CUBES, SIDE LENGTHS AND CENTRES

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Abstract. It is known that in $\mathbb{R}^n, n \geq 2$, a compact set which contains $n-1$ spheres with all radii in $[1/2, 1]$ or with all possible centres in $[0,1]^n$ has full Hausdorff dimension. In fact the later set has positive Lebesgue measure. In this paper we consider a similar problem with sphere replacing by fractal cubes. The radii set and the centre set are also considered to be fractal sets. In addition we discuss the exceptional set in the setting of general largeness. In the end, an Furstenberg type exmaple is discussed which can be somehow considered as the Furstenberg $\times 2$, $\times 3$ set conjecture (now theorem) in the setting of cubes/circles sets considered here.

1. Introduction and results of this paper

It is known by Kolasa and Wolff [WK99] and Wolff [Wol97] that for any $n \geq 2$, any compact subset of $\mathbb{R}^n$ which contains sphere of all radii in $[1/2, 1]$ has full Hausdorff dimension. Recently, Keleti, Nagy, Shmerkin [KNS14], Thornton [Tho17] and Chang, Csörnyei, Héra, Keleti [CCHK17] considered a similar problem also introduced in [Wol97] (originally considered by Bourgain [Bou86] and Marstrand [Mar87]) but with spheres replaced by cubes. In particular in $\mathbb{R}^2$ they showed that a set which contains cubes with all centres in $[0,1] \times [0,1]$ has Hausdorff dimension at least 1 and lower box dimension at least $7/4$. Their result is sharp. The crucial difference between circles and squares is the curvature, in fact the curvature property was heavily exploited by Kolasa and Wolff [WK99] and Wolff [Wol97].

Inspired by [KNS14], we consider sets in $\mathbb{R}^n$ which contain cubes of all side length in $[1/2, 1]$. Formally we have the following definition:

Definition 1.1. *Let G be a compact subset of $\mathbb{R}^n$. Then we require that for any $r \in [1/2, 1]$, there exists a cube of side length r contained in G.*

Here we fix an orthogonal coordinate system and our cubes are aligned cubes with respect with the coordinate lines.

We shall call such set G a cube-Wolff *set.*

The situation appeared in [KNS14] can be put in the following definition:

Definition 1.2. *Let G be a compact subset of $\mathbb{R}^n$. Then we require that for any $x \in [0,1]^n$, there exists a cube centred at x contained in G.*

Here we fix an orthogonal coordinate system and our cubes are aligned cubes with respect with the coordinate lines.

We shall call such set G a cube-Wolffff *set(with four 'f' indicating the difference).*

For dimension results of cube-Wolff sets we have the following theorem.

Theorem 1.3. *For any $n \geq 1$, let $G \subset \mathbb{R}^n$ be a cube-Wolff set, then:*

$$\dim_{\mathrm{A}} G \geq \underline{\dim}_{\mathrm{B}} G \geq n - 1/2.$$

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$$\dim_{\mathrm{H}} G \geq n-1.$$

We shall construct cube-Wolff sets with Hausdorff dimension $n-1$, lower box dimension $n-1/2$ and Assouad dimension arbitrarily close to $n-1/2$ (not simultaneously). In this sense, our results above are sharp.

Despite the above results and those in [KNS14] which shows that cube-Wolff(ff) sets may not be as large as we expect even if we consider the Assouad dimension. However, we propose here another sense of largeness which can be described as 'large in general'. The idea is that we can mildly 'change' any cube-Wolff(ff) sets a little bit to achieve large dimension. Here 'mildly' is crucial, for example take any $\epsilon>0$ if we allow each cube to either translate within distance ϵ or rescale with ratio within $(1-\epsilon, 1+\epsilon)$ then we can achieve generically a large set (full upper box dimension). The point here is that do not allow too many freedoms of manipulating the set.

Towards this direction we consider the original motivation of circle maximal problems. Given a wave equation, the solution can be presented as 'wave-fronts', if the source set is given, then how large is the 'wave-front' set in relation with time? Then we can consider similar problem with 'cube-front' and precise formulation is included in the following theorems.

Theorem 1.4. *Let G be a cube-Wolff set in $\mathbb{R}^n, n\geq 1$. In particular, for all $r\in[1/2,1]$ there exists a cube $C(r)$ of side length r contained in G. If such choice is not unique, we choose any one of them. For a number $t\in[0,1]$. Denote $C_t(r)$ be the cube co-centred with $C(r)$ but the side length is scaled to tr. Denote $G_t=\bigcup_{r\in[1/2,1]} C_t(r)$. Then denote the following set:*

$$V(\sigma)=\{t\in[0,1]:\overline{\dim}_{\mathrm{B}} G_t\leq n-\sigma\}.$$

Then $\dim_{\mathrm{H}} V(\sigma)\leq 1-\sigma$.

Theorem 1.5. *Let G be a cube-Wolffff set in $\mathbb{R}^n, n\geq 1$. In particular, for all $x\in[0,1]^n$ there exists a cube $C(x)$ centred at x contained in G. If such choice is not unique, we choose any one of them. For a number $t\in[0,1]$. Denote $C_t(x)$ be the cube co-centred with $C(x)$ but the side length is scaled by multiplying t. Denote $G_t=\bigcup_{x\in[0,1]^n} C_t(x)$. Then denote the following set:*

$$V(\sigma)=\{t\in[0,1]:\overline{\dim}_{\mathrm{B}} G_t\leq n-\sigma\}.$$

Then $\dim_{\mathrm{H}} V(\sigma)\leq 1-\sigma$.

For lower box dimension we have the following result.

Theorem 1.6. *Let G be a cube-Wolff(ff) set in $\mathbb{R}^n, n\geq 1$. Let $\sigma>0$*

$$V(\sigma)=\{t\in[0,1]:\underline{\dim}_{\mathrm{B}} G_t\leq n-\sigma\}.$$

Then $\dim_{\mathrm{H}} V(\sigma)<1$.

We can compare the above results as some sort of Marstrand projection theorem although the situation here is closer to slicing.

One possible further problem is to replace the full cube by some fractal sets and study the dimension results and largeness in general. In order to study those problems we need to define some special sets in Euclidean spaces which were included in [HKM17] as special cases.

1. Fractal Kakeya book:

Let $n \geq 2$ be an integer. Consider $S^{n-1} \subset \mathbb{R}^n$ to be identified with the set of directions in $\mathbb{R}^n$. Let $C \subset S^{n-1}$ be any compact smooth curve. For any $t \in C$, we consider the hyperplane passing through the origin and orthogonal with t:

$$H(t) = \{x \in \mathbb{R}^n :< x, t >= 0\}.$$

Here $< .,. >$ denotes the Euclidean inner product in $\mathbb{R}^n$. For any $a \in \mathbb{R}^n$ we denote $H(t,a)$ to be the affine hyperplane passing through a and parallel with $H(t)$.

Definition 1.7. *For a smooth simple curve $C \subset S^{n-1} \subset \mathbb{R}^n$ and a real number $\beta \in [0, n-1]$. We say a Borel subset $K \subset \mathbb{R}^n$ to be a C, β Kakeya book if there exists a constant $c > 0$ such that:*

$$\inf_{t \in C} \sup_{a \in \mathbb{R}^n} \mathcal{H}^\beta(K \cap H(t,a)) \geq c.$$

Sometimes the dependence of curve C is not important and we simply call a set β Kakeya book if it is a C, β Kakeya book for a smooth simple curve C.

2. Fractal grass set

Definition 1.8. *Let $n \geq 2$ be an integer, consider a smooth submanifold $C \subset \mathbb{R}^n$ and a real number $\beta \in [0,1]$. A bounded Borel set $G \subset \mathbb{R}^n$ is called a C, β-grass set (or β-grass set on C) if there exists a constant c_1 such that:*

$$\forall x \in C$$

G intersects a unit line segment l_x centered at x with β-Hausdorff measure greater than c_1.

Further we require the following transverse condition with a positive constant $0 < c_2 < 1$, denote the direction of l_x as t_x:

$$\sup_{\gamma \in T_x C} \frac{< t_x, \gamma >}{|\gamma|} \leq c_2.$$

Here $T_x C$ is the tangent space of C at x and $< .,. >$ denotes the Euclidean inner product.

We state the following two conjectures. The optimistic guess is probably not true but this gives us an best possible result to achieve, maybe with some additional conditions. One of those possibilities is considered in this paper.

Conjecture 1.9 (Fractal Kakeya book)**.** *Let $K \subset \mathbb{R}^n, n \geq 3$ be a β Kakeya book, if $\beta \in (n-2, n-1)$ then:*

$$\dim_{\mathrm{H}} K \geq \beta + 1. \textit{ Optimistic guess}$$

$$\dim_{\mathrm{H}} K \geq n - 2 + \frac{3}{2}(\beta - (n-1)) + \frac{1}{2}. \textit{ Rational guess}$$

Remark 1.10. *It is known that the conclusion* $\dim_{\mathrm{H}} K \geq \beta + 1$ *is not true for $n = 2$ in its full generality by* [Wol99].

Conjecture 1.11 (Fractal grass)**.** *Let $\beta \in [0,1]$, $G \subset \mathbb{R}^2$ be a $[0,1] \times \{0\}, \beta$-grass set then:*

$$\dim_{\mathrm{H}} G \geq \beta + 1. \textit{ Optimistic guess}$$

$$\dim_{\mathrm{H}} G \geq \frac{3}{2}\beta + \frac{1}{2}. \textit{ Rational guess}$$

Remark 1.12. *Later we shall show that for $\beta = 1$ the conjecture holds in $\mathbb{R}^2$ and in $\mathbb{R}^3$ with $\beta = 1$, we can probably use Wolff's hairbrush method to show a lower bound 2.5 for the three dimensional version of grass sets.*

The relation between the above conjectures and cube-Wolff(ff) can be stated in the following meta-theorems:

Theorem 1.13 (Meta-theorem, optimistic largeness in general)**.** *We have the following statements for any $\sigma > 0$:*

1. Rational conjectures 1.9,1.11 $\implies$ $V(\sigma)$ appeared in theorem 1.3,1.4 has Hausdorff dimension $\leq 1 - 2\sigma$.

2. Optimistic conjecture 1.9,1.11 $\implies$ $V(\sigma)$ appeared in theorem 1.3,1.4 has 0 Hausdorff dimension.

Remark 1.14. *The reason that for cube-Wolff sets $1 - 2\sigma$ is a reasonable upper bound for $\dim_{\mathrm{H}} V(\sigma)$ is because $\underline{\dim}_{\mathrm{B}} G \geq n - 1/2$ for any cube-Wolff set in $\mathbb{R}^n$. While this is not the case for cube-Wolffff sets, however, from the proof we shall see that our argument for largeness in general holds for cube-Wolffff sets with all cubes replaced by its vertices.*

Now we come back to cube problems. In $\mathbb{R}^2$, if instead of the full square we only have a β-set imbedded in each square in the definition of cube-Wolff(ff) sets, then what are the dimention results?

Further for cube-Wolff sets instead of all radius in $[1/2, 1]$ we only require all radius in a α-set contained in $[1/2, 1]$. For cube-Wolffff sets instead of requiring the whole $[0, 1]^2$ to be the center set, we can also consider cubes centred within an α-set contained in $[0, 1]^2$. Here β, α are suitable real numbers.

Then we will refer those sets as (α, β)-cube-Wolff(ff) sets. It is also natural to consider (α, β)-circle-Wolff(ff) sets which can be similarly defined. We refer [Sch97] for a comprehensive discussion of circle maximal operator.

An upper bound of any dimension of (α, β)-cube(circle)-Wolff(ff) sets would be just $\min\{\alpha + \beta, 2\}$ and follows the idea of the classical Marstrand projection we would expect the value $\min\{\alpha + \beta, 2\}$ is attained in general in a similar way as in theorem 1.3, 1.4.

Theorem 1.15. *For $\alpha \in (0, 1], \beta \in (0, 1]$. Let G be a (α, β)-cube-Wolff(ff) set in $\mathbb{R}^2$. Denote*

$$V(\sigma) = \{t \in [0, 1] : \underline{\dim}_{\mathrm{B}} G_t \leq \alpha + \beta - \sigma\}.$$

Then $\forall \sigma > 0, \dim_{\mathrm{H}} V(\sigma) < 1$. This implies that for Lebesgue almost every $t \in [0, 1]$, $\underline{\dim}_{\mathrm{B}} G_t = \alpha + \beta$.

Furthermore denote:

$$W(\sigma) = \{t \in [0, 1] : \overline{\dim}_{\mathrm{B}} G_t \leq \alpha + \beta - \sigma\}.$$

Then $\forall \sigma > 0, \dim_{\mathrm{H}} W(\sigma) < \alpha$. This implies that apart from a set of Hausdorff dimension small or equal to α, G_t has upper box dimension $\alpha + \beta$.

Remark 1.16. *The above result holds for (α, β)-circle-Wolff(ff) sets as well. Since circles have non vanishing curvature, in this case we have a bit better bound of the exception set. For example the exceptional set for upper box dimension is no greater than 0.5α. From existing result on circle maximal operators for example in* [Wol97] *we can in fact show that (α, β)-circle-Wolff(ff) sets has Hausdorff dimension at least $3\beta + \alpha - 2 = \alpha + \beta + 2(\beta - 1)$. We will not discuss circle-Wolff(ff) sets in much detail. The central difference is that two 'cones' intersect on a parabola which also has non vanishing curvature.*

2. Notations and preliminaries

Here we list some notions of dimensions we shall use in this paper. All the notions of dimensions here are quite standard (except for the sliced box dimension), for Hausdorff dimension and box dimensions we refer [Mat99, chapter 4,5], [Fal04, chapter 2,3] for more details.

Here we shall use $N_r(F)$ for the minimal covering number of a set F in $\mathbb{R}^n$ with balls (sometimes with cubes but the dimension result will not differ) of radius $r > 0$.

Here we shall discuss the definitions of the dimensions of a set $F \subset \mathbb{R}^n$.

Hausdorff dimension: For any $s \in \mathbb{R}^+$, for any $\delta > 0$ define the following quantity:

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam}U_i)^s : \bigcup_i U_i \supset F, \mathrm{U_i} < \delta\right\}.$$

Then the s-Hausdorff measure of F is:

$$\mathcal{H}^s(F) = \lim_{\delta\to 0}\mathcal{H}^s_\delta(F).$$

The Hausdorff dimension of F is:

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}.$$

For $s \in \mathbb{R}^+$ An s-set in $\mathbb{R}^n$ is a Borel subset of $\mathbb{R}^n$ with positive and finite s-Hausdorff measure.

Box dimensions: The upper/lower box dimension of F is:

$$\overline{\dim_{\mathrm{B}}}/\underline{\dim_{\mathrm{B}}}(F) = \limsup_{r\to 0} / \liminf_{r\to 0}\left(-\frac{\log N_r(F)}{\log r}\right).$$

If the $limsup$ and $liminf$ are equal we call this value the box dimension of F.

Through out this paper, we shall discuss cubes or squares. To be precise when we have an Euclidean space R^n, we fix a Cartesian coordinate system. A cube centred at $x \in \mathbb{R}^n$ with side length r is the following 'layer' set of the supreme norm in Euclidean space:

$$\{y \in \mathbb{R}^n : \|y - x\|_\infty = r/2\}.$$

So we see that such a cube is aligned with the coordinate axis.

Sliced box dimensions: In this paper the notion of sliced box dimension is introduced in order to study the largeness in general property of cube-Wolff(ff) sets. We will return to this topic in a later chapter.

Let $n \geq 2$ be an integer. We consider $A \subset \mathbb{R}^n$. Let $t \in S^{n-1}$ be a directional vector, we shall consider slices of A with hyperplanes orthogonal with t.

Without loss of generality, we assume $t = (1, 0, 0, \dots, 0)$. For any $y \in \mathbb{R}$, we consider the stripe of width $\epsilon > 0$:

$$S(y, \epsilon) = \left\{X \in \mathbb{R}^n : |\pi_1 X - y| \leq \frac{\epsilon}{2}\right\}.$$

Here π_1 is the first coordinate function: $\pi_1((x_1, \dots, x_n)) = x_1$.

We also denote $A(y)$ the slice of A with first coordinate y:

$$A(y) = \pi_1^{-1}(y) \cap A.$$

Definition 2.1. *In the above setting. The sliced box dimension at $y \in \mathbb{R}$ is defined to be the following quantity:*

$$\dim^y_{\mathrm{B}} A = -\lim_{\epsilon\to 0}\frac{\log N_\epsilon(S(y,\epsilon)\cap A)}{\log \epsilon},$$

if the above limit exists. Otherwise we define the upper/lower sliced box dimension $\overline{\dim_{\mathrm{B}}}^y/\underline{\dim_{\mathrm{B}}}^y$ *by taking* $\limsup/\liminf$.

Assouad dimension:
The Assouad dimension of F is

$$\dim_{\mathrm{A}} F = \inf \Bigg\{ s \geq 0 \,:\, (\exists C > 0)\,(\forall R > 0)\,(\forall r \in (0, R))\,(\forall x \in F)$$
$$N_r(B(x, R) \cap F) \leq C \left(\frac{R}{r}\right)^s \Bigg\}$$

where $B(x, R)$ denotes the closed ball of centre x and radius R.

Later we shall discuss dyadic cubes. For $\alpha \in (0, 1]$. By saying dyadic cubes of side length $2^{-\alpha k}$ (for a fixed k) we actually mean the collection of cubes with disjoint interiors whose closure covers the whole $\mathbb{R}^n$. There are many possibilities of such collections, by convention we require the cube centred at the origin to be inside this collection and each pair of adjacent cubes share a common boundary.

When we consider Hausdorff dimension, we shall consider covering by dyadic cubes of different side length (a net with different sizes of holes). In that situation, our collection of cubes have side length 2^{-k}(with multiple choices of $k \in \mathbb{N}$) centred at dyadic rational points.

3. Proof of theorem 1.3

We shall focus on $\mathbb{R}^2$, in this situation we have a clear picture of what is going on. The arguments work for other cases after some modification. From now on we fix $n = 2$ and let G be a cube-Wolff set.

The result of Hausdorff dimension is obvious since any cube-Wolff set contains a cube and therefore has Hausdorff dimension at least 1.

To see the box dimension result. Let $\delta > 0$ be a small positive number. Then we shall find approximately $\frac{1}{100\delta}$ many 100δ-separeted points in $[1/2, 1]$. We denote those points from small to large as:

$$r_1 < r_2 < \cdots < r_K, K \approx (100\delta)^{-1}$$

For each $r_i, i \in [1, K]$, there is a cube C_{r_i} of side length r_i contained in G. Any cube contains 4 sides, therefore there are $\approx 4K$ many sides of length in $[1/2, 2]$. Now we can focus on for example the right most side I_{r_i} of each cube C_{r_i}. We know that the obstruction of G having high dimension is the heavy overlap of sides. Then we consider δ-neighbourhood $C^{\delta}_{r_i}$ of each C_{r_i}. Consider the characteristic function $\chi_{I^{\delta}_{r_i}}$, if there exist a $x \in \mathbb{R}^2$ and an integer $M > 0$ such that:

$$\sum_i \chi_{I^{\delta}_{r_i}}(x) \geq M,$$

then there are M cubes whose right sides are 2δ close to each other. Because different cubes have side length at least 100δ difference, we see that the left sides of those M cubes stays at least 96δ away from each other. This implies that we can find M sides which are 96δ away from each other therefore union of the δ-neighbourhood of those sides takes area at least:

$$0.5M\delta.$$

On the other hand, we can find an integer M such that:

$$\left\|\sum_i \chi_{I^{\delta}_{r_i}}\right\|_{\infty}=M,$$

we see that:

$$\left\|\chi_{\bigcup_i I^{\delta}_{r_i}}\right\|_{L^1}\geq\frac{1}{M}\left\|\sum_i \chi_{I^{\delta}_{r_i}}\right\|_{L^1}\geq\frac{0.5\delta\times\frac{1}{100\delta}}{M}=\frac{1}{200M}.$$

From the above argument we see that G^{δ} takes area at least:

$$\max\left\{0.5M\delta,\frac{1}{200M}\right\}\geq\frac{1}{40}\sqrt{\delta}.$$

This gives us the lower bound of the lower box dimension of G:

$$\underline{\dim_{\mathrm{B}}}G\geq 0.5.$$

For other cases, "sides" of cubes are replaced by "faces" which are subsets of $n-1$ dimensional affine hyperplanes (here we allow the case $n=1$). Arguing as above we see that there is a constant $c>0$:

$$|G^{\delta}|\geq c\max\left\{M\delta,\frac{1}{M}\right\}.$$

Therefore we established the lower bound for lower box dimension, the Assouad dimension result follows because the Assouad dimension is always greater or equal to the lower box dimension.

4. Sharpness of theorem 1.3

Dimension 1: There exists subset F of $[0,1]$ whose distance set $|F-F|$ contains $[0,1]$. Among those subsets, we can find F_1,F_2 such that $\dim_{\mathrm{H}} F_1=0$, $\dim_{\mathrm{B}} F_2=0.5$. And for every $\sigma>0$ we can find F_3 such that $\dim_{\mathrm{A}} F_3\leq 1/2+\sigma$.

The construction of F_1,F_2 can be found in [DHL+13]. For Assouad dimension, we shall use a result in [Nat92] by Nathanson which says that for all integer $n>1$ there exists a subset $B\subset\{0,\dots,n-1\}$ such that $B+B\mod n=\{0,\dots,n-1\}$ and $|B|\leq 2(n\log n)^{1/2}+2$.

Now for any integer $n>1$, we find such set B, then we let $C=B\cup(n-B\mod n)$. We can now construct Cantor set in $[0,1]$ by restricting n-ary digital expansions. Precisely, we let

$$C_n=\{x\in[0,1]: n-\text{ary expansion of x contains only digits in } C\}$$

Then it is easy to see that $|C_n-C_n|$ contains $[0,1]$. We also have the following result concerning the Assouad dimension of C_n:

$$\dim_{\mathrm{A}} C_n=\dim_{\mathrm{H}} C_n=\frac{\log|B|}{\log n}\leq\frac{\log(4(n\log n)^{1/2}+4)}{\log n}.$$

Let $F_3=C_n$ for a large enough n we see that $\dim_{\mathrm{A}} F_3\leq 1/2+\sigma$.

Remark 4.1. *For set F_3, we can also apply Solomyak's result* [Sol97] *on Palis conjecture if we only require $|F_3-F_3|$ to have positive measure.*

Higher dimensional cases: Having settled down the one dimensional case, we shall see that we can extend the one dimensional case to any higher dimensional cases. Again we focus here on $\mathbb{R}^2$ and similar arguments lead us the corresponding results in $\mathbb{R}^n, n \geq 3$.

Pick $F \subset [0,1] \times \{0\} \subset \mathbb{R}^2$ whose property we shall require later. Then for $x \in F$ we construct two lines passing through x with slope ± 1. Denote $L(F)$ to be the union of all such lines. Let $r \in |F-F|$, then there exist $x_1, x_2 \in F$ with $|x_1 - x_2| = r$. Then the lines passing through x_1, x_2 with slope ± 1 (there are four of them) will enclose a cube of side length $r/\sqrt{2}$.

Now it is easy to see that $L(F)$ can be written as a union of two subsets with lines of slope 1 or -1. Each of those two sets can be viewed as the Cartesian product of F and $\mathbb{R}$ with a certain affine transformation. From this fact it is easy to see that:

$$\dim(L(F)) = \dim F + 1,$$

here dim can be any dimension we considered above. For example we see that:

$$\dim_H L(F_1) = 1$$

and $L(F_1)$ contains cubes of all side length in $[0, \sqrt{2}/2]$. After some rescaling, we can obtain a cube-Wolff set of Hausdorff dimension 1.

The box and Assouad dimension results follow in a similar way.

5. Results concerning fractal Kakeya books and grass sets

The following results can be found in [HKM17] with greater level of generalities. In this paper we will only include detailed discussions for cases which were not included in [HKM17]. Along the lines we will give some heuristic argument how to obtain those results for readers' convenience.

Lemma 5.1. *Let $K \subset \mathbb{R}^n, n \geq 3$ be a β Kakeya book, if $\beta \in (n-2, n-1)$ then:*

$$\dim_{\mathrm{H}} K \geq n - 2 + 2(\beta - (n-1)).$$

Lemma 5.2. *Let $\beta \in [0,1]$, $G \subset \mathbb{R}^2$ be a $[0,1] \times \{0\}, \beta$-grass set then:*

$$\dim_{\mathrm{H}} G \geq 2\beta.$$

Lemma 5.3. *Let $G \subset \mathbb{R}^n, n \geq 3$ be a $([0,1]^{n-1} \times \{0\}, \beta)$-grass set. Furthermore for all*

$$(x_2, \dots, x_{n-1}) \in [0,1]^{n-2},$$

we require that $G \cap ([0,1] \times \{(x_2, \dots, x_{n-1})\} \times \mathbb{R})$ is a $([0,1] \times \{(x_2, \dots, x_{n-1}, 0)\}, \beta)$-grass set.

Then $\dim_{\mathrm{H}} G \geq n - 2 + 2\beta$.

We will show the following combinatorial version of lemma 5.2 to illustrate how the number 2β is obtained. In fact the method appeared in [Wol99].

Lemma 5.4. *Let $\rho < 1$ be a number close to 1. Let $\delta > 0$ be a small number. Let $A \subset [0,1] \times \{0\}$ be a δ^ρ-separated set with cardinality $\delta^{-\rho}$ (assumed to be an integer). The for any $a \in A$ there is an unit segment l_a passing through a with angle at least 60 degree against x-axis, for each unit segment l_a, there are $\delta^{-\beta}$ many disjoint δ-balls centred on l_a.*

Then for an absolute constant $C > 0$, the Lebesgue measure of the union of all δ-balls is greater than:

$$C\delta^{-\rho-\beta}\delta^2 - C\delta^{-2\rho}\delta^2 \log \delta^{-\rho}. \qquad (*)$$

Remark 5.5. *In order that the inequality is not trivial we need $\rho < \beta$, and that is why in the end a 2β bound on the dimension appears.*

Proof. There are $\delta^{-\rho-\beta}$ many δ-balls in total. This provide us the first term of (*). But they could intersect each other.

For two points s_1, s_2 in $[0,1]\times\{0\}$, the δ neighbourhood of l_{s_1}, l_{s_2} intersects with area at most(lemma 5.6 below):

$$0.001\delta^2 \frac{2}{|s_1 - s_2|}.$$

Then the area of intersection of all δ neighbourhoods of line segments is at most:

$$\sum_{s_i, s_j \in A} 0.001\delta^2 \frac{2}{|s_1 - s_2|} \geq 0.002\delta^2\delta^{-\rho} \sum_{i,j=1, i\neq j}^{\delta^{-\rho}} \frac{1}{|i-j|}$$

The later sum is bounded from below by $\delta^{-\rho}\log\delta^{-\rho}$. The result follows by the Bonferroni inequality. $\square$

Lemma 5.6. *Let $t_1, t_2 \in [0,1]$, consider two unit line segments l_1, l_2 centred at $(t_1, 0), (t_2, 0)$ respectively. Furthermore, the slope of l_1, l_2 is not in $[-\sqrt{3}, \sqrt{3}]$. Denote the extended line of l_1, l_2 as L_1, L_2 respectively. For any positive number $\epsilon > 0$, if $L_1^\epsilon \cap L_1^\epsilon \cap ([0,1]\times[-1,1]) \neq \emptyset$ then the angle θ between l_1, l_2 satisfies:*

$$\tan(\theta/2) \geq \frac{|t_1 - t_2|}{2} - 10\epsilon.$$

Proof. If L_1 and L_2 intersect with angle θ we see that the condition of this lemma implies $\theta \in (0, \pi/3)$. Then the intersection of their ϵ neighbourhood is contained in a ball of radius

$$\frac{10\epsilon}{\tan(\theta/2)}.$$

However, the intersecting point of L_1 and L_2 must be not very near to the line $[0,1]\times\{0\}$. In fact, if we draw a circle passing through $(t_1, 0)$ and $(t_2, 0)$ such that the angle of minor arc between $(t_1, 0)$ and $(t_2, 0)$ is precisely θ. Then we see that all possible intersecting points of L_1, L_2 are on this circle. We also know that the slope of L_1, L_2 is not in $[-\sqrt{3}, \sqrt{3}]$. In particular this implies that the distance between $L_1 \cap L_2$ and the line $[0,1]\times\{0\}$ must be more than:

$$\frac{1}{\tan(\theta/2)} \frac{|t_1 - t_2|}{2}.$$

So from the condition that

$$L_1^\epsilon \cap L_1^\epsilon \cap ([0,1]\times[-1,1]) \neq \emptyset,$$

we see that

$$\max\left\{\frac{1}{\tan(\theta/2)}\frac{|t_1 - t_2|}{2} - \frac{10\epsilon}{\tan(\theta/2)}, 0\right\} \leq 1.$$

Then we see that:

$$\tan(\theta/2) \geq \frac{|t_1 - t_2|}{2} - 10\epsilon.$$

$\square$

Lemma 5.3 stated here was not actually proved in [HKM17], for simplicity we shall show a special case when $\beta = 1$ then for general β, we can either modify the following proof with Bonferroni inequality or we use similar argument discussion in [HKM17, section 5]. We note that the L^2 argument used there was introduce by Cordoba [Cor77] and our Bonferroni argument is actually a combinatorial version of the same argument.

proof of lemma 5.3 with $\beta = 1$. Fix arbitrarily a $k_0 > 0$. Suppose we cover G with dyadic cubes of disjoint interiors of radius not greater than 2^{-k_0}. We see that for any point $x = (x_1, x_2, \dots, x_{n-1}, 0) \in [0,1]^{n-1} \times \{0\}$ there exits at least one $k \geq k_0$ such that the line segment l_x passing through x is covered significantly by cubes of side length 2^{-k}.

Formally, let

$$\mathcal{F}_k = \{B_i : i \in \mathbb{N},\ B_i \text{ has radius in } [2^{-k-1}, 2^{-k}]\} \text{ and } F_k = \bigcup_{i \in \mathcal{F}_k} B_i.$$

Then for any x as mentioned above, there exist a $k \geq k_0$ such that:

$$\mathcal{H}^1(l_x \cap F_k) \geq \frac{3}{\pi^2}\frac{1}{k^2}.$$

Now for each $(x_2, \dots, x_{n-1}) \in [0,1]^{n-2}$, we see that $G \cap ([0,1] \times \{(x_2, \dots, x_{n-1})\} \times \mathbb{R})$ is a $[0,1] \times \{(x_2, \dots, x_{n-1}, 0)\}$-grass set. We denote the following sets:

$$C_k(x_2, \dots, x_{n-1}) = \left\{ x_1 \in [0,1] : x = (x_1, \dots, x_{n-1}, 0), \mathcal{H}^1(l_x \cap F_k) > \frac{3}{\pi^2}\frac{1}{k^2} \right\}.$$

The point here is that l_x is not only passing through x, it is actually inside

$$[0,1] \times \{(x_2, \dots, x_{n-1})\} \times \mathbb{R}.$$

For each k, consider the dyadic decomposition of $[0,1]$ with disjoint intervals of length $2^{-\alpha k}$. Here we consider $\alpha < 1$. We denote the number of such intervals intersecting $C_k(x_2, \dots, x_{n-1})$ as $N_k(x_2, \dots, x_{n-1})$.

Then we see that $\mathcal{H}^1(C_k(x_2, \dots, x_{n-1})) \leq 2^{-\alpha k} N_k$. Then it is easy to see that:

$$\sum_k 2^{-\alpha k} N_k(x_2, \dots, x_{n-1}) \geq 1.$$

Therefore there exist at least one $k \geq k_0$ such that $2^{-\alpha k} N_k(x_2, \dots, x_{n-1}) \geq \frac{6}{\pi^2}\frac{1}{k^2}$.

Finally, we denote the following sets:

$$D_k = \left\{ (x_2, \dots, x_{n-1}) \in [0,1]^{n-2} : 2^{-\alpha k} N_k(x_2, \dots, x_{n-1}) \geq \frac{6}{\pi^2}\frac{1}{k^2} \right\}.$$

Let M_k denote the number of $2^{\alpha k}$ dyadic cubes intersecting D_k, we see that:

$$\sum_k 2^{-\alpha(n-2)k} M_k \geq 1,$$

therefore there exists at least one $k \geq k_0$ such that $2^{-\alpha(n-2)k} M_k \geq \frac{6}{\pi^2}\frac{1}{k^2}$.

To summarize, we obtained here a $k \geq k_0$, such that there are at least M_k $2^{-\alpha k}$-separated points in D_k. Suppose $(x_2 \dots, x_{n-2})$ is one of them. We see that there are at least $N_k(x_2, \dots, x_{n-2})$ $2^{-\alpha k}$-separated points in $C_k(x_2, \dots, x_{n-2})$. Let x_1 be one of them, then we see that for $x = (x_1, x_2, \dots, x_{n-2}, 0)$, the line segment l_x is covered significantly:

$$\mathcal{H}^1(l_x \cap F_k) > \frac{3}{\pi^2}\frac{1}{k^2}.$$

We can also obtain a cardinality version of the above inequality:

$$\#|\mathcal{F}_k \text{ intersecting } l_x| \geq \frac{3}{\pi^2 k^2} 2^{-(n-1)k} \frac{1}{2^{-nk}}.$$

Then overall we see that for some constant $c_1, c_2 > 0$ (Bonferroni):

$$\#|\mathcal{F}_k| \geq c_1 k^{-6} 2^{(1+\alpha(n-1))k} - c_2 k^{-4} \log(2^{\alpha k}) 2^{\alpha k n}.$$

For any $\alpha < 1$ we see that:

$$\#|\mathcal{F}_k| \gtrsim k^{-6} 2^{1+\alpha(n-1)}.$$

The above inequality holds for infinitely many k by choosing different $k_0 \to \infty$. We see that:

$$\dim_{\mathrm{H}} G \geq 1 + \alpha(n-1),$$

as α can be arbitrarily close to 1 we see that:

$$\dim_{\mathrm{H}} G = n.$$

□

6. Sliced box dimension

Before we prove largeness in general results, some geometric notions are needed. In this section we introduce what we shall call the *sliced box dimension* and discuss some of its properties. We will only need lemma 6.4 and lemma 6.5.

Let $n \geq 2$ be an integer. We consider $A \subset \mathbb{R}^n$. Let $t \in S^{n-1}$ be a directional vector, we shall consider slices of A with hyperplanes orthogonal with t.

Without loss of generality, we assume $t = (1, 0, 0, \ldots, 0)$. For any $y \in \mathbb{R}$, we consider the stripe of width $\epsilon > 0$:

$$S(y, \epsilon) = \left\{ X \in \mathbb{R}^n : |\pi_1 X - y| \leq \frac{\epsilon}{2} \right\}.$$

Here π_1 is the first coordinate function: $\pi_1((x_1, \ldots, x_n)) = x_1$.

We also denote $A(y)$ the slice of A with first coordinate y:

$$A(y) = \pi_1^{-1}(y) \cap A.$$

Definition 6.1. *In the above setting. The sliced box dimension at $y \in \mathbb{R}$ is defined to be the following quantity:*

$$\dim_{\mathrm{B}}^{y} A = -\lim_{\epsilon \to 0} \frac{\log N_\epsilon(S(y, \epsilon) \cap A)}{\log \epsilon},$$

if the above limit exists. Otherwise we define the upper/lower sliced box dimension $\overline{\dim_{\mathrm{B}}}^{y}/\underline{\dim_{\mathrm{B}}}^{y}$ by taking $\limsup / \liminf$.

One simple observation is that $\dim_{\mathrm{B}}^{y} A \geq \dim_{\mathrm{B}} A(y)$ for all y. It is also simple to see that such inequality can be strict.

We now show the following property of sliced box dimension.

Lemma 6.2. *Let $A \subset \mathbb{R}^n$ be a bounded Borel measurable set. Then $\forall \sigma > 0$ we can find a $y \in \mathbb{R}$, such that:*

$$\underline{\dim_{\mathrm{B}}}^{y} A \geq \dim_{\mathrm{H}} A - 1 - \sigma.$$

Proof. We shall first show that the result holds when we replace $\underline{\dim_{\mathrm{B}}}^y$ by $\overline{\dim_{\mathrm{B}}}^y$. With out loss of generality we assume that $\pi_1(A) \subset [0,1]$.

Suppose that for a $\sigma > 0$ we have for all $y \in [0,1]$ such that:

$$\overline{\dim_{\mathrm{B}}}^y A < \dim_{\mathrm{H}} A - 1 - \sigma.$$

By definition, for all $y \in [0,1]$, there exists a $r(y) > 0$ such that:

$$(*) \qquad r < r(y) \implies N_r(S(y,r) \cap A) \leq \left(\frac{1}{r}\right)^{\dim_{\mathrm{H}} A - 1 - 0.5\sigma}$$

By Besicovich covering theorem (see [Mat99, theorem 2.7]) we see that for all $\delta > 0$, we can find a countable collection of intervals $I_i = (y_i - 0.5r_i, y_i + 0.5r_i), i \in \mathbb{N}$ and a positive number M (depends on n) such that:

$$\forall i \in \mathbb{N}, r_i \leq \delta.$$

$$\forall i \in \mathbb{N}, r_i \leq r(y_i).$$

$$\left\| \sum_i \chi_{I_i} \right\|_\infty < M$$

Further more $I_i, i \in \mathbb{N}$ form a covering of $\pi_1(A)$.

In particular we can find a covering of A with collection of $N_r(S(y_i, r_i) \cap A)$ many balls of radius r_i with i ranging over $\mathbb{N}$. Therefore we see that for $s \in \mathbb{R}$:

$$\mathcal{H}^s_\delta(A) \leq \sum_i N_r(S(y_i, r_i) \cap A)(r_i)^s \leq \sum_i (r_i)^s \left(\frac{1}{r_i}\right)^{\dim_{\mathrm{H}} A - 1 - 0.5\sigma}.$$

Then it is easy to see that whenever $s > \dim_{\mathrm{H}} A - 0.4\sigma$:

$$\mathcal{H}^s_\delta(A) \leq \sum_i (r_i)^{1+0.1\sigma} \leq \delta^{0.1\sigma} \sum_i r_i \leq M\delta^{0.1\sigma}.$$

Therefore we see that:

$$\mathcal{H}^s(A) = 0.$$

The above argument shows that $\dim_{\mathrm{H}} A \leq s$ for all such s and therefore:

$$\dim_{\mathrm{H}} A \leq \dim_{\mathrm{H}} A - 0.4\sigma.$$

This is a contradiction.

For lower sliced box dimension. We see that (*) is no longer true. However we still have the following statement:

$$\forall y \in \mathbb{R}, \forall \epsilon > 0, \exists r < \epsilon \text{ such that } N_r(S(y,r) \cap A) \leq \left(\frac{1}{r}\right)^{\dim_{\mathrm{H}} A - 1 - 0.5\sigma}$$

Now with y ranging over $[0,1]$ and for each y we can find a sequence $r_i(y) \to 0$ such that the conclusion of the above statement holds. We see that the conditions of Besicovich covering theorem ([Mat99, theorem 2.7]) are still satisfied and we can repeat the above argument to obtain the result concerning lower sliced box dimension. $\square$

We recall here the Vitali covering theorem for Lebesgue measure ([Mat99, theorem 2.8] for more general Radon measure result)

Theorem (Vitali)**.** *Let $\mathcal{B}$ be a Vitali convering system for $E \subset [0,1]$, namely, for all $x \in E$ there exist infinitely many $B \in \mathcal{B}$ with arbitrarily small radius such that:*

$$x \text{ is the center of } B.$$

Then there exists a countable sub-collection $\mathcal{B}_1 \subset \mathcal{B}$ such that $\mathcal{B}_1$ consists disjoint intervals and

$$E \setminus \bigcup_{B \in \mathcal{B}_1} B \text{ is a Lebesgue null set.}$$

From the above theorem, we can obtain the following result which has stronger conclusion but requires stronger conditions at the same time.

Lemma 6.3. *Let $A \subset \mathbb{R}^n$ be a bounded Borel measurable set. Suppose that $\pi_1(A) = [0,1]$, if there exists a positive number $t > 0$ such that whenever $E \subset [0,1]$ has positive Lebesgue measure we have the following bound:*

$$\dim_{\mathrm{H}}(A \cap \pi_1^{-1}(E)) \geq t,$$

then:

$$\left\{y \in [0,1] : \underline{\dim_{\mathrm{B}}}^y A \geq t-1\right\} \text{ has full Lebesgue measure.}$$

Proof. Denote for each $\sigma > 0$ the following set:

$$I_\sigma = \left\{y \in [0,1] : \underline{\dim_{\mathrm{B}}}^y A \geq t-1-\sigma\right\}.$$

Denote μ to be the Lebesgue measure. Fix an arbitrary $\sigma > 0$, suppose that:

$$\mu(I_\sigma) < 1.$$

Then $\mu(I^c_\sigma) > 0$. We have the following statement:

$$y \in I^c_\sigma \implies \underline{\dim_{\mathrm{B}}}^y A < t-1-\sigma.$$

Therefore for all $y \in I^c_\sigma$, there exists a countable sequence of positive numbers $r_i(y) \to 0$ as $i \to \infty$ such that:

$$N_{r_i(y)}(S(y, r_i(y)) \cap A) \leq \left(\frac{1}{r_i(y)}\right)^{t-1-0.5\sigma}.$$

We see that $r_i(y)$ with i ranging over $\mathbb{N}$ and y ranging over I^c_σ is a Vitali covering system for I^c_σ.

By Vitali covering theorem for Lebesgue measure we see that there exists a countable collection of disjoint intervals B_i such that:

$$\mu(I^c_\sigma \setminus \bigcup_i B_i) = 0.$$

Let us fix an integer k by the above argument we can find a collection of disjoint intervals $B_{k,i}$ for such that:

$$\mu(B_{k,i}) \leq 2^{-k},$$
$$\mu(I^c_\sigma \setminus \bigcup_i B_{k,i}) = 0.$$

Denote the centre of $B_{k,i}$ by $y_{k,i} \in I^c_\sigma$ and the length by $r_{k,i}$ ($r_{k,i} \leq 2^{-k}$). Then we see that for each k there exists a covering of $A \cap \pi_1^{-1}(\bigcup_i B_{k,i})$ with collections of $N_{r_{k,i}}(S(y_{k,i}, r_{k,i}) \cap A)$ many balls of radius $r_{k,i}$ and i ranging over $\mathbb{N}$. We can perform the above step for any $k \geq k_0$ with an arbitrarily fixed integer k_0.

Let A' be the following set:

$$A' = A \bigcap_{k\geq k_0} \pi_1^{-1}\left(\bigcup_i B_{k,i}\right) \subset A\cap \pi_1^{-1}\left(\bigcup_i B_{k_0,i}\right).$$

From the above argument we see that:

$$\mu\left(I_\sigma^c \setminus \bigcap_{k\geq k_0}\left(\bigcup_i B_{k,i}\right)\right) = 0$$

therefore we see that:

(**) $$\dim_{\mathrm{H}}(A') \geq t.$$

Now we see that (k_0 is fixed and $\delta = 2^{-k_0}$):

$$\begin{aligned}\mathcal{H}^s_\delta(A') &\leq \mathcal{H}^s_\delta\left(A\cap \pi_1^{-1}\left(\bigcup_i B_{k_0,i}\right)\right)\\ &\leq \sum_i N_{r_{k_0,i}}(S(y_{k_0,i}, r_{k,i})\cap A)(r_{k_0,i})^s \leq \sum_i (r_{k_0,i})^s\left(\frac{1}{r_{k_0,i}}\right)^{t-1-0.5\sigma}.\end{aligned}$$

Now if $s > t-0.4\sigma$ then:

$$\mathcal{H}^s_\delta(A') \leq \sum_i r_{k_0,i}^{1+0.1\sigma} \leq 2^{-0.1\sigma k}\sum_i r_{k_0,i} \leq 2^{-0.1\sigma k_0} = \delta^{0.1\sigma}.$$

Where the last inequality follows from the fact that $B_{k_0,i}, i\in\mathbb{N}$ are disjoint. As $k_0\to\infty$ we see that $\delta\to 0$ and hence we see that:

$$\mathcal{H}^s(A') = 0.$$

This implies that for all $s > t-0.4\sigma$:

$$\dim_{\mathrm{H}} A' \leq s.$$

Together with (**) we see that:

$$t \leq \dim_{\mathrm{H}} A' \leq t - 0.4\sigma.$$

This is impossible and therefore we see that:

$$\forall \sigma > 0, \mu(I_\sigma) = 1.$$

Then we see that:

$$\left\{y\in[0,1] : \underline{\dim_{\mathrm{B}}}^y A \geq t-1\right\} = \bigcap_{k\in\mathbb{N}} I_{k^{-1}}$$

the above set has full measure and this concludes the proof. □

Lemma 6.4. *Let $A\subset\mathbb{R}^n$ be a bounded Borel measurable set. Let $h : [0,1]\to[0,1]$ be a continuous function such that for all $E\subset\pi(A)$ with:*

$$\dim_{\mathrm{H}} E \geq \tau \implies \dim_{\mathrm{H}}(\pi_1^{-1}(E)\cap A) \geq h(\tau).$$

Then we have the following:

$$\dim_{\mathrm{H}} E \geq \tau \implies \sup_{y\in E} \overline{\dim_{\mathrm{B}}}^y A \geq h(\tau) - \tau.$$

Proof. Let $\dim_{\mathrm{H}} E \geq \tau$. Then let $0 < \tau_1 < \tau$, we see that $\mathcal{H}^{\tau_1}(E) = \infty$.

By taking a subset if necessary we can assume that $\mathcal{H}^{\tau_1}(E) = 1$. Further, by Egorov theorem, after dropping out a small $\mathcal{H}^{\tau_1}$ measure subset of E if necessary we can assume that the lim sup in the definition of lower sliced box dimension along $r_i = i^{-1}$ is uniform on E:

$$\overline{\dim_{\mathrm{B}}}^y A = -\limsup_{i\to\infty} \frac{\log N_{i^{-1}}(S(y, i^{-1}) \cap A)}{\log i^{-1}}.$$

This means that for any $\delta > 0$ there exists a uniform constant C_δ which does not depend on $y \in E$ such that:

$$\forall i \in \mathbb{N}, N_{i^{-1}}(S(y, i^{-1}) \cap A) \leq C_\delta \left(\frac{1}{i}\right)^{\overline{\dim_{\mathrm{B}}}^y A + \delta}.$$

Because $\mathcal{H}^{\tau_1}(E) = 1$, for all small enough $\epsilon > 0$ we can find a countable covering $B_k, k \in \mathbb{N}$ of E such that:

$$\mathrm{diam}(\tilde{B}_k) \leq \epsilon, \sum_k \mathrm{diam}(\tilde{B}_k)^{\tau_1} < 2.$$

In particular for all $y \in E$ there is a $B(y)$ among those $B_k, k \in \mathbb{N}$ such that $y \in B(y)$. $B(y)$ may not be centred at y, however it is simple to see that:

$$B(y) \subset \tilde{B}(y)$$

where $\tilde{B}(y)$ is the interval centred at y with length 2 times that of $B(y)$.

Then we obtain a covering of $\pi_1^{-1}(E) \cap A$ so that:

$$\mathcal{H}^d_\epsilon(\pi_1^{-1}(E) \cap A) \leq \sum_k C_\delta \mathrm{diam}(B_k)^d \mathrm{diam}(B_k)^{-(\overline{\dim_{\mathrm{B}}}^y A + \delta)}.$$

If $d > \sup_{y\in E} \overline{\dim_{\mathrm{B}}}^y A + 2\delta + \tau_1$ then we see that:

$$\mathcal{H}^d_\epsilon(\pi_1^{-1}(E) \cap A) \leq C_\delta \epsilon^\delta \sum_k \mathrm{diam}(B_k)^{\tau_1} \leq 2C_\delta \epsilon^\delta.$$

This implies that for all $\delta > 0, \tau_1 < \tau$:

$$\dim_{\mathrm{H}}(\pi_1^{-1}(E) \cap A) \leq \sup_{y\in E} \overline{\dim_{\mathrm{B}}}^y A + 2\delta + \tau_1.$$

This implies further that:

$$\dim_{\mathrm{H}}(\pi_1^{-1}(E) \cap A) \leq \sup_{y\in E} \overline{\dim_{\mathrm{B}}}^y A + \tau.$$

On the other hand the condition of this lemma implies that:

$$\dim_{\mathrm{H}}(\pi_1^{-1}(E) \cap A) \geq h(\tau).$$

Therefore we see that:

$$\sup_{y\in E} \overline{\dim_{\mathrm{B}}}^y A \geq h(\tau) - \tau.$$

□

For the end point $\dim_{\mathrm{H}} E = 1$, we can obtain the following slightly stronger lemma:

Lemma 6.5. *Let $A \subset \mathbb{R}^n$ be a bounded Borel measurable set and $h(1)$ be a positive number such that for Borel subsets E of $\mathbb{R}$:*

$$\dim_{\mathrm{H}} E = 1 \implies \dim_{\mathrm{H}}(\pi_1^{-1}(E) \cap A) \geq h(1).$$

Then we have the following:

$$\dim_{\mathrm{H}} E = 1 \implies \sup_{y \in E} \underline{\dim_{\mathrm{B}}}^y A \geq h(1) - 1.$$

Proof. Without loss of generality we assume that $E = \pi_1(A)$ with Hausdorff dimension 1.

Let $\sigma > 0$, then by definition we see that:

$$\forall y \in \pi_1(A), \forall \epsilon > 0, \exists r < \epsilon \text{ such that } N_r(S(y,r) \cap A) \leq \left(\frac{1}{r}\right)^{\sup_{y \in \pi_1(A)} \underline{\dim_{\mathrm{B}}}^y A + \sigma}$$

By Besicovich covering theorem (see [Mat99, theorem 2.7]) we see that for all $\delta > 0$, we can find a countable collection of intervals $I_i = (y_i - 0.5r_i, y_i + 0.5r_i), i \in \mathbb{N}$ and a positive number M (depends on n) such that:

$$\forall i \in \mathbb{N}, r_i \leq \delta.$$

$$\forall i \in \mathbb{N}, r_i \leq r(y_i).$$

$$\left\| \sum_i \chi_{I_i} \right\|_\infty < M$$

Further more $I_i, i \in \mathbb{N}$ form a covering of $\pi_1(A)$.

In particular we can find a covering of A with collection of $N_r(S(y_i, r_i) \cap A)$ many balls of radius r_i with i ranging over $\mathbb{N}$. Therefore we see that for $s \in \mathbb{R}$:

$$\mathcal{H}^s_\delta(A) \leq \sum_i N_r(S(y_i, r_i) \cap A)(r_i)^s \leq \sum_i (r_i)^s \left(\frac{1}{r_i}\right)^{\sup_{y \in \pi_1(A)} \underline{\dim_{\mathrm{B}}}^y A + \sigma}.$$

Then it is easy to see that whenever $s > \sup_{y \in \pi_1(A)} \underline{\dim_{\mathrm{B}}}^y A + 1 + 1.1\sigma$:

$$\mathcal{H}^s_\delta(A) \leq \sum_i (r_i)^{1+0.1\sigma} \leq \delta^{0.1\sigma} \sum_i r_i \leq M\delta^{0.1\sigma}.$$

Therefore we see that:

$$\mathcal{H}^s(A) = 0.$$

The above argument shows that $\dim_{\mathrm{H}} A \leq s$ for all such s and therefore:

$$h(1) \leq \dim_{\mathrm{H}} A \leq \sup_{y \in \pi_1(A)} \underline{\dim_{\mathrm{B}}}^y A + 1.$$

Thus the result follows;.

$\square$

7. Largeness in general

Now we can finally prove theorem 1.4. We first show that the exceptional set has Lebesgue zero then we point out how to obtain the dimension results.

Let $n \geq 1$ be an integer. Let $G \subset \mathbb{R}^n$ be a cube-Wolff set. We shall define the 'cone'-set of G as follows:

$$\mathrm{Cone}(G) = \left\{(t, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : t \in [0,1], (x_2, \dots, x_{n+1}) \in G_t\right\}.$$

Intuitively the 'cone'-set $\mathrm{Cone}(G)$ looks like a union of pyramids.

From the construction we see that:

$$(\mathrm{Cone}(G))(t) = G_t,$$

recall that $A(t)$ denotes the section $\pi_1^{-1}(t) \cap A$.

It is not hard to see that for any $E \subset [0,1]$ with positive measure, the following set:

$$\pi_1^{-1}(E) \cap \mathrm{Cone}(G)$$

is a Kakeya book in $\mathbb{R}^{n+1}$.

Therefore by lemma 5.1 with $\beta = n$ we see that $\dim_{\mathrm{H}}(\pi_1^{-1}(E) \cap \mathrm{Cone}(G)) = n + 1$.

By lemma 6.3 we see that for Lebesgue a.e $t \in [0,1]$:

$$\underline{\dim_{\mathrm{B}}}^t \mathrm{Cone}(G) = n.$$

Fix such a t. Let $\sigma > 0$ be an arbitrarily chosen positive number. We see that for all small enough $r > 0$:

$$N_r(S(t,r) \cap \mathrm{Cone}(G)) > \left(\frac{1}{r}\right)^{n-\sigma}.$$

Now consider the set

$$S(t,r) \cap \mathrm{Cone}(G),$$

covering this set with r-balls is essentially the same as covering $\bigcup_{y \in (t-r/2, y+r/2)} G_y$ with r-balls. From the construction of 'cone'-set we see that there exists an absolute constant $C > 0$ such that:

$$\bigcup_{y \in (t-r/2, y+r/2)} G_y \subset G_t^{Cr},$$

here G_t^{Cr} is the Cr-neighbourhood of G_t. Then form the above arguments we see that for all small enough $r > 0$:

$$N_r(G_t^{Cr}) \geq \left(\frac{1}{r}\right)^{n-\sigma}.$$

This implies that $\underline{\dim_{\mathrm{B}}} G_t \geq n - \sigma$. As $\sigma > 0$ is arbitrary we see that:

$$\underline{\dim_{\mathrm{B}}} G_t = n.$$

The above equality holds for Lebesgue a.e $t \in [0,1]$.

Now we want to estimate the size of exceptional set by dimension. Again by lemma 5.1 with $\beta = n - 1 + \tau$, we see that G satisfies lemma 6.4 with $h(\tau) = n - 1 + 2\tau$.

Then let

$$\tilde{V}(\sigma) = \{t \in [0,1] : \overline{\dim_{\mathrm{B}}}^t \mathrm{Cone}(G) \leq n - \sigma\}.$$

We see that:

$$n - \sigma \geq \sup_{y \in \tilde{V}(\sigma)} \overline{\dim_{\mathrm{B}}}^y \mathrm{Cone}(G) \geq h(\dim_{\mathrm{H}} \tilde{V}(\sigma)) - \dim_{\mathrm{H}} \tilde{V}(\sigma) = n - 1 + \dim_{\mathrm{H}} \tilde{V}(\sigma).$$

Therefore we see that:

$$\tilde{V}(\sigma) \le 1-\sigma.$$

For the end point case let

$$\hat{V}(\sigma)=\{t\in[0,1]:\underline{\dim_{\mathrm{B}}}^t\mathrm{Cone}(G)\le n-\sigma\}.$$

Suppose $\dim_{\mathrm{H}}\hat{V}=1$, then we can use lemma 5.3, lemma 6.5:

$$n-\sigma\ge\sup_{y\in\hat{V}(\sigma)}\underline{\dim_{\mathrm{B}}}^y\mathrm{Cone}(G)\ge n-1+2-1,$$

this is not possible when $\sigma>0$.

The proof for cube-Wolffff sets is the same, we shall use lemma 5.3 instead of lemma 5.1 in the above argument. The reason is that if we focus on a vertex (upper left one) of each cube, then we obtain a grass set as a subset of $\mathrm{Cone}(G)$. Then we can use the results related with grass sets.

8. (α,β)-cube-Wolffff sets, proof of theorem 1.15

In this section we discuss the general largeness of α,β-cube-Wolffff sets. For the correspoding cube-Wolff sets, the arguments are similar and we shall not give too much details here.

Now consider an α set contained in $[0,1]^2$. Let $G\subset\mathbb{R}^2$ be such that for all point $a\in A$ there exists a β-square contained in G and centred at a whose side length is within $[1/2,1]$. Here a β-square is a β set contained in a square.

We can construct $\mathrm{Cone}(G)$ in a similar way considered in previous sections. Now $\mathrm{Cone}(G)\subset\mathbb{R}^3$ is an union of 'fractal'-pyramid. We want to show that whenever E is a subset of interval of full Hausdorff dimension:

$$\dim_{\mathrm{H}}\mathrm{Cone}(G)\cap\pi_1^{-1}(E)=\min\{3,\alpha+\beta+1\}. \tag{**}$$

Then we can use lemma 6.5 to show that the sliced box dimension of $\mathrm{Cone}(G)$ is $\min\{2,\alpha+\beta\}$ with perhaps an exceptional set of Hausdorff dimension smaller than 1. But now for a generic $t\in[0,1]$ and a positive number $\delta>0$ the box counting $N_\delta(\pi_1^{-1}(B(t,\delta))\cap\mathrm{Cone}(G))$ is somehow the same as box counting of G_t^δ: there exists an absolute constant $C>1$ such that:

$$C^{-1}N_\delta(G_t^{C^{-1}\delta})\le N_\delta(\pi_1^{-1}(B(t,\delta))\cap\mathrm{Cone}(G))\le CN_\delta(G_t^{C\delta}).$$

Now we are left with proving (**) for $\alpha\le1$ to conclude the theorem.

We shall show the box dimension result and the Hausdorff dimension result follows by a further pigeonhole principle. Now let $A\subset[0,1]^2$ be an α-set. Then there exist a direction θ such that $\pi_\theta(A)$ has Hausdorff dimension $\min\{\alpha,1\}$ by Marstrand's projection theorem. We can require θ to be within angle $[44,45]$ degree against the x-axis.

We shall now assume $\alpha\le1$. The projection $\pi_\theta(A)$ might not be a α set, but we can assume it to be an α^--set by taking a subset if necessary and since α^- can be made arbitrarily close to α we can see in the end it is no harm to assume further that $\pi_\theta(A)$ is an α-set. With exact the same reasoning we can also assume that E is an 1-set. And the intersection with $\pi_1^{-1}(E)$ is implicitly assume in the following discussion.

For any two points in $\pi_\theta(A)$ there exist two corresponding points on the fibers in A. More precisely, take

$$\pi_\theta((x,y))=x\cos\theta+y\sin\theta,$$

then for any two points $a_1, a_2 \in \pi_\theta(A)$ we can choose two elements b_1, b_2

$$b_1 \in \pi_\theta^{-1}(a_1), b_2 \in \pi_\theta^{-1}(a_2).$$

If there are multiple choices we only need to chose one of them. In the end we obtain a section of $\pi_\theta(A)$ in A. We denote this section as set S. For each $s \in S$ there exist a pyramid with the vertex at s. We can take one of the four sides of each pyramid, for convenience we choose the left face of each pyramid which is orthogonal with x-axis. For a point $s \in S$, we call the left face orthogonal with x-axis of the pyramid with vertex at s F_s, then F_s is a triangle and for any two $s_1, s_2 \in S$, F_{s_1} and F_{s_2} are parallel.

Now we shall assume all the α-sets and β-sets in considerations are actually of box dimension β, α in a uniform way. Namely for $\delta > 0$ which is small enough all β-sets have δ box counting number equal to $\delta^{-\beta}$ and all α-sets have δ box counting number equal to $\delta^{-\beta}$.

Because θ is transverse to y-axis, we see that for two points s_1, s_2, the intersection of δ-neighbourhoods $F^\delta_{s_1} \cap F^\delta_{s_2}$ is at most:

$$C\delta\delta^{1-\beta}\frac{1}{|\pi_1(s_1) - \pi_1(s_2)|}$$

for a constant $C > 0$(see lemma 5.6 for details). Here the power $1 - \beta$ comes from the fact that each F_s is essentially a Cartesian product of a β-set with the unit interval $[0, 1]$.

Now fix a $\rho < 1$ and let $\delta > 0$ be of form 2^{-m} with integer $m \to \infty$. For each such δ choose a finite subset K of S which is δ^ρ separated and with cardinality within $[0.5\delta^{-\rho\alpha}, \delta^{-\rho\alpha}]$. Then each $F^\delta_k, k \in K$ has box counting number within $[0.5\delta^{-\beta-1}, \delta^{-\beta-1}]$. They sum up to at least

$$0.25\delta^{-\rho\alpha-\beta-1}.$$

We can then use Bonferroni inequality to obtain a lower bound of the δ box counting number of $\bigcup_k F^\delta_k$:

$$N_\delta\left(\bigcup_k F^\delta_k\right) \geq 0.25\delta^{-\rho\alpha-\beta-1} - C\sum_{i,j\in K}\frac{\delta^{-\beta}}{|i-j|\delta^{\rho\alpha}},$$

the sum in the above inequality can be bounded from above by:

$$C\delta^{-\rho\alpha-\beta}\delta^{-\rho\alpha}\log\delta^{-\rho\alpha}$$

because $\alpha \leq 1$ we see that for all small enough δ:

$$N_\delta\left(\bigcup_k F^\delta_k\right) \geq 0.1\delta^{-\rho\alpha-\beta-1}.$$

Then as we can also choose ρ close to one, we see that the lower box dimension of $\mathrm{Cone}(G)$ is at least $\alpha + \beta + 1$ if $\alpha \leq 1$ at least if all the fractal sets attain their dimension uniformly as mentioned before.

Now we shall pursue a pigeonhole principle for Hausdorff dimension.Fix a number $\rho \in (0, 1)$. For each $s \in S$, F_s is a $\beta + 1$ set, and therefore:

$$\lim_{\delta\to 0}\mathcal{H}^{\beta+1}_\delta(F_s) > 0.$$

Because S has positive α-Hausdorff measure, we can find a subset S' of S with positive α-Hausdorff measure and a positive number $\epsilon > 0$:

$$s \in S' \implies \lim_{\delta\to 0}\mathcal{H}^{\beta+1}_\delta(F_s) > \epsilon.$$

Further by Egorov's theorem we can assume the above limit holds uniformly within S'.

For convenience we denote S' with the original notation S.

Choose a large enough integer m and consider any covering with dyadic cubes of side length smaller than 2^{-m}. Since every F_s is a $\beta+1$ set (in an uniform way as discussed above), consider the following:

$$C(s,k) = \{\text{dyadic cubes of side length } 2^{-k} \text{ intersecting } F_s\},$$

$$|C(s,k)| = \bigcup_{\text{Cube}\in C(s,k)} \text{Cube},$$

$$\#C(s,k) = \text{Cardinality of } C(s,k).$$

Then we see that:

$$\sum_{k\geq m} \#C(s,k)2^{-(\beta+1)k} \geq \epsilon.$$

Then there exists a $k\geq m$ such that:

$$\#C(s,k) \geq \frac{\epsilon\pi^2}{6k^2}2^{-(\beta+1)k}.$$

Now for any $k\geq m$, consider the following:

$$D(k) = \left\{s\in S : C(s,k) \geq \frac{\epsilon\pi^2}{6k^2}2^{-(\beta+1)k}\right\}.$$

$$\#D(k) = \text{cardinality of a maximal } 2^{-\rho k} \text{ separeted subset of } D(k)$$

Then we see that by choosing another larger m and smaller ϵ if necessary:

$$\sum_{k\geq m} \#D(k)2^{-\rho\alpha k} \geq \epsilon.$$

Then there exists a $k\geq m$ such that:

$$\#D(k) \geq \frac{\epsilon\pi^2}{6k^2}2^{-\rho\alpha k}.$$

Choose such a k, dropping all other dyadic cubes. Then we are in the situation such that we can apply the Bonferroni argument discussed before. From here we see that:

$$\dim_{\mathrm{H}} \mathrm{Cone}(G)\cap\pi_1^{-1}(E) = \alpha+\beta+1$$

whenever $\dim_{\mathrm{H}} E = 1$ and $\alpha\leq 1$. From here the frist part of this theorem concludes.

For the upper dimension result, we should work with lemma 6.4. Let E be a τ-set and by the same argument as before:

$$N_\delta\left(\bigcup_k F_k^\delta\right) \geq 0.25\delta^{-\rho\alpha-\beta-\tau} - C\delta^{-\rho\alpha-\beta}\delta^{-\rho\alpha}\log\delta^{-\rho\alpha}.$$

Then if $\tau\geq\alpha$

$$N_\delta\left(\bigcup_k F_k^\delta\right) \geq 0.1\delta^{-\rho\alpha-\beta-\tau}.$$

This implies that

$$\dim_{\mathrm{H}} \mathrm{Cone}(G)\cap\pi_1^{-1}(E) = \alpha+\beta+\tau.$$

And the second part of this theorem concludes.

9. An example with Assouad dimension being always maximal

With the largeness in general results at hand. We can construct a class of (α, β)-cube(circle)-Wolffff sets in $\mathbb{R}^2$. A long the lines of the celebrate Furstenberg conjectures see [Shm17] and [Wu16] together with proofs of one of the conjectures. For original reference see [Fur70]. The idea is that if the centres of cubes/circles form a self-similar set or other sort set invariant under some dynamics and the sizes of cubes/circles are also assigned in a dynamical way which is 'independent' of the dynamics of the centres then we should expect the 'largeness in general' actually implies 'largeness for all'.

Given a family of self-similarities $f_1, \dots, f_k$ for an integer $k \geq 2$. Let the contraction ratio be $r_i, i \in \{1, \dots, k\}$ and translations be $a_i, i \in \{1, \dots, k\}$. Then given any string $S = s_1 s_2 \dots s_N$ of $\{1, \dots, k\}$ of length N we denote the map:

$$P_S = f_{s_N} \circ \cdots \circ f_{s_1}.$$

Suppose the attractor K of the self-similarities has open set condition. We can assume $(0,0) \in K$.

We put a β-cube(circle) centred at $(0,0)$ with radius 1. Given positive numbers $\rho_i, i \in \{1, \dots, k\}$. For any string S, we put a β-cube(circle) centred at $P_S((0,0))$ with radius $\prod_{i=1}^{N} \rho_{s_i}$. For each integer N we denote G^S to be the union of all above mentioned circles centred at $P_{S'}((0,0))$ with S' ranging over all strings with initial(prefix) S. This notation will not conflict the previous notion of G_t with $t \in [0,1]$.

The resulting set is not quite a cube(circle)-Wolffff set because it based on a countable set with Hausdorff dimension 0. But we should now discuss the Assouad dimension of such a set. For convenience for each $a \in A$ we denote C_a as the circle centred at A which is contained in G.

Assume now that there exist $i, j \in \{1, \dots, k\}$ such that:

$$\frac{\rho_i}{r_i} < 1 < \frac{\rho_j}{r_j}$$

and

$$\frac{\log(\rho_i/r_i)}{\log(\rho_j/r_j)} \notin \mathbb{Q}.$$

Let G be the union of all the circles constructed above. Then $\dim_{\mathrm{A}} G_t = \min\{\alpha + \beta, 2\}$ for all $t \in (0,1]$.

Without loss of generality we assume that $\alpha + \beta \leq 2$ and $\dim_{\mathrm{A}} G < \alpha + \beta$. Let κ be a number smaller than $\alpha + \beta$ but greater than $\dim_{\mathrm{A}} G$, let $\delta > 0$ be a small positive number such that for any two positive numbers $r, R : 0 < r < R < 1, R/r > \delta^{-1}$ we have the following inequality:

$$N_r(B(x,R) \cap G) < \left(\frac{R}{r}\right)^{\kappa}. \tag{*}$$

For convenience let $\rho_{1,2}, r_{1,2}$ be such that:

$$\frac{\rho_1}{r_1} < 1 < \frac{\rho_2}{r_2}$$

and

$$\frac{\log(\rho_1/r_1)}{\log(\rho_2/r_2)} \notin \mathbb{Q}.$$

If we consider G^1 and perform the inverse map f_1^{-1} restricted on the centres of circles that build up G^1 and at the same time rescale each circle with a factor r_1^{-1} then the resulting set we obtain is a union of circles with centres exactly the same as G, namely A. For all $a \in A$ the circle centred at A has radius ρ_1/r_1 times that of C_a. Which means we obtained G_{ρ_1/r_1}. Then we see that:

$$N_\delta(G_{\rho_1/r_1}) = N_{r_1\delta}G^1.$$

The same equality holds for G^2 as well. And in general for any string S consists only $1, 2$ of length N. For example 121122. Then we see that:

$$N_\delta(G_{\prod_i \rho_{s_i}/r_{s_i}}) = N_{\delta \prod_i r_{s_i}} G^S.$$

Now we shall carefully choose sequences S such that:

$$\prod_i \frac{\rho_{s_i}}{r_{s_i}} \approx 1.$$

Indeed, if a sequence S has m $1's$ and n $2's$ then

$$\prod_i \frac{\rho_{s_i}}{r_{s_i}} = \left(\frac{\rho_{s_1}}{r_{s_1}}\right)^m \left(\frac{\rho_{s_2}}{r_{s_2}}\right)^n.$$

Then we see that:

$$\log \prod_i \frac{\rho_{s_i}}{r_{s_i}} = m \log(\rho_1/r_1) + n \log(\rho_2/r_2),$$

now because $\log(\rho_1/r_1) < 0 < \log(\rho_2/r_2)$ and they are rational independent, we can see that

$$\left\{m \log(\rho_1/r_1) + n \log(\rho_2/r_2) : (m, n) \in \mathbb{Z}_+^2\right\}$$

is dense in $\mathbb{R}$ in particular it is dense in $[-1/2, 1/2]$, we only choose (m, n) such that:

$$m \log(\rho_1/r_1) + n \log(\rho_2/r_2) \in [-1/2, 0].$$

There are infinitely many different choices and we see that for the corresponding strings S:

$$\prod_i \frac{\rho_{s_i}}{r_{s_i}} = \left(\frac{\rho_{s_1}}{r_{s_1}}\right)^m \left(\frac{\rho_{s_2}}{r_{s_2}}\right)^n \in [e^{-1/2}, 1].$$

Then for those S we see that if G is contained in a ball of radius $M > 1$ then G^S is contained in a ball of radius $M \prod_i r_{s_i}$ and therefore we see that:

$$N_\delta(G_{\prod_i \rho_{s_i}/r_{s_i}}) = N_{\delta \prod_i r_i} G^S \leq \left(\frac{M \prod_i r_{s_i}}{\delta \prod_i r_{s_i}}\right)^\kappa.$$

Now for any $t \in [e^{-1/2}, 1]$ we choose a sequence of strings S_j such that $\lim_{j\to\infty} \prod_i \frac{\rho_{s_{j,i}}}{r_{s_{j,i}}} = t$, then we see that there exists a constant C and for all large enough j:

$$N_\delta(G_t) \leq C N_\delta(G_{\prod_i \rho_{s_{j,i}}/r_{s_{j,i}}}) \leq C M^2 \delta^{-\kappa}.$$

The above argument holds for all $t \in [e^{-1/2}, 1]$ and we see that $\overline{\dim_{\mathrm{B}}} G_t < \kappa$ (uniformly) for $t \in [e^{-1/2}, 1]$. This contradicts with theorem 1.15 because the exception set should not have positive measure.

10. FURTHER QUESTIONS

Apart from the fractal grass set/fractal Kakeya book conjectures stated in the first section of this paper, there are still some questions to ask about.

Largeness in general for Hausdorff or packing dimension One natural question to ask in concerning with theorem 1.4, 1.5 is that can we obtain Hausdorff or packing dimension result. If we allow some more variations of cub-Wolff(ff) sets then by using the classical Marstrand slicing theorem [Mat99, chaper 10] we can obtain some results.

By the approach here, we used sliced box dimension to obtain the results. It seems unlikely that such strategy works for Hausdorff or packing dimension.

About (α, β)-circle-Wolff(ff) sets

We omit the proof of largeness in general results for circle sets in this paper because it is similar to that of cube sets. However, because the non-vanishing of the curvature, two β-circle intersect in a mild way without really heavy overlaps. The curvature corresponds naturally to dimension $1/2$. And we think that for $\beta > 1/2$, any (α, β)-circle-Wolff(ff) set should have Hausdorff/Box/Assouad dimension equal to $\min\{\alpha+\beta, 2\}$ at least for $\alpha \leq 1$.

Wolff's original result implies the lower bound $3\beta + \alpha - 2$. Which establishes the case when $\beta = 1$.

About (α, β)-cube-Wolff(ff) sets

We do not think (α, β)-cube-Wolff(ff) sets always have dimension $\min\{\alpha+\beta, 2\}$. We can obtain some sort of lower bounds on box dimension. For example let G be an (α, β)-cube-Wolffff set. We assume that $\alpha \leq 1$ and by Marstrand projection theorem we can in fact consider the α-set A is embedded in a unit segment which is of angle within $[44, 45]$ degree against x-axis.

Further more we require all β-cubes have box dimension β in an uniform way together with the α-set. For any $\delta > 0$, we can choose in A a 100δ separated finite set F with cardinality $\approx \delta^{-\alpha}$.

For each $a \in F$ there is a cube centred at a, and δ-neighbourhood of this cube can be seen as a disjoint union with $\approx \delta^{-\beta}$ many δ cubes.

There are now $\approx \delta^{-\alpha-\beta}$ many δ-cubes, but some of them may have large multiplicity. If there is a δ cube c which is counted M times then there are at least M cubes with a side within 2δ distance from c. Then as the centres of the cubes are 100δ separated we see that we can find at least $M/2$ many 96δ separated sides of cubes. This gives us a lower bound of box counting:

$$M\delta^{-\beta}.$$

On the other hand if no δ-cubes have more than M multiplicity then we obtain another lower bound of box counting:

$$M^{-1}\delta^{-\alpha-\beta}.$$

Balancing the above two lower bounds we see that there is in fact an lower bound of box dimension:

$$\beta + \alpha/2.$$

To show that this is in fact sharp we consider $\alpha < 1/2$ and a α-set $A \subset [0, 1]$. Further we assume that A has box dimension equal to α and $A + A$ has box dimension equal to 2α.

Then we embed A into $0 \times [0, 1]$, and for each $(0, a)$ we construct two lines $l_{\pm 1}(a)$ passing through a with slope ± 1.

For each line, we embed a middle-third Cantor set (by restricting a homogeneous linear map from $[0, 1]$ to the line $l_{\pm}(a)$) and extend it with period 1. Then as a result the union of the lines has box dimension $\alpha+\beta$. The union of lines contains β-cube centred every where in $\frac{1}{2}(A+A)$ which has dimension 2α. Then we obtain a $(2\alpha,\beta)$-cube-Wolffff set with dimension $\alpha+\beta$.

Similar consideration holds for (α,β)-cube-Wolff sets.

However, we assumed $\alpha\leq 1$, but what happens if $\alpha>1$? Repeating the argument above will lead us to a lower bound $\beta+1$.

Some further questions about cube sets

So far we have only considered cube sets with so-called coordinate cubes. There are also other configurations for example orientations of the cubes. For circle sets, orientation does not matter. In this more general situation, how to formulate the largeness in general. A possible guess is still keeping all the centres and orientations fixed while shrinking the side length. In this case the largeness in general just comes for free by using the dimension result about the special grass sets as in lemma 5.3. An interesting generalization is to only consider vertices of the cubes, then we still have a grass set with the cone set construction but it is not necessary to be of any special types. For this situation, we really need to proof the grass set conjecture.

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