x(t-1))}=\overline{(t-1)^{-1}\,dt}=\omega_1,\\ \tau_2(\omega_1)&=\omega_2,\\ \tau_2(\omega_2)&=\omega_0.\end{aligned}$$

and $\tau_2^2(\omega_0)=\omega_2$, $\tau_2^2(\omega_1)=\omega_0$, $\tau_2^2(\omega_2)=\omega_1$. As a consequence $\chi_{\Omega_{R_a}/dR_a}(\tau_2)=0=\chi_{\Omega_{R_a}/dR_a}(\tau_2^2)$. Thus the character of Ω_{R_a}/dR_a is the same as the three dimensional irreducible representation for A_4 and thus it must be isomorphic to it. □

4.4. **Decomposition of Ω_{R_a}/dR_a under A_5.**

Lemma 4.11 ([Kle56], [Tot02] and [Shu97], [CGLZ14] Appendix)**.** *Let $\zeta=\exp(2\pi\imath/5)$ and*

$$a_0=0,\ \ a_i=\zeta^{i-1}(\zeta+\zeta^4),\ \ 1\le i\le 5,\quad a_i=\zeta^{i-6}(\zeta^2+\zeta^3),\ \ 6\le i\le 10.$$

The automorphism group of $\mathcal{V}_{(0,a_1,\dots,a_{10})}$ is A_5 where the automorphisms are

$$t\mapsto \zeta^j t,\quad t\mapsto -\frac{1}{\zeta^j t},$$
$$t\mapsto \zeta^j\frac{-(\zeta-\zeta^4)\zeta^l t+(\zeta^2-\zeta^3)}{(\zeta^2-\zeta^3)\zeta^l t+(\zeta-\zeta^4)}$$
$$t\mapsto \zeta^j\frac{(\zeta^2-\zeta^3)\zeta^l t+(\zeta-\zeta^4)}{(\zeta-\zeta^4)\zeta^l t-(\zeta^2-\zeta^3)},$$

for $j,l=0,\dots 4$.

As usual set $\omega_0=\overline{t^{-1}\,dt}$ and $\omega_i=\overline{(t-a_i)^{-1}\,dt}$ for $1\le i\le 10$.

Proposition 4.12. The $Aut(R_a)$ module Ω_{R_a}/dR_a decomposes into a direct sum of irreducible submodules as

$$\Omega_{R_a}/dR_a=U_3\oplus U_3'\oplus U_5$$

where U_3 and U_3' are the two distinct irreducible representations of A_5 of dimension 3 and U_5 is the unique irreducible representation of dimension 5. A basis of U_5 is $\{-\omega_0+\omega_i+\omega_{i+5}\,|\,1\le i\le 5\}$ and bases of U_3 and U_3' are respectively

$$\begin{aligned}&\sqrt{5}\omega_0+\sum_{i=1}^{5}(\omega_i-\omega_{i+5}),\\ &\xi\omega_0+\xi\omega_1+\omega_2+\omega_5-\xi\omega_6-\omega_7-\omega_{10},\\ &\xi\omega_0+\omega_1+\xi\omega_2+\omega_3-\omega_6-\xi\omega_7-\omega_8,\end{aligned}$$

and

$$\begin{aligned}&\sqrt{5}\omega_0-\sum_{i=1}^{5}(\omega_i-\omega_{i+5}),\\ &-\bar\xi\omega_0-\bar\xi\omega_1-\omega_2-\omega_5+\bar\xi\omega_6+\omega_7+\omega_{10},\\ &-\bar\xi\omega_0-\omega_1-\bar\xi\omega_2-\omega_3+\omega_6+\bar\xi\omega_7+\omega_8,\end{aligned}$$

where $\xi=\dfrac{1+\sqrt{5}}{2}$ and $\bar\xi=\dfrac{1-\sqrt{5}}{2}$.

Proof. Let ϕ be the automorphism given by $\phi(t)=\zeta t$. Then a straightforward calculation yields

$$\begin{aligned}&\phi(\omega_0)=\omega_0,\quad \phi(\omega_1)=\omega_5,\quad \phi(\omega_2)=\omega_1,\quad \phi(\omega_3)=\omega_2,\quad \phi(\omega_4)=\omega_3,\quad \phi(\omega_5)=\omega_4,\\ &\phi(\omega_6)=\omega_{10},\quad \phi(\omega_7)=\omega_6,\quad \phi(\omega_8)=\omega_7,\quad \phi(\omega_9)=\omega_8,\quad \phi(\omega_{10})=\omega_9.\end{aligned}$$

Thus the $\chi_{\Omega_{R_a}/dR_a}(\phi^l)=1$ for $1\le l\le 4$.

Let ψ be the automorphism given by $\psi(t)=-\dfrac{1}{\zeta t}$. In the calculations below note that $(\zeta+\zeta^4)(\zeta^2+\zeta^3)=-1$. Then we have

$$\begin{aligned}&\psi(\omega_1)=-\omega_0+\omega_{10}\quad \psi(\omega_2)=-\omega_0+\omega_9,\quad \psi(\omega_3)=-\omega_0+\omega_8\\ &\psi(\omega_4)=-\omega_0+\omega_7,\quad \psi(\omega_5)=-\omega_0+\omega_6,\quad \psi(\omega_6)=-\omega_0+\omega_5\\ &\psi(\omega_7)=-\omega_0+\omega_4,\quad \psi(\omega_8)=-\omega_0+\omega_3,\quad \psi(\omega_9)=-\omega_0+\omega_2\\ &\psi(\omega_{10})=-\omega_0+\omega_1,\quad \psi(\omega_0)=-\omega_0\end{aligned}$$

Thus $\chi_{\Omega_{R_a}/dR_a}(\psi)=-1$ and $\psi^2=1$.

Next we observe that

$$\frac{\zeta-\zeta^4}{\zeta^2-\zeta^3}=-(\zeta^2+\zeta^3)$$
$$\frac{\zeta^2-\zeta^3}{\zeta-\zeta^4}=\zeta+\zeta^4,$$

so that

$$\zeta^j\frac{-(\zeta-\zeta^4)\zeta^{1-l}t+(\zeta^2-\zeta^3)}{(\zeta^2-\zeta^3)\zeta^{1-l}t+(\zeta-\zeta^4)}=\zeta^j\frac{(\zeta^2+\zeta^3)t+\zeta^{l-1}}{t-\zeta^{l-1}(\zeta^2+\zeta^3)}$$
$$\zeta^j\frac{(\zeta^2-\zeta^3)\zeta^{1-l}t+(\zeta-\zeta^4)}{(\zeta-\zeta^4)\zeta^{1-l}t-(\zeta^2-\zeta^3)}=\zeta^j\frac{(\zeta+\zeta^4)t+\zeta^{l-1}}{t-\zeta^{l-1}(\zeta+\zeta^4)}$$

for $j,l=1,\dots 5$. Thus if we set

$$\beta(t):=\frac{(\zeta^2+\zeta^3)t+1}{t-(\zeta^2+\zeta^3)},$$

then

$$\phi^j\circ\beta\circ\phi^{1-l}(t)=\zeta^j\frac{(\zeta^2+\zeta^3)t+\zeta^{l-1}}{t-\zeta^{l-1}(\zeta^2+\zeta^3)},$$
$$\psi\circ\beta(t)=-\zeta^4\frac{(\zeta+\zeta^4)t+1}{t-(\zeta+\zeta^4)}$$

and

$$\beta(t)-\zeta^r(\zeta+\zeta^4)=\frac{(1-\zeta^r)(\zeta^2+\zeta^3)t+1+\zeta^r(\zeta^2+\zeta^3)^2}{t-(\zeta^2+\zeta^3)}$$

So we will only calculate β evaluated on the basis elements

$$\beta(\omega_0)=\omega_1-\omega_6$$
$$\beta(\omega_1)=\omega_0-\omega_6$$
$$\beta(\omega_2)=\omega_5-\omega_6$$
$$\beta(\omega_3)=\omega_8-\omega_6$$

$$\beta(\omega_4)=\omega_9-\omega_6$$
$$\beta(\omega_5)=\omega_2-\omega_6$$
$$\beta(\omega_6)=-\omega_6$$
$$\beta(\omega_7)=\omega_{10}-\omega_6$$

$$\beta(\omega_8)=\omega_3-\omega_6$$
$$\beta(\omega_9)=\omega_4-\omega_6$$
$$\beta(\omega_{10})=\omega_7-\omega_6.$$

$$\beta^2(\omega_{10})=\beta(\omega_7-\omega_6)=\omega_{10}$$

The automorphism $\beta\circ\phi$ has order 3 and thus is in the conjugacy class of (123). Moreover its trace is -1. The character table of A_5 is the following

Table 4. Character of A_5.

	I	(123)	(12)(34)	(12345)	(21345)
$\chi_{\Omega_{R_a}/dR_a}$	11	-1	-1	1	1
I	1	1	1	1	1
χ_2	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

Solving

$$
\begin{aligned}
11 &= a + 3b + 3c + 4d + 5e \\
-1 &= a + d - e \\
-1 &= a - b - c + e \\
1 &= a + \frac{1+\sqrt{5}}{2}b + \frac{1-\sqrt{5}}{2}c - d \\
1 &= a + \frac{1-\sqrt{5}}{2}b + \frac{1+\sqrt{5}}{2}c - d
\end{aligned}
$$

we get $a = d = 0$ and $b = c = e = 1$. That is to say $\chi_{\Omega_{R_a}/dR_a} = \chi_2 + \chi_3 + \chi_5$.

The conjugacy class of $\beta \circ \phi$ has 20 elements in it. The conjugacy class of the two cycle ψ has 15 elements in it. There are two conjugacy classes that have 5-cycles in them and they both have 12 elements in them.

The matrices for the above linear transformations with respect to the ordered basis $\{\omega_0, \dots, \omega_{10}\}$ The linear transformations ϕ, β and ψ have with respect to this basis have matrix representation

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\psi = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

To get bases for the three irreducible subrepresentations we need to use the projection formulae and sum over the conjugacy classes (using Mathematica).

$$\pi_{\chi_5} = \frac{1}{|A_5|}\sum_{g\in A_5}\chi_5(g)\rho(g) = \frac{1}{60}\sum_{g\in A_5}\chi_5(g)\rho(g)$$
$$= \frac{1}{60}\begin{pmatrix} 0 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & -6 & -6 \\ 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 \\ 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}.$$

□

Thus a basis of the irreducible 5 dimensional subrepresentation is $\{-\omega_0 + \omega_i + \omega_{i+5} \,|\, 1 \le i \le 5\}$. For one of the two three dimensional irreducible subrepresentations we have projection operator:

$$\pi_{\chi_2} = \frac{1}{|A_5|}\sum_{g\in A_5}\chi_2(g)\rho(g) = \frac{1}{60}\sum_{g\in A_5}\chi_2(g)\rho(g)$$
$$= \frac{\sqrt{5}}{30}\begin{pmatrix} \sqrt{5} & \xi & \xi & \xi & \xi & \xi & -\bar{\xi} & -\bar{\xi} & -\bar{\xi} & -\bar{\xi} & -\bar{\xi} \\ 1 & \xi & 1 & 0 & 0 & 1 & \bar{\xi} & 0 & 1 & 1 & 0 \\ 1 & 1 & \xi & 1 & 0 & 0 & 0 & \bar{\xi} & 0 & 1 & 1 \\ 1 & 0 & 1 & \xi & 1 & 0 & 1 & 0 & \bar{\xi} & 0 & 1 \\ 1 & 0 & 0 & 1 & \xi & 1 & 1 & 1 & 0 & \bar{\xi} & 0 \\ 1 & 1 & 0 & 0 & 1 & \xi & 0 & 1 & 1 & 0 & \bar{\xi} \\ -1 & -\xi & -1 & 0 & 0 & -1 & -\bar{\xi} & 0 & -1 & -1 & 0 \\ -1 & -1 & -\xi & -1 & 0 & 0 & 0 & -\bar{\xi} & 0 & -1 & -1 \\ -1 & 0 & -1 & -\xi & -1 & 0 & -1 & 0 & -\bar{\xi} & 0 & -1 \\ -1 & 0 & 0 & -1 & -\xi & -1 & -1 & -1 & 0 & -\bar{\xi} & 0 \\ -1 & -1 & 0 & 0 & -1 & -\xi & 0 & -1 & -1 & 0 & -\bar{\xi} \end{pmatrix}.$$

This means that a basis for U_3 is

$$\sqrt{5}\omega_0 + \sum_{i=1}^{5}(\omega_i - \omega_{i+5}),$$
$$\xi\omega_0 + \xi\omega_1 + \omega_2 + \omega_5 - \xi\omega_6 - \omega_7 - \omega_{10},$$
$$\xi\omega_0 + \omega_1 + \xi\omega_2 + \omega_3 - \omega_6 - \xi\omega_7 - \omega_8.$$

The projection operator for the other three dimensional subrepresentation is

$$
\begin{aligned}
\pi_{\chi_3} &= \frac{1}{|A_5|}\sum_{g\in A_5}\chi_3(g)\rho(g) = \frac{1}{60}\sum_{g\in A_5}\chi_3(g)\rho(g)\\
&= \frac{\sqrt{5}}{30}\begin{pmatrix}
\sqrt{5} & -\bar\xi & -\bar\xi & -\bar\xi & -\bar\xi & -\bar\xi & \xi & \xi & \xi & \xi & \xi\\
-1 & -\bar\xi & -1 & 0 & 0 & -1 & -\xi & 0 & -1 & -1 & 0\\
-1 & -1 & -\bar\xi & -1 & 0 & 0 & 0 & -\xi & 0 & -1 & -1\\
-1 & 0 & -1 & -\bar\xi & -1 & 0 & -1 & 0 & -\xi & 0 & -1\\
-1 & 0 & 0 & -1 & -\bar\xi & -1 & -1 & -1 & 0 & -\xi & 0\\
-1 & -1 & 0 & 0 & -1 & -\bar\xi & 0 & -1 & -1 & 0 & -\xi\\
1 & \bar\xi & 1 & 0 & 0 & 1 & \xi & 0 & 1 & 1 & 0\\
1 & 1 & \bar\xi & 1 & 0 & 0 & 0 & \xi & 0 & 1 & 1\\
1 & 0 & 1 & \bar\xi & 1 & 0 & 1 & 0 & \xi & 0 & 1\\
1 & 0 & 0 & 1 & \bar\xi & 1 & 1 & 1 & 0 & \xi & 0\\
1 & 1 & 0 & 0 & 1 & \bar\xi & 0 & 1 & 1 & 0 & \xi
\end{pmatrix}.
\end{aligned}
$$

So a basis for U_3' is

$$
\begin{aligned}
&\sqrt{5}\omega_0 - \sum_{i=1}^{5}(\omega_i - \omega_{i+5}),\\
&-\bar\xi\omega_0 - \bar\xi\omega_1 - \omega_2 - \omega_5 + \bar\xi\omega_6 + \omega_7 + \omega_{10},\\
&-\bar\xi\omega_0 - \omega_1 - \bar\xi\omega_2 - \omega_3 + \omega_6 + \bar\xi\omega_7 + \omega_8.
\end{aligned}
$$

5. Acknowledgements

This work was partially supported by a grant from the Simons Foundation (#319261). The author would also like to thank the Mittag-Leffler Institute and its organizers for its hospitality during part of the spring semester of 2015 where a portion of this work was done. We would also like to thank Kaiming Zhao for useful discussions.

6. Concluding Remark.

There are other rings R that arise as rings of meromorphic functions on a Riemann surface with a finite number of points removed. We plan to investigate how Ω_R/dR decomposes under the action their automorphism group of R.

References

[Blo81] Spencer Bloch. The dilogarithm and extensions of Lie algebras. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 1–23. Springer, Berlin-New York, 1981.

[Bre94] Murray Bremner. Generalized affine Kac-Moody Lie algebras over localizations of the polynomial ring in one variable. *Canad. Math. Bull.*, 37(1):21–28, 1994.

[Bre00] Thomas Breuer. *Characters and automorphism groups of compact Riemann surfaces*, volume 280 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2000.

[CGLZ14] Ben Cox, Xiangqian Guo, Rencai Lu, and Kaiming Zhao. n-point Virasoro algebras and their modules of densities. *Commun. Contemp. Math.*, 16(3):1350047, 27, 2014.

[FBZ01] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[Kas84] Christian Kassel. Kähler differentials and coverings of complex simple Lie algebras extended over a commutative algebra. In *Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983)*, volume 34, pages 265–275, 1984.

[KL82] C. Kassel and J.-L. Loday. Extensions centrales d'algèbres de Lie. *Ann. Inst. Fourier (Grenoble)*, 32(4):119–142 (1983), 1982.

[KL91] David Kazhdan and George Lusztig. Affine Lie algebras and quantum groups. *Internat. Math. Res. Notices*, (2):21–29, 1991.

[KL93] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. I, II. *J. Amer. Math. Soc.*, 6(4):905–947, 949–1011, 1993.

[Kle56] Felix Klein. *Lectures on the icosahedron and the solution of equations of the fifth degree*. Dover Publications, Inc., New York, N.Y., revised edition, 1956. Translated into English by George Gavin Morrice.

[KN87a] Igor Moiseevich Krichever and S. P. Novikov. Algebras of Virasoro type, Riemann surfaces and strings in Minkowski space. *Funktsional. Anal. i Prilozhen.*, 21(4):47–61, 96, 1987.

[KN87b] Igor Moiseevich Krichever and S. P. Novikov. Algebras of Virasoro type, Riemann surfaces and the structures of soliton theory. *Funktsional. Anal. i Prilozhen.*, 21(2):46–63, 1987.

[KN89] Igor Moiseevich Krichever and S. P. Novikov. Algebras of Virasoro type, the energy-momentum tensor, and operator expansions on Riemann surfaces. *Funktsional. Anal. i Prilozhen.*, 23(1):24–40, 1989.

[Sch14] Martin Schlichenmaier. *Krichever-Novikov type algebras*, volume 53 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, 2014. Theory and applications.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[She12] Oleg K. Sheinman. *Current algebras on Riemann surfaces*, volume 58 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, 2012. New results and applications.

[Shu97] Jerry Shurman. *Geometry of the quintic*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1997.

[Tay04] Richard Taylor. Galois representations. *Ann. Fac. Sci. Toulouse Math. (6)*, 13(1):73–119, 2004.

[Tot02] Gabor Toth. *Finite Möbius groups, minimal immersions of spheres, and moduli*. Universitext. Springer-Verlag, New York, 2002.

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