

KOSZUL DUALITY FOR SEMIDIRECT PRODUCTS AND GENERALIZED TAKIFF ALGEBRAS

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ABSTRACT. We obtain Koszul-type dualities for categories of graded modules over a graded associative algebra which can be realized as the semidirect product of a bialgebra coinciding with its degree zero part and a graded module algebra for the latter. In particular, this applies to graded representations of the universal enveloping algebra of the Takiff Lie algebra (or the truncated current algebra) and its (super)analogues, and also to semidirect products of quantum groups with braided symmetric and exterior module algebras in case the latter are flat deformations of classical ones.

1. INTRODUCTION

Koszul property, as defined in [17], plays an important role in modern representation and structural theory of graded associative algebras. It typically occurs in the following setting. Let A be a \mathbb{Z} -graded associative algebra over a field \mathbb{k} whose non-zero homogeneous components appear only in non-negative degrees and are finite dimensional and whose degree zero part A_0 is semisimple. Consider the category of locally finite dimensional graded A -modules, which can be regarded as a non-semisimple deformation of the semisimple category of finite dimensional A_0 -modules. Koszulity of A is then formulated via the requirement that simple graded A -modules have so-called *linear* projective resolutions. One consequence of Koszulity is a derived equivalence between the bounded derived category of finitely generated graded A -modules and a similar category for the quadratic dual of A (see [2]). This classical Koszul duality has numerous generalizations and extensions (see [12, 14, 16] and references therein).

However, it often happens that one needs to consider graded modules over a graded algebra whose degree zero part is not semisimple. A typical example is the current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$ of a finite dimensional simple Lie algebra \mathfrak{g} over \mathbb{C} which is intimately connected to, in particular, the quantum affine algebra corresponding to \mathfrak{g} . The universal enveloping algebra of $\mathfrak{g}[t]$ is naturally \mathbb{Z} -graded, with the degree zero part being isomorphic to the enveloping algebra of \mathfrak{g} . The latter is very far from being semisimple. However, the category of finite dimensional \mathfrak{g} -modules is semisimple. This allows one to associate a Koszul algebra with the full subcategory of

the category of finitely generated graded locally finite dimensional $\mathfrak{g}[t]$ -modules whose objects are annihilated by the Lie ideal $\mathfrak{g} \otimes t^2\mathbb{C}[t]$ of $\mathfrak{g}[t]$ (cf. [6, 7]). Alternatively, such modules can be regarded as modules over the Takiff Lie algebra $\mathfrak{g} \rtimes \mathfrak{g}$ ([18]) which is naturally isomorphic to $\mathfrak{g} \otimes \mathbb{C}[t]/(t^2)$. The interest in that category stems from the observation that $q = 1$ limits of celebrated Kirillov-Reshetikhin modules over quantum affine algebras ([13]) and also of certain generalizations of Kirillov-Reshetikhin modules known as minimal affinizations ([5]) are its objects, provided that \mathfrak{g} is of a classical type. Once one has a Koszul algebra, its quadratic dual is also Koszul, and the natural question is whether one can find a Lie-theoretic background for a representation theory of that quadratic dual.

In the present paper we answer that question and establish the most general framework in which such a question can be answered. Namely, it turns out that for the Takiff Lie algebra and its generalizations, the “Koszul graded dual” is, morally, a certain Lie superalgebra which is isomorphic to our initial Lie algebra as a \mathfrak{g} -module. This is a special case of the following setup. Suppose that we have a \mathbb{Z} -graded algebra A such that A_0 is a bialgebra and A is a semidirect product of A_0 with a \mathbb{Z} -graded right A_0 -module algebra H . We consider a category of \mathbb{Z} -graded left A -modules whose graded pieces are in a suitable “underlying” category \mathcal{C} of A_0 -modules (for example, a semisimple category of finite dimensional A_0 -modules if it is available). Assuming that H is Koszul, we establish (see [Theorem 4.5](#) and its Corollary) a Koszul-type duality between that category and a category of \mathbb{Z} -graded left modules with graded pieces in \mathcal{C} over the algebra A^\circledast , which is the semidirect product of the quadratic dual of H with the bialgebra A^{cop} which coincides with A as an algebra and has the opposite comultiplication. The case of the Takiff algebra described above corresponds to taking A_0 to be the enveloping algebra of \mathfrak{g} , H to be the symmetric algebra of the adjoint representation of \mathfrak{g} and \mathcal{C} to be the category of finite dimensional \mathfrak{g} -modules.

The paper is organized as follows. In [Section 2](#) we collected basic generalities on graded algebras and categories of graded modules. [Section 3](#) contains results pertaining to module algebras and semidirect products that are needed for our construction. In [Section 4](#) we establish Koszul duality in the setting of semidirect products of bialgebras with their module algebras. In particular, it contains the main results of the paper ([Theorem 4.5](#) and [Corollary 4.5](#)). Finally, [Section 5](#) provides examples illustrating applications of our main result.

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2. GENERALITIES

2.1. All algebras considered in this paper are unital and over a fixed base field \mathbb{k} . If A is an algebra and M is a left (respectively, right) A -module, then we denote the action of $a \in A$ on $m \in M$ by $a \triangleright m$ (respectively $m \triangleleft a$). Given a category \mathcal{A} , we write $X \in \mathcal{A}$ as a shortcut for X being an object in \mathcal{A} .

2.2. Let \mathcal{B} be an additive, \mathbb{k} -linear and idempotent split category. The *projective abelianization* $\overline{\mathcal{B}}$ of \mathcal{B} is the category defined as follows:

- objects of $\overline{\mathcal{B}}$ are all diagrams of the form

$$X \xrightarrow{\alpha} Y, \quad (2.1)$$

where $X, Y \in \mathcal{B}$ and $\alpha \in \text{Hom}_{\mathcal{B}}(X, Y)$;

- for objects $X \xrightarrow{\alpha} Y$ and $X' \xrightarrow{\alpha'} Y'$, the set $\text{Hom}_{\overline{\mathcal{B}}}(X \xrightarrow{\alpha} Y, X' \xrightarrow{\alpha'} Y')$ is the quotient of the vector space formed by all solid diagrams of the form

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ f \downarrow & \theta \text{ (dotted)} & \downarrow g \\ X' & \xrightarrow{\alpha'} & Y' \end{array} \quad (2.2)$$

by the subspace generated by all such solid diagrams for which there exists a morphism θ , depicted by the dotted arrow, such that $\alpha'\theta = g$.

Dually, the *injective abelianization* $\underline{\mathcal{B}}$ of \mathcal{B} is the category whose objects are diagrams of the form (2.1) and whose corresponding morphisms are given as quotients of the vector space formed by all solid diagrams of the form (2.2) by the subspace generated by all such solid diagrams for which there exists a morphism θ as depicted by the dotted arrow such that $\theta\alpha = f$. We refer the reader to [10] and [15, §3.1] for more details on abelianizations.

2.3. Let \mathcal{A} be an idempotent split exact category. Denote by $\text{Proj}(\mathcal{A})$ (respectively, $\text{Inj}(\mathcal{A})$) the strictly full subcategory of \mathcal{A} consisting of projective (respectively, injective) objects of \mathcal{A} .

The category $\text{Gr } \mathcal{A}$ of \mathbb{Z} -graded objects over \mathcal{A} is an additive category whose objects are $\mathbf{X} = \bigoplus_{i \in \mathbb{Z}} \mathbf{X}_i$, where $\mathbf{X}_i \in \mathcal{A}$, and

$$\text{Hom}_{\text{Gr } \mathcal{A}}(\mathbf{X}, \mathbf{Y}) := \prod_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(\mathbf{X}_i, \mathbf{Y}_i).$$

Given $r \in \mathbb{Z}$ and $\mathbf{X} \in \text{Gr } \mathcal{A}$, we denote by $\mathbf{X}\langle r \rangle$ the object in $\text{Gr } \mathcal{A}$ such that $(\mathbf{X}\langle r \rangle)_i = \mathbf{X}_{i+r}$, $i \in \mathbb{Z}$. This defines the degree shift endofunctor $\langle r \rangle$ of $\text{Gr } \mathcal{A}$. We say that $\mathbf{X} \in \text{Gr } \mathcal{A}$ is *bounded above* (respectively, *below*) if $\mathbf{X}_i = 0$ for $i \gg 0$ (respectively, $i \ll 0$). The full subcategories of bounded above (respectively, below) objects in $\text{Gr } \mathcal{A}$ are denoted by $\text{Gr}^- \mathcal{A}$ (respectively, $\text{Gr}^+ \mathcal{A}$).

For $j \in \mathbb{Z}$, define a functor $\Pi_j : \text{Gr } \mathcal{A} \rightarrow \mathcal{A}$ by $\Pi_j(\mathbf{X}) = \mathbf{X}_j$ for all $\mathbf{X} \in \text{Gr } \mathcal{A}$ and $\Pi_j(f) = f_j$ for all $f = (f_i)_{i \in \mathbb{Z}} \in \text{Hom}_{\text{Gr } \mathcal{A}}(\mathbf{X}, \mathbf{Y})$ and all $\mathbf{X}, \mathbf{Y} \in \text{Gr } \mathcal{A}$. Clearly, Π_j is exact. On the other hand, define a functor $\text{gr}_j : \mathcal{A} \rightarrow \text{Gr } \mathcal{A}$ by $\text{gr}_j(X)_i = 0$ if $i \neq j$ and $\text{gr}_j(X)_j = X$, for $X \in \mathcal{A}$, while $\text{gr}_j(f)_j = f$, $\text{gr}_j(f)_i = 0$, $i \neq j$, for all $f \in \text{Hom}_{\mathcal{A}}(X, Y)$ and $X, Y \in \mathcal{A}$. Then gr_j is a full exact embedding. It is immediate that $\Pi_j \circ \text{gr}_j$ is isomorphic to the identify functor on \mathcal{A} for all $j \in \mathbb{Z}$.

We say that a non-zero object \mathbf{X} of $\text{Gr } \mathcal{A}$ is concentrated in degree i provided that $\Pi_j(\mathbf{X}) = 0$ unless $j = i$. Clearly, if \mathbf{X} is concentrated in degree j then $\mathbf{X}\langle r \rangle$ is concentrated in degree $j-r$, $\Pi_{j+r} = \Pi_j \circ \langle r \rangle$ and $\text{gr}_{j+r} = \langle r \rangle \circ \text{gr}_j$, for all $j, r \in \mathbb{Z}$.

2.4. A pair $(\mathbf{X}^\bullet, d_{\mathbf{X}})$ where $\mathbf{X}^\bullet = \bigoplus_{i \in \mathbb{Z}} \mathbf{X}^i$ with $\mathbf{X}^i \in \mathcal{A}$, and $d_{\mathbf{X}} \in \prod_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{A}}(\mathbf{X}^j, \mathbf{X}^{j+1})$ satisfies $d_{\mathbf{X}} \circ d_{\mathbf{X}} = 0$ is called a *complex* over \mathcal{A} ; $d_{\mathbf{X}}$ is called the differential. The object \mathbf{X}^i , $i \in \mathbb{Z}$ is called the component of $(\mathbf{X}^\bullet, d_{\mathbf{X}})$ of homological degree i , and an object X in \mathcal{A} identifies with the complex with zero differential whose only non-zero component is X in homological degree 0 (the trivial complex of X). A complex $(\mathbf{X}^\bullet, d_{\mathbf{X}})$ is said to be *bounded above* (respectively, *bounded below*) if $\mathbf{X}^i = 0$ for all $i \gg 0$ (respectively, $i \ll 0$). The homotopy category of bounded below (respectively, bounded above) complexes over \mathcal{A} is denoted $\mathcal{K}^+(\mathcal{A})$ (respectively, $\mathcal{K}^-(\mathcal{A})$).

If \mathcal{A} is abelian, we denote by $\mathcal{D}^+(\mathcal{A})$ (respectively, $\mathcal{D}^-(\mathcal{A})$) the corresponding bounded below (respectively, bounded above) derived category.

2.5. Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a \mathbb{Z} -graded unital \mathbb{k} -algebra. We will always assume that $A_i = 0$ for all $i < 0$. In particular, this implies that $A_0 \cong A / \bigoplus_{i > 0} A_i$ as an algebra. Clearly, each A_i is an A_0 -bimodule.

Given an A_0 -bimodule V , denote by $T_{A_0}^i(V)$ the i -fold tensor product $V \otimes_{A_0} \cdots \otimes_{A_0} V$. Set $(K_A)_i = 0$, for $i < 0$, and $(K_A)_i = A_i$, for $i = 0, 1$, and define for $i > 1$

$$(K_A)_i = \bigcap_{j=0}^{i-2} T_{A_0}^j(A_1) \otimes_{A_0} (\ker m_A|_{A_1 \otimes_{A_0} A_1}) \otimes_{A_0} T_{A_0}^{i-j-2}(A_1),$$

where $m_A : A \otimes_{A_0} A \rightarrow A$ is the multiplication map (cf. [2, §2.6]). Clearly, all $(K_A)_i$, for $i \geq 0$, are A_0 -bimodules. In particular, $A_i \otimes_{A_0} -$, $(K_A)_i \otimes_{A_0} -$, $\text{Hom}_{A_0}(A_i, -)$ and $\text{Hom}_{A_0}((K_A)_i, -)$ are endofunctors of the category of left A_0 -modules.

For $i, j \in \mathbb{Z}_{\geq 0}$, denote by m_{A_i, A_j} (respectively, m_{A, A_j} , $m_{A_i, A}$) the restriction of m_A to $A_i \otimes_{A_0} A_j$ (respectively, to $A \otimes_{A_0} A_j$, $A_i \otimes_{A_0} A$). Let $K_A = \bigoplus_{i \geq 0} (K_A)_i$. The following Lemma is rather standard (cf. [2]).

Lemma. *The object $K_A \otimes_{A_0} A$ (respectively, $A \otimes_{A_0} K_A$) can be equipped with the structure of a bounded above complex of \mathbb{Z} -graded right (respectively, left) A -modules, called the right (respectively, left) Koszul complex of A , with*

$$(K_A \otimes_{A_0} A)^{-i}_j := (K_A)_i \otimes_{A_0} A_{j-i} \quad \text{and} \quad (A \otimes_{A_0} K_A)^{-i}_j := A_{j-i} \otimes_{A_0} (K_A)_i,$$

for $i, j \in \mathbb{Z}_{\geq 0}$, and with the differential d^A (respectively, ${}^A d$) defined by

$$d^A_{-i} := \text{id}_{A_1}^{\otimes(i-1)} \otimes m_{A_1, A} \quad \text{and} \quad {}^A d_{-i} := m_{A, A_1} \otimes \text{id}_{A_1}^{\otimes(i-1)}, \quad \text{for } i > 0.$$

Proof. We only prove the statement for $K_A \otimes_{A_0} A$, the proof for $A \otimes_{A_0} K_A$ being similar. Since $(K_A)_i \subset (K_A)_{i-1} \otimes_{A_0} A_1$, the map d^A_{-i} is well-defined, is evidently a homomorphism of \mathbb{Z} -graded right A -modules and maps $(K_A \otimes_{A_0} A)^{-i}_j$ to $(K_A \otimes_{A_0} A)^{-i+1}_j$. It remains to observe that

$$d^A_{-i+1} \circ d^A_{-i} = \text{id}_{A_1}^{\otimes(i-2)} \otimes m_{A_1, A} (\text{id}_{A_1} \otimes m_{A_1, A}) = \text{id}_{A_1}^{\otimes(i-2)} \otimes m_{A_1, A} (m_{A_1, A_1} \otimes \text{id}_A)$$

which equals zero on $(K_A)_i \otimes_{A_0} A$ since $(K_A)_i \subset T_{A_0}^{i-2}(A_1) \otimes_{A_0} \ker m_{A_1, A_1}$. \square

2.6. Fix a strictly full, additive, idempotent split subcategory \mathcal{C} of the category of left A_0 -modules. We will always assume that \mathcal{C} is contained in a strictly full abelian subcategory $\widehat{\mathcal{C}}$ of the category of left A_0 -modules such that the following conditions hold for all $i > 0$ and for all $X, X' \in \mathcal{C}$:

- (I) $\dim_{\mathbb{k}} \text{Hom}_{A_0}(X, X') < \infty$;
- (II) $\text{Ext}_{\widehat{\mathcal{C}}}^i(X, X') = 0$;
- (III) $A_i \otimes_{A_0} X$ and $(K_A)_i \otimes_{A_0} X$ are objects in \mathcal{C} ;
- (IV) $\text{Hom}_{A_0}(A_i, X)$ and $\text{Hom}_{A_0}((K_A)_i, X)$ are objects in \mathcal{C} .

In particular, the category \mathcal{C} is exact with all short exact sequences being split. This implies that \mathcal{C} is abelian if and only if it is semisimple.

Denote by $\text{Gr}_A \mathcal{C}$ the subcategory of $\text{Gr} \mathcal{C}$ whose objects are graded A -modules and morphisms are those morphisms in $\text{Gr} \mathcal{C}$ which are also homomorphisms of A -modules. Thus, for $M \in \text{Gr}_A \mathcal{C}$, we have

$$M = \bigoplus_{i \in \mathbb{Z}} M_i, \quad M_i \in \mathcal{C}, \quad A_i M_j \subset M_{i+j}$$

and for objects M, N in $\text{Gr}_A \mathcal{C}$ the space of morphisms from M to N is

$$\text{hom}_A(M, N) := \text{Hom}_A(M, N) \cap \text{Hom}_{\text{Gr} \mathcal{C}}(M, N).$$

We note that $\text{Gr}_A \mathcal{C}$ is neither full nor isomorphism closed in $\text{Gr} \mathcal{C}$, in general. Let $\text{Gr}_A^- \mathcal{C}$ (respectively, $\text{Gr}_A^+ \mathcal{C}$) be the full subcategory of bounded above (respectively, bounded below) objects in $\text{Gr}_A \mathcal{C}$. Both $\text{Gr}_A^- \mathcal{C}$ and $\text{Gr}_A^+ \mathcal{C}$ are exact categories and hence we may consider projective objects in these categories relative to the corresponding exact structures.

2.7. Given an object X in \mathcal{C} , let $P^0(X) := A \otimes_{A_0} X$. By (III), $P^0(X)$ is an object in $\text{Gr}_A^+ \mathcal{C}$ with $P^0(X)_i = A_i \otimes_{A_0} X$, and there is a canonical epimorphism

$$P^0(X) \twoheadrightarrow \text{gr}_0(X), \quad a \otimes x \mapsto a \triangleright x.$$

Dually, set $I^0(X) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A_0}(A_i, X)$ which is naturally a left A -submodule of $\text{Hom}_{A_0}(A, X)$. By (IV), $I^0(X)$ is an object in $\text{Gr}_A^- \mathcal{C}$ with $I^0(X)_{-i} = \text{Hom}_{A_0}(A_i, X)$, and we have a natural monomorphism $\text{gr}_0(X) \rightarrow I^0(X)$ given by the isomorphism $X \cong \text{Hom}_{A_0}(A_0, X)$ of left A_0 -modules.

Lemma. (a) *The assignment $X \mapsto P^0(X)$, where $X \in \mathcal{C}$, defines an additive functor $P^0 : \mathcal{C} \rightarrow \text{Proj}(\text{Gr}_A^+ \mathcal{C})$.*

(b) *Similarly, the assignment $X \mapsto I^0(X)$, where $X \in \mathcal{C}$, defines an additive functor $I^0 : \mathcal{C} \rightarrow \text{Inj}(\text{Gr}_A^- \mathcal{C})$.*

Proof. Clearly, $X \mapsto P^0(X)$ defines a functor $P^0 : \mathcal{C} \rightarrow \text{Gr}_A \mathcal{C}$ (in fact, $\mathcal{C} \rightarrow \text{Gr}_A^+ \mathcal{C}$). Let $X \in \mathcal{C}$ and $Y \in \text{Gr}_A \mathcal{C}$. Since

$$\text{Hom}_A(A \otimes_{A_0} X, Y) \cong \text{Hom}_{A_0}(X, \text{Hom}_A(A, Y)) \cong \text{Hom}_{A_0}(X, Y),$$

we conclude that $\text{hom}_A(P^0(X), Y) \cong \text{Hom}_{A_0}(X, \Pi_0(Y))$, that is, the functor P^0 is left adjoint to Π_0 . Since a left adjoint of an exact functor maps projectives to projectives and every object in \mathcal{C} is projective, part (a) follows.

Similarly, $X \mapsto \mathbb{I}^0(X)$ defines a functor $\mathbb{I}^0 : \mathcal{C} \rightarrow \text{Gr}_A^- \mathcal{C}$. Since, for any $Y \in \text{Gr}_A \mathcal{C}$, we have

$$\text{Hom}_A(Y, \text{Hom}_{A_0}(A, X)) \cong \text{Hom}_{A_0}(A \otimes_A Y, X) \cong \text{Hom}_{A_0}(Y, X),$$

we conclude that $\text{hom}_A(Y, \mathbb{I}^0(X)) \cong \text{Hom}_{A_0}(\Pi_0(Y), X)$. Thus, \mathbb{I}^0 is right adjoint to the exact functor Π and hence maps injectives to injectives. Since every object of \mathcal{C} is injective, this proves part (b). \square

2.8. Our present goal is to construct functors

$$\text{Gr}_A^- \mathcal{C} \rightarrow \mathcal{K}^+(\text{Inj}(\text{Gr}_A^- \mathcal{C})) \quad \text{and} \quad \text{Gr}_A^+ \mathcal{C} \rightarrow \mathcal{K}^-(\text{Proj}(\text{Gr}_A^+ \mathcal{C})).$$

Given an object $X = \bigoplus_{j \in \mathbb{Z}} X_j$ in $\text{Gr}_A^- \mathcal{C}$, set, for all $r \geq 0$,

$$\begin{aligned} \mathbb{I}^r(X) &= \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A_0}((K_A)_r \otimes_{A_0} A_i, X) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A_0}(A_i, \text{Hom}_{A_0}((K_A)_r, X)) \\ &= \bigoplus_{j \in \mathbb{Z}} \mathbb{I}^0(\text{Hom}_{A_0}((K_A)_r, \Pi_j(X)) \langle r - j \rangle), \end{aligned} \quad (2.3)$$

which is an object in $\text{Inj}(\text{Gr}_A^- \mathcal{C})$ by (IV) and Lemma 2.7(b). In particular,

$$\mathbb{I}^r(X)_s = \bigoplus_{i, j \in \mathbb{Z}: j - i = r + s} \text{Hom}_{A_0}((K_A)_r \otimes_{A_0} A_i, X_j), \quad s \in \mathbb{Z}. \quad (2.4)$$

Define $d_{X,r}, \rho_{X,r} : \mathbb{I}^r(X) \rightarrow \mathbb{I}^{r+1}(X)$ by

$$d_{X,r}(f) = f \circ d_{-r-1}^A, \quad \rho_{X,r}(f) = \triangleright_X \circ (\text{id}_{A_1} \otimes f), \quad f \in \mathbb{I}^r(X),$$

where $\triangleright_X : A \otimes_{A_0} X \rightarrow X$ is the action map $a \otimes x \mapsto a \triangleright x$. Note that $\rho_{X,r}$ is well-defined since $(K_A)_{r+1} \subset A_1 \otimes_{A_0} (K_A)_r$.

Lemma. *For all $r \in \mathbb{Z}_{\geq 0}$ and $X \in \text{Gr}_A \mathcal{C}$, the map $\rho_{X,r}$ is a morphism in $\text{Gr}_A \mathcal{C}$ satisfying $\rho_{X,r+1} \circ \rho_{X,r} = 0$ and $\rho_{X,r+1} \circ d_{X,r} = d_{X,r+1} \circ \rho_{X,r}$. In particular, $(\mathbb{I}^\bullet(X), \partial_X^+)$, where $\partial_{X,r}^+ := d_{X,r} + (-1)^r \rho_{X,r}$, is an object in $\mathcal{K}^+(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$.*

Proof. Clearly, $\rho_{X,r}$ is a homomorphism of A -modules and

$$\rho_{X,r}(\text{Hom}_{A_0}((K_A)_r \otimes_{A_0} A_i, X_j)) \subset \text{Hom}_{A_0}((K_A)_{r+1} \otimes_{A_0} A_i, X_{j+1}),$$

hence $\rho_{X,r} \in \text{hom}_A(\mathbb{I}^r(X), \mathbb{I}^{r+1}(X))$ by (2.4). Furthermore, for

$$f \in \text{Hom}_{A_0}((K_A)_r \otimes_{A_0} A_i, X),$$

we have

$$\rho_{X,r+1} \circ \rho_{X,r}(f) = \triangleright_X \circ (\text{id}_{A_1} \otimes \triangleright_X) \circ (\text{id}_{A_1}^{\otimes 2} \otimes f) = \triangleright_X \circ (m_{A_1, A_1} \otimes \text{id}_X) \circ (\text{id}_{A_1}^{\otimes 2} \otimes f) = 0$$

since $(K_A)_{r+1} \subset \ker m_{A_1, A_1} \otimes_{A_0} T_{A_0}^{r-1}(A_1)$. Finally,

$$\begin{aligned} \rho_{\mathbf{X}, r+1} \circ d_{\mathbf{X}, r}(f) &= \rho_{\mathbf{X}, r+1}(f \circ \text{id}_{A_1}^{\otimes(r-1)} \otimes m_{A_1, A}) = \triangleright_{\mathbf{X}} \circ (\text{id}_{A_1} \otimes f \circ \text{id}_{A_1}^{\otimes(r-1)} \otimes m_{A_1, A}) \\ &= \triangleright_{\mathbf{X}} \circ (\text{id}_{A_1} \otimes f) \circ (\text{id}_{A_1}^{\otimes r} \otimes m_{A_1, A}) = d_{\mathbf{X}, r+1} \circ \rho_{\mathbf{X}, r}(f). \end{aligned}$$

The remaining assertion is obvious. \square

Proposition. *The assignment $\mathbf{X} \mapsto (\mathbf{l}^\bullet(\mathbf{X}), \partial_{\mathbf{X}}^+)$, $\mathbf{X} \in \text{Gr}_A^- \mathcal{C}$ defines a functor*

$$\Phi^+ : \text{Gr}_A^- \mathcal{C} \rightarrow \mathcal{K}^+(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$$

which extends to an endofunctor Φ^+ of $\mathcal{K}^+(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$.

Proof. For the first assertion, we only need to check that a morphism $\phi : \mathbf{X} \rightarrow \mathbf{Y}$ in $\text{Gr}_A^- \mathcal{C}$ gives rise to a morphism of complexes $\phi : \mathbf{l}^\bullet(\mathbf{X}) \rightarrow \mathbf{l}^\bullet(\mathbf{Y})$. Indeed, ϕ clearly induces morphisms $\phi_r : \mathbf{l}^r(\mathbf{X}) \rightarrow \mathbf{l}^r(\mathbf{Y})$, $f \mapsto \phi \circ f$, for $f \in \text{Hom}_{A_0}((K_A)_r \otimes A_i, \mathbf{X}_j)$, and it is easy to see that $\partial_{\mathbf{X}, r}^+ \phi = \phi \partial_{\mathbf{Y}, r}^+$. Furthermore, given a bounded below complex $(\mathbf{X}^\bullet, \delta)$ over $\text{Gr}_A^- \mathcal{C}$, define $\Phi^+((\mathbf{X}^\bullet, \delta))$ to be the total complex

$$\left(\bigoplus_{r+s=p} \mathbf{l}^r(\mathbf{X}^s), \sum_{r+s=p} \partial_{\mathbf{X}, r}^+ + (-1)^s \delta_s \right).$$

Since $\mathbf{X}^s = 0$ if $s \ll 0$ and $\mathbf{l}^r(\mathbf{X}^s) = 0$ if $r < 0$, we conclude that the above complex is bounded from below and, in fact, for each p the direct sum is finite. In particular, this implies that every term is an object in $\text{Inj}(\text{Gr}_A^- \mathcal{C})$. This yields the desired endofunctor on $\mathcal{K}^+(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$. \square

2.9. The functor $\Phi^- : \text{Gr}_A^+ \mathcal{C} \rightarrow \mathcal{K}^-(\text{Proj}(\text{Gr}_A^+ \mathcal{C}))$ is constructed similarly. Indeed, given $\mathbf{X} = \bigoplus_{i \in \mathbb{Z}} \mathbf{X}_i \in \text{Gr}_A^+ \mathcal{C}$, we set, for all $r \geq 0$,

$$\mathbf{P}^{-r}(\mathbf{X}) = A \otimes_{A_0} (K_A)_r \otimes_{A_0} \mathbf{X} = \bigoplus_{j \in \mathbb{Z}} \mathbf{P}^0((K_A)_r \otimes_{A_0} \Pi_j(\mathbf{X})) \langle -j - r \rangle. \quad (2.5)$$

This implies that $\mathbf{P}^{-r}(\mathbf{X}) \in \text{Proj}(\text{Gr}_A^+ \mathcal{C})$ and

$$\mathbf{P}^{-r}(\mathbf{X})_s = \bigoplus_{i, j \in \mathbb{Z} : i+j=s-r} A_i \otimes_{A_0} (K_A)_r \otimes_{A_0} \mathbf{X}_j. \quad (2.6)$$

Define $\partial_{\mathbf{X}, -r}^- : \mathbf{P}^{-r}(\mathbf{X}) \rightarrow \mathbf{P}^{-r+1}(\mathbf{X})$ by

$$\partial_{\mathbf{X}, -r}^- = (-1)^r A d_{-r} \otimes \text{id}_{\mathbf{X}} + \text{id}_A \otimes \text{id}_{A_1}^{\otimes(r-1)} \otimes \triangleright_{\mathbf{X}}.$$

Note that $\partial_{\mathbf{X}}^-$ is well-defined since $(K_A)_r \subset A_1 \otimes_{A_0} (K_A)_{r-1} \cap (K_A)_{r-1} \otimes_{A_0} A_1$.

Lemma. *We have $(\mathbf{P}^\bullet(\mathbf{X}), \partial_{\mathbf{X}}^-) \in \mathcal{K}^-(\text{Proj}(\text{Gr}_A^+ \mathcal{C}))$ for all $\mathbf{X} \in \text{Gr}_A^+ \mathcal{C}$.*

Proof. It is obvious that $\partial_{\mathbf{X},-r}^-$ is a homomorphism of \mathbb{Z} -graded A -modules. Setting $\lambda_{\mathbf{X},-r} := \text{id}_A \otimes \text{id}_{A_1}^{\otimes(r-1)} \otimes \triangleright_{\mathbf{X}}$, we obtain

$$\begin{aligned} \lambda_{\mathbf{X},-r+1} \circ \lambda_{\mathbf{X},-r} &= \text{id}_A \otimes \text{id}_{A_1}^{\otimes(r-2)} \otimes (\triangleright_{\mathbf{X}} \circ (\text{id}_{A_1} \otimes \triangleright_{\mathbf{X}})) = \\ &= \text{id}_A \otimes \text{id}_{A_1}^{\otimes(r-2)} \otimes \triangleright_{\mathbf{X}} \circ (m_{A_1, A_1} \otimes \text{id}_{\mathbf{X}}) = 0 \end{aligned}$$

since $(K_A)_r \subset T_{A_0}^{r-2}(A_1) \otimes \ker m_{A_1, A_1}$. Furthermore, as we clearly have

$$({}^A d_{-r+1} \otimes \text{id}_{\mathbf{X}}) \circ \lambda_{\mathbf{X},-r} = \lambda_{\mathbf{X},-r+1} \circ ({}^A d_{-r} \otimes \text{id}_{\mathbf{X}}),$$

it follows that $\partial_{\mathbf{X},-r+1}^- \circ \partial_{\mathbf{X},-r}^- = 0$. \square

The following statement is proved similarly to [Proposition 2.5](#).

Proposition. *The assignment $\mathbf{X} \mapsto (\mathbf{P}^\bullet(\mathbf{X}), \partial_{\mathbf{X}}^-)$, for $\mathbf{X} \in \text{Gr}_A^+ \mathcal{C}$, defines a functor $\Phi^- : \text{Gr}_A^+ \mathcal{C} \rightarrow \mathcal{K}^-(\text{Proj}(\text{Gr}_A^+ \mathcal{C}))$ which further extends to an endofunctor Φ^- of $\mathcal{K}^-(\text{Proj}(\text{Gr}_A^+ \mathcal{C}))$.*

2.10. Given a complex $(\mathbf{X}^\bullet, d_{\mathbf{X}})$ over $\text{Gr}_A \mathcal{C}$, its i -th homology $\mathbf{H}_i(\mathbf{X}^\bullet)$ is defined as $\mathbf{H}_i(\mathbf{X}^\bullet) := \ker d_{\mathbf{X},i} / \text{Im } d_{\mathbf{X},i-1}$ which, a priori, is an object in $\text{Gr}_A \widehat{\mathcal{C}}$ (see [§2.6](#)). We will need some properties of \mathbf{I}^\bullet and \mathbf{P}^\bullet which are collected in the following Lemma.

Lemma. *Suppose that A is generated by A_1 over A_0 . Then we have:*

(a) $\mathbf{H}_0(\mathbf{I}^\bullet(\mathbf{X})) \cong \mathbf{X}$ (respectively, $\mathbf{H}_0(\mathbf{P}^\bullet(\mathbf{Y})) \cong \mathbf{Y}$) for any $\mathbf{X} \in \text{Gr}_A^- \mathcal{C}$ (respectively, $\mathbf{Y} \in \text{Gr}_A^+ \mathcal{C}$).

Furthermore, if \mathbf{X} is concentrated at degree zero then

- (b) $\mathbf{I}^r(\mathbf{X}) = \bigoplus_{\mathbb{Z}} \mathbf{I}^0(\mathbf{Z})\langle r \rangle$, i.e. \mathbf{I}^\bullet is diagonal.
- (c) $\mathbf{P}^{-r}(\mathbf{X}) = \bigoplus_{\mathbb{Z}} \mathbf{P}^0(\mathbf{Z})\langle -r \rangle$, i.e. \mathbf{P}^\bullet is diagonal.
- (d) $\mathbf{H}_r(\mathbf{I}^\bullet(\mathbf{X}))_{-r} = 0 = \mathbf{H}_{-r}(\mathbf{P}^\bullet(\mathbf{X}))_r$, for $r > 0$.

Proof. To prove part (a), note that $\partial_{\mathbf{X},-1}^+ = 0$ and $f \in \mathbf{I}^0(\mathbf{X}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A_0}(A_i, \mathbf{X})$ is in $\ker \partial_{\mathbf{X},0}^+$ if and only if $f(a_1 a) = a_1 \triangleright f(a)$ for all $a_1 \in A_1$, $a \in A$, and $i \geq 0$. Since A is generated by A_1 , we have that $\ker \partial_{\mathbf{X},0}^+$ is a \mathbb{Z} -graded A -submodule of $\mathbf{I}^0(\mathbf{X})$ and $f \in \ker \partial_{\mathbf{X},0}^+$ is uniquely determined by $f(1)$, hence the map $\ker \partial_{\mathbf{X},0}^+ \rightarrow \mathbf{X}$, $f \mapsto f(1)$, is an isomorphism of \mathbb{Z} -graded A -modules. Similarly, $\partial_{\mathbf{X},0}^- = 0$ and the image of $\partial_{\mathbf{X},-1}^-$ is generated by elements of the form $aa_1 \otimes x - a \otimes a_1 \triangleright x$, where $a \in A$, $a_1 \in A_1$, and $x \in \mathbf{X}$. Since A is generated by A_1 , we have that $\mathbf{H}_0(\mathbf{P}^\bullet(\mathbf{X}))$ is isomorphic to $A \otimes_A \mathbf{X} \cong \mathbf{X}$.

Parts (b) and (c) follow from (2.3) and (2.5), respectively. Further, by (2.4), we have

$$\begin{aligned} \Gamma^r(\mathbf{X})_{-r} &= \mathrm{Hom}_{A_0}((K_A)_r, \mathbf{X}), & \Gamma^{r-1}(\mathbf{X})_{-r} &= \mathrm{Hom}_{A_0}((K_A)_{r-1} \otimes_{A_0} A_1, \mathbf{X}), \\ \Gamma^{r+1}(\mathbf{X})_{-r} &= 0. \end{aligned}$$

Thus, for $r > 0$ it suffices to prove that $\partial_{\mathbf{X}, r-1}^+(\Gamma^{r-1}(\mathbf{X})_{-r}) = \Gamma^r(\mathbf{X})_{-r}$. But this is immediate since $(K_A)_r \subset (K_A)_{r-1} \otimes A_1$, the map $\partial_{\mathbf{X}, r-1}^+|_{\Gamma^{r-1}(\mathbf{X})_{-r}}$ identifies with the restriction and \mathbf{X} is injective in \mathcal{C} . Similarly, by (2.6)

$$\mathbf{P}^{-r-1}(\mathbf{X})_r = 0, \quad \mathbf{P}^{-r}(\mathbf{X})_r = (K_A)_r \otimes_{A_0} \mathbf{X}, \quad \mathbf{P}^{-r+1}(\mathbf{X})_r = A_1 \otimes_{A_0} (K_A)_{r-1} \otimes_{A_0} \mathbf{X}.$$

Thus, we need to prove that $\partial_{\mathbf{X}, -r}^-|_{\mathbf{P}^{-r}(\mathbf{X})_r}$ is injective. This follows since $\partial_{\mathbf{X}, -r}^-|_{\mathbf{P}^{-r}(\mathbf{X})_r}$ is induced by the natural inclusion $(K_A)_r \rightarrow A_1 \otimes_{A_0} (K_A)_{r-1}$ and \mathbf{X} is projective in \mathcal{C} . This completes the proof of part (d). \square

2.11. Given an object \mathbf{X} in $\mathrm{Gr}_A^- \mathcal{C}$, define a complex $(\mathrm{soc} \mathbf{l})^\bullet(\mathbf{X})$ of left A_0 -modules, and hence A -modules via the canonical homomorphism of algebras $A \twoheadrightarrow A_0$, by

$$(\mathrm{soc} \mathbf{l})^r(\mathbf{X}) = \mathrm{Hom}_{A_0}((K_A)_r, \mathbf{X}).$$

Observe that $\partial_{\mathbf{X}}^+$ restricts to $(\mathrm{soc} \mathbf{l})^\bullet(\mathbf{X})$ since, given $f \in \mathrm{Hom}_{A_0}((K_A)_r, \mathbf{X})$, we have

$$\partial_{\mathbf{X}, r}^+(f) = (-1)^{r+1} \triangleright_{\mathbf{X}} \circ (\mathrm{id}_{A_1} \otimes f) \in \mathrm{Hom}_{A_0}((K_A)_{r+1}, \mathbf{X}).$$

We can regard $(\mathrm{soc} \mathbf{l})^\bullet(\mathbf{X})$ as a subcomplex of $\Gamma^\bullet(\mathbf{X})$. Clearly, this construction is functorial in \mathbf{X} .

Dually, define $(\mathrm{top} \mathbf{P})^\bullet(\mathbf{X})$ by

$$(\mathrm{top} \mathbf{P})^{-r}(\mathbf{X}) = (K_A)_r \otimes_{A_0} \mathbf{X}.$$

Again, the differential $\partial_{\mathbf{X}}^-$ restricts to a morphism $(\mathrm{top} \mathbf{P})^{-r} \rightarrow (\mathrm{top} \mathbf{P})^{-r+1}$ which is given by $\partial_{\mathbf{X}, -r}^- = \mathrm{id}_{A_1}^{\otimes r-1} \otimes \triangleright_{\mathbf{X}}$. The canonical epimorphism $A \twoheadrightarrow A_0$ yields a commutative diagram

$$\begin{array}{ccc} \mathbf{P}^{-r}(\mathbf{X}) & \twoheadrightarrow & (\mathrm{top} \mathbf{P})^{-r}(\mathbf{X}) \\ \partial_{\mathbf{X}, -r}^- \downarrow & & \downarrow \partial_{\mathbf{X}, -r}^- \\ \mathbf{P}^{-r+1}(\mathbf{X}) & \twoheadrightarrow & (\mathrm{top} \mathbf{P})^{-r+1}(\mathbf{X}). \end{array}$$

That is, $(\mathrm{top} \mathbf{P})^\bullet(\mathbf{X})$ identifies with a quotient of $\mathbf{P}^\bullet(\mathbf{X})$.

3. MODULE ALGEBRAS AND SEMIDIRECT PRODUCTS

3.1. Let B be a bialgebra over \mathbb{k} with the comultiplication $\Delta_B : B \rightarrow B \otimes_{\mathbb{k}} B$ and the counit $\varepsilon_B : B \rightarrow \mathbb{k}$. Henceforth we write $\Delta_B(b) = b_{(1)} \otimes b_{(2)}$ in Sweedler's notation. Let H_B be a right B -module algebra, that is H_B is an associative \mathbb{k} -algebra and a right B -module, while the multiplication map $m_H : H_B \otimes_{\mathbb{k}} H_B \rightarrow H_B$ is a morphism of right B -modules and B acts on 1_{H_B} by the counit. Thus, for all $b \in B$ and $h, h' \in H_B$ we have

$$(hh') \triangleleft b = (h \triangleleft b_{(1)})(h' \triangleleft b_{(2)}), \quad 1_{H_B} \triangleleft b = \varepsilon_B(b)1_{H_B}.$$

Then we can form the semidirect product $B \rtimes H_B$ as follows. As a \mathbb{k} -vector space, $B \rtimes H_B = B \otimes_{\mathbb{k}} H_B$, with the multiplication given by

$$(b \otimes h) \cdot (b' \otimes h') = bb'_{(1)} \otimes (h \triangleleft b'_{(2)})h', \quad b, b' \in B, h, h' \in H_B.$$

Let M be a right B -module. Then $M \otimes_{\mathbb{k}} H_B$ acquires a natural structure of a right $B \rtimes H_B$ -module via

$$(m \otimes h) \triangleleft (b \otimes h') = m \triangleleft b_{(1)} \otimes (h \triangleleft b_{(2)})h'.$$

Suppose that $H_B = \bigoplus_{i \in \mathbb{Z}} (H_B)_i$ is a \mathbb{Z} -graded algebra and that each $(H_B)_i$ is a right B -submodule of H_B . Then $B \rtimes H_B$ is naturally a \mathbb{Z} -graded algebra with $(B \rtimes H_B)_i = B \otimes_{\mathbb{k}} (H_B)_i$ as a \mathbb{k} -vector space. Clearly, $M \otimes_{\mathbb{k}} H_B$ is a \mathbb{Z} -graded $B \rtimes H_B$ -module with $(M \otimes_{\mathbb{k}} H_B)_i = M \otimes (H_B)_i$, where $i \in \mathbb{Z}$.

Similarly, if ${}_B H$ is a left B -module algebra, we define ${}_B H \rtimes B$ as ${}_B H \otimes_{\mathbb{k}} B$ with the multiplication defined by

$$(h \otimes b) \cdot (h' \otimes b') = h(b_{(1)} \triangleright h') \otimes b_{(2)}b', \quad b, b' \in B, h, h' \in {}_B H.$$

Let M be a left B -module. Then ${}_B H \otimes_{\mathbb{k}} M$ acquires the natural structure of a left ${}_B H \rtimes B$ -module via

$$(h \otimes b) \triangleright (h' \otimes m) = h(b_{(1)} \triangleright h') \otimes b_{(2)} \triangleright m, \quad h, h' \in {}_B H, b \in B, m \in M.$$

Finally, if ${}_B H = \bigoplus_{i \in \mathbb{Z}} ({}_B H)_i$ is a \mathbb{Z} -graded algebra such that each $({}_B H)_i$ is a left B -submodule of ${}_B H$, then ${}_B H \rtimes B$ has the natural structure of a \mathbb{Z} -graded algebra given by $({}_B H \rtimes B)_i = ({}_B H)_i \otimes_{\mathbb{k}} B$. Likewise, ${}_B H \otimes_{\mathbb{k}} M$ is a graded ${}_B H \rtimes B$ -module with $({}_B H \otimes_{\mathbb{k}} M)_i = ({}_B H)_i \otimes_{\mathbb{k}} M$.

3.2. Suppose now that our graded algebra A from §2.5 is isomorphic to $A_0 \rtimes H$ where A_0 is a bialgebra with the comultiplication Δ and the counit ε as above and H is a \mathbb{Z} -graded *right* A_0 -module algebra with $H_0 = \mathbb{k}$ and $H_i = 0$, $i < 0$. Thus, in particular, $A_i \cong A_0 \otimes_{\mathbb{k}} H_i$ as an A_0 -bimodule, where

$$a \triangleright (a' \otimes h) \triangleleft a'' = aa'a''_{(1)} \otimes h \triangleleft a''_{(2)}$$

for all $a, a', a'' \in A_0$, $h \in H$. Clearly, we have

$$(K_A)_i \cong A_0 \otimes_{\mathbb{k}} (K_H)_i, \quad (K_H)_i = \bigcap_{j=0}^{i-2} T_{\mathbb{k}}^j(H_1) \otimes_{\mathbb{k}} \ker m_{H_1, H_1} \otimes_{\mathbb{k}} T_{\mathbb{k}}^{i-j-2}(H_1)$$

as A_0 -bimodules. The action of A_0 can be written explicitly as

$$a'' \triangleright (a' \otimes h_1 \otimes \cdots \otimes h_r) \triangleleft a = a'' a' a_{(1)} \otimes h_1 \triangleleft a_{(2)} \otimes \cdots \otimes h_r \triangleleft a_{(r+1)},$$

where $a, a', a'' \in A_0$, $(\Delta \otimes 1^{\otimes r-1}) \circ \cdots \circ (\Delta \otimes 1) \circ \Delta(a) = a_{(1)} \otimes \cdots \otimes a_{(r+1)}$ and, finally, $h_1 \otimes \cdots \otimes h_r \in (K_H)_r$, with $h_i \in H_1$, in Sweedler-like notation with summation understood.

3.3. Our basic example of this construction is provided by the following generalization of the Takiff Lie algebra (cf. [18]) and is motivated by results of [7]. Let \mathfrak{g} be a Lie algebra and V be a \mathfrak{g} -module. Then we can consider the Lie algebra $\mathfrak{g} \times V$, which is isomorphic to $\mathfrak{g} \oplus V$ as a vector space, with the Lie bracket defined by

$$[(x, v), (x', v')] = ([x, x']_{\mathfrak{g}}, xv' - x'v), \quad x, x' \in \mathfrak{g}, v, v' \in V.$$

Then $H = S(V)$ is a (right) $U(\mathfrak{g})$ -module algebra. Let $A_0 = U(\mathfrak{g})$. Then it is easy to see, using the Poincaré-Birkhoff-Witt theorem, that $U(\mathfrak{g} \times V) \cong U(\mathfrak{g}) \times S(V)$. The classical Takiff Lie algebra is obtained by taking V to be the adjoint representation of \mathfrak{g} .

Similarly, we can consider a generalized *Takiff Lie superalgebra* $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$ with $\mathfrak{s}_0 = \mathfrak{g}$, $\mathfrak{s}_1 = V$ and the (super) Lie bracket defined by

$$[(x, v), (x', v')] = ([x, x']_{\mathfrak{g}}, xv' + x'v), \quad x, x' \in \mathfrak{s}_0, v, v' \in \mathfrak{s}_1.$$

In this case, regarding $\bigwedge V$ as a left $U(\mathfrak{g})$ -module, we have

$$U(\mathfrak{s}) \cong \bigwedge V \rtimes U(\mathfrak{g}).$$

3.4. Let R be a unital ring and K a unital subring of its center. Let V be a left R -module. Then $V^* = \text{Hom}_K(V, K)$ is naturally a right R -module via

$$(f \triangleleft r)(v) = f(rv), \quad \text{for } f \in V^*, v \in V, r \in R.$$

Similarly, if W is a right R -module, $W^* = \text{Hom}_K(W, K)$ is a left R -module via $(r \triangleright f)(w) = f(wr)$, where $f \in W^*$, $w \in W$, $r \in R$. The following lemma is standard.

Lemma. *Assume that $R \cong R^*$ as an R -bimodule. Suppose that W is a finitely generated projective left R -module and V is an R -bimodule. Then*

$$(V \otimes_R W)^* \cong W^* \otimes_R V^*$$

Proof. We have

$(V \otimes_R W)^* = \text{Hom}_K(V \otimes_R W, K) \cong \text{Hom}_R(W, \text{Hom}_K(V, K)) \cong \text{Hom}_R(W, R) \otimes_R V^*$
since W is projective and finitely generated. Since $R \cong \text{Hom}_K(R, K)$ as an R -bimodule,

$$\text{Hom}_R(W, R) \cong \text{Hom}_R(W, \text{Hom}_K(R, K)) \cong \text{Hom}_K(R \otimes_R W, K) = W^*. \quad \square$$

3.5. Assume that H_1 is finite dimensional over $H_0 = \mathbb{k}$. In particular, this implies that $(H_1 \otimes_{\mathbb{k}} H_1)^* \cong H_1^* \otimes_{\mathbb{k}} H_1^*$ where $H_1^* = \text{Hom}_{\mathbb{k}}(H_1, \mathbb{k})$ (cf. [Lemma 3.4](#)). Define the quadratic dual $H^!$ of H by

$$H^! = T_{\mathbb{k}}(H_1^*) / \langle \text{Im } \mu^* \rangle$$

where $\mu^* : H_2^* \rightarrow H_1^* \otimes_{\mathbb{k}} H_1^*$ is induced by the multiplication map, namely

$$\mu^*(\xi)(h \otimes h') = \xi(hh'), \quad h, h' \in H_1, \xi \in H_2^*.$$

By construction, $H^!$ is \mathbb{Z} -graded with $H^!_i$ being the canonical image of $T_{\mathbb{k}}^i(H_1^*)$. In particular, $H^!_i = 0$ if $i < 0$.

Remark. After [Lemma 3.4](#), we can consider a more general situation. Namely, we can assume that $H_0 \cong H_0^*$ as an H_0 -bimodule and that H_1 is finitely generated and projective as an H_0 -module. Then $A_0 = B \rtimes H_0$ and we still have

$$(H_1 \otimes_{H_0} H_1)^* \cong H_1^* \otimes_{H_0} H_1^*.$$

Given a bialgebra B , denote by B^{cop} the vector space B endowed with the same multiplication and with the opposite comultiplication.

Lemma. $H^!$ is a graded left A_0^{cop} -module algebra, that is, $a \triangleright (\xi \xi') = (a_{(2)} \triangleright \xi)(a_{(1)} \triangleright \xi')$.

Proof. Using [§3.4](#), we obtain

$$\begin{aligned} (a \triangleright (\xi \otimes \xi'))(h' \otimes h) &= (\xi \otimes \xi')((h' \otimes h) \triangleleft a) = (\xi \otimes \xi')(h' \triangleleft a_{(1)} \otimes h \triangleleft a_{(2)}) \\ &= \xi(h \triangleleft a_{(2)}) \xi'(h' \triangleleft a_{(1)}) = (a_{(2)} \triangleright \xi)(h)(a_{(1)} \triangleright \xi')(h') \\ &= (a_{(2)} \triangleright \xi \otimes a_{(1)} \triangleright \xi')(h' \otimes h), \end{aligned}$$

for all $\xi \in T_{\mathbb{k}}^r(H_1^*)$, $\xi' \in T_{\mathbb{k}}^s(H_1^*)$, $h \in T_{\mathbb{k}}^r(H_1)$ and $h' \in T_{\mathbb{k}}^s(H_1)$. This shows that $T_{\mathbb{k}}(H_1^*)$ is a left A_0^{cop} -module algebra and the action is obviously compatible with the grading. Thus, it remains to check that $\text{Im } m_H^* \subset H_1^* \otimes_{\mathbb{k}} H_1^*$ is a left A_0^{cop} -submodule of $H_1^* \otimes_{\mathbb{k}} H_1^*$. But this is immediate since m_H is a morphism of right A_0 -modules. \square

3.6. Assume that H is quadratic, that is, H is a quotient of $T_{\mathbb{k}}(H_1)$ by the ideal generated by $\ker m_{H_1, H_1}$. Note that $H^!$ is always quadratic. Since H_1 is finite dimensional, it follows that H_i is finite dimensional for all $i \geq 0$. Then $(H^!)^!$ naturally identifies with H . Moreover, by [2, §2.8] we have $(K_H)_i^* \cong H^!_i$ and $(K_{H^!})_i \cong H^*_i$. By Lemma 3.4, there is a natural isomorphism

$$\psi_{i,j} : (T_{\mathbb{k}}^j(H_1) \otimes_{\mathbb{k}} T_{\mathbb{k}}^i(H_1))^* \rightarrow (T_{\mathbb{k}}^i(H_1))^* \otimes_{\mathbb{k}} (T_{\mathbb{k}}^j(H_1))^* \cong T_{\mathbb{k}}^i(H_1^*) \otimes_{\mathbb{k}} T_{\mathbb{k}}^j(H_1^*)$$

which yields an isomorphism

$$\bar{\psi}_{i,j} : ((K_H)_j \otimes_{\mathbb{k}} H_i)^* \rightarrow (H_i)^* \otimes_{\mathbb{k}} ((K_H)_j)^* \rightarrow (K_{H^!})_i \otimes_{\mathbb{k}} H^!_j.$$

Let $m_{j,i} : T_{\mathbb{k}}^j(H_1) \otimes_{\mathbb{k}} H_i \rightarrow T_{\mathbb{k}}^{j-1}(H_1) \otimes_{\mathbb{k}} H_{i+1}$ be the map induced by the natural isomorphism $T_{\mathbb{k}}^j(H_1) \otimes T_{\mathbb{k}}^i(H_1) \rightarrow T_{\mathbb{k}}^{j-1}(H_1) \otimes T_{\mathbb{k}}^{i+1}(H_1)$ and denote by $\bar{m}_{j,i}$ its restriction to $(K_H)_j \otimes_{\mathbb{k}} H_i$. Since $(K_H)_j \subset (K_H)_{j-1} \otimes_{\mathbb{k}} H_1$, it follows that $\bar{m}_{j,i}((K_H)_j \otimes_{\mathbb{k}} H_i) \subset (K_H)_{j-1} \otimes_{\mathbb{k}} H_{i+1}$. The corresponding maps for $H^!$ are denoted by $\bar{m}^!_{j,i}$.

Lemma. *We have $\bar{\psi}_{i-1,j+1}(f \circ \bar{m}_{j+1,i-1}) = \bar{m}^!_{i,j} \circ \bar{\psi}_{i,j}(f)$ for all $f \in ((K_H)_j \otimes_{\mathbb{k}} H_i)^*$, $i > 0$, and $j \geq 0$.*

Proof. Take $u \otimes h \in (K_H)_{j+1} \otimes H_{i-1}$. Since $(K_H)_{j+1} \subset (K_H)_j \otimes_{\mathbb{k}} H_1$, we can write $u = u_1 \otimes u_2$ in Sweedler-like notation, where $u_1 \in (K_H)_j$, $u_2 \in H_1$. Then we have $\bar{m}_{j+1,i-1}(u \otimes h) = u_1 \otimes u_2 h$ and, for any $f \in ((K_H)_j \otimes_{\mathbb{k}} H_i)^*$, we have

$$\psi_{i-1,j+1}(f \circ \bar{m}_{j+1,i-1})(u \otimes h) = f_1(u_1) f_2(u_2 h),$$

where $\bar{\psi}_{i,j}(f) = f_2 \otimes f_1$ in Sweedler-like notation, with $f_1 \in (K_H)_j^*$ and $f_2 \in H_i^*$. Furthermore, identifying H_i^* with $(K_{H^!})_i \subset (K_{H^!})_{i-1} \otimes H^!_1 \cong H_{i-1}^* \otimes H^!_1$, we can write $\bar{\psi}_{i,j}(f) = f_3 \otimes f_2 \otimes f_1$, where $f_3 \in H_{i-1}^*$, $f_2 \in H^!_1$ and $f_1 \in (K_H)_j^*$, whence $\psi_{i-1,j+1}(f \circ \bar{m}_{j+1,i-1})(u \otimes h) = f_1(u_1) f_2(u_2) f_3(h)$.

On the other hand, $\bar{m}^!_{i,j}$ maps $(K_{H^!})_i \otimes_{\mathbb{k}} H^!_j \cong H_i^* \otimes_{\mathbb{k}} (K_H)_j^*$ to

$$(K_{H^!})_{i-1} \otimes H^!_{j+1} \cong H_{i-1}^* \otimes (K_H)_{j+1}^*.$$

Thus,

$$\bar{m}^!_{i,j} \circ \bar{\psi}_{i,j}(f)(u \otimes h) = (f_3 \otimes f_2 \otimes f_1)(u_1 \otimes u_2 \otimes h) = f_1(u_1) f_2(u_2) f_3(h). \quad \square$$

3.7. The algebra $T_{\mathbb{k}}(H_1^*)$ acts naturally on $T_{\mathbb{k}}(H_1)$ by right contractions. Namely, $\xi \in H_1^*$ acts trivially on $T_{\mathbb{k}}^0(H_1)$ and by $\text{id}_{H_1}^{\otimes r-1} \otimes \xi$, for $\xi \in H_1^*$, on $T_{\mathbb{k}}^r(H_1)$. Since the algebra $T_{\mathbb{k}}(H_1^*)$ is freely generated by H_1^* , this extends to a right action of $T_{\mathbb{k}}(H_1^*)$ which we will denote by $(u, \xi) \mapsto u \triangleleft \xi$, where $u \in T_{\mathbb{k}}(H_1)$, $\xi \in T_{\mathbb{k}}(H_1^*)$.

Similarly, the algebra $T_{\mathbb{k}}(H_1)$ acts on $T_{\mathbb{k}}(H_1^*)$ by left contractions. Namely, H_1 acts trivially on $T_{\mathbb{k}}^0(H_1^*)$ and by $\langle h, \cdot \rangle \otimes \text{id}_{H_1^*} \otimes (r-1)$, for $h \in H_1$, on $T_{\mathbb{k}}^r(H_1^*)$, where we have $\langle h, \xi \rangle = \xi(h)$, for $h \in H_1$, $\xi \in H_1^*$. Since $T_{\mathbb{k}}(H_1)$ is freely generated by H_1 , this extends to a left $T_{\mathbb{k}}(H_1)$ -action on $T_{\mathbb{k}}(H_1^*)$.

Lemma. (a) *The right action of $T_{\mathbb{k}}(H_1^*)$ on $T_{\mathbb{k}}(H_1)$ by right contractions induces a right action of the algebra $H^!$ on $K_H := \bigoplus_{r \in \mathbb{Z}} (K_H)_r$.*

(b) *If H is quadratic, then the left action of $T_{\mathbb{k}}(H_1)$ on $T_{\mathbb{k}}(H_1^*)$ by left contractions induces a left action of H on $K_{H^!}$.*

Proof. To prove part (a), observe that we have $(1^{\otimes r-1} \otimes \xi)(u) \in (K_H)_{r-1}$ for all elements $u \in (K_H)_r$. Thus, the algebra $T_{\mathbb{k}}(H_1^*)$ acts on K_H . It remains to prove that the defining ideal of $H^!$ acts trivially. For this, choose any $\xi \in H_2^*$ and observe that $m_H^*(\xi)(h \otimes h') = \xi(hh')$, hence $m_H^*(\xi)(\ker m_H) = 0$. Part (b) is proved similarly. \square

4. KOSZUL DUALITY FOR SEMIDIRECT PRODUCTS

4.1. Retain the assumptions of §3.5. Using Lemma 3.5, define $A^{\otimes} := H^! \rtimes A_0^{\text{cop}}$. Our aim now is to construct an action of A^{\otimes} on the complex $(\text{soc l})^{\bullet}(\mathbf{X})$, $\mathbf{X} \in \text{Gr}_A^- \mathcal{C}$ where \mathcal{C} is a category of left A_0 -modules satisfying (I)–(IV). First, we would like to identify the left $A = A_0 \rtimes H$ -module

$$\Gamma^r(\mathbf{X}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A_0}((K_A)_r \otimes_{A_0} A_i, \mathbf{X})$$

with a more manageable object. We have

$$(K_A)_r \otimes_{A_0} A_i \cong A_0 \otimes_{\mathbb{k}} (K_H)_r \otimes_{A_0} A_0 \otimes_{\mathbb{k}} H_i \cong A_0 \otimes_{\mathbb{k}} (K_H)_r \otimes_{\mathbb{k}} H_i$$

as A_0 -bimodules, where A_0 acts on $A_0 \otimes_{\mathbb{k}} (K_H)_r \otimes_{\mathbb{k}} H_i$ via

$$a \triangleright (a'' \otimes u \otimes h) \triangleleft a' = aa''a'_{(1)} \otimes u \triangleleft a'_{(2)} \otimes h \triangleleft a'_{(3)}, \quad a, a', a'' \in A_0, u \in (K_H)_r, h \in H_i. \quad (4.1)$$

Then

$$\text{Hom}_{A_0}((K_A)_r \otimes_{A_0} A_i, \mathbf{X}) \cong \text{Hom}_{\mathbb{k}}((K_H)_r \otimes_{\mathbb{k}} H_i, \mathbf{X}).$$

In particular, $(\text{soc l})^{\bullet}(\mathbf{X})$ identifies with

$$\bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\mathbb{k}}((K_H)_r, \mathbf{X}).$$

The right action of $H^!$ on K_H by contractions introduced in §3.7 yields a left action of $H^!$ on $\text{Hom}_{\mathbb{k}}(K_H, \mathbf{X})$ which we denote by \blacktriangleright ; thus, $(\xi \blacktriangleright f)(u_1 \otimes u_2) = \xi(u_2)f(u_1)$, where $u_1 \otimes u_2 \in K_H$ in Sweedler-like notation, with $u_2 \in H_1$. Clearly, $(\text{soc l})^{\bullet}(\mathbf{X})$

identifies with a left H^1 -submodule of $\text{Hom}_{\mathbb{k}}(K_H, \mathbf{X})$. Consider the natural left action of A_0^{cop} on $\text{Hom}_{\mathbb{k}}(K_H, \mathbf{X})$ given by

$$(a \blacktriangleright f)(u) = a_{(1)}f(u \triangleleft a_{(2)}), \quad f \in \text{Hom}_{\mathbb{k}}(K_H, \mathbf{X}), u \in K_H, a \in A_0,$$

where the right action of A_0 on K_H is defined by (4.1).

Lemma. *The left actions of A_0^{cop} and H^1 on $\text{Hom}_{\mathbb{k}}(K_H, \mathbf{X})$ defined above yield a left action of A^{\otimes} on $\text{Hom}_{\mathbb{k}}(K_H, \mathbf{X})$. Moreover, $(\text{soc l})^{\bullet}(\mathbf{X})$ is a left A^{\otimes} -submodule of $\text{Hom}_{\mathbb{k}}(K_H, \mathbf{X})$ and the A^{\otimes} action commutes with the differential $\partial_{\mathbf{X}}^+$.*

Proof. To prove the first statement, it suffices to show that

$$a \blacktriangleright (\xi \blacktriangleright f) = (a_{(2)} \triangleright \xi) \blacktriangleright (a_{(1)} \blacktriangleright f),$$

for all $f \in \text{Hom}_{\mathbb{k}}(K_H, \mathbf{X})$, $a \in A_0$ and $\xi \in H_1^*$.

Take $u \in K_H$ and write $u = u_1 \otimes u_2 \in K_H$ in Sweedler-like notation, with $u_2 \in H_1$. Then we have

$$\begin{aligned} (a \blacktriangleright (\xi \blacktriangleright f))(u) &= a_{(1)}(\xi \blacktriangleright f)(u_1 \triangleleft a_{(2)} \otimes u_2 \triangleleft a_{(3)}) = \xi(u_2 \triangleleft a_{(3)})a_{(1)}f(u_1 \triangleleft a_{(2)}) \\ &= (a_{(3)} \triangleright \xi)(u_2)a_{(1)}f(u_1 \triangleleft a_{(2)}) = (a_{(2)} \triangleright \xi)(u_2)(a_{(1)} \blacktriangleright f)(u_1) \\ &= ((a_{(2)} \triangleright \xi) \blacktriangleright (a_{(1)} \blacktriangleright f))(u). \end{aligned}$$

Since all actions involved are locally finite, it follows that $(\text{soc l})^{\bullet}(\mathbf{X})$ is a left A^{\otimes} -submodule of $\text{Hom}_{\mathbb{k}}(K_H, \mathbf{X})$. For the second assertion, we need to show that

$$\xi \blacktriangleright (a \blacktriangleright \partial_{\mathbf{X}}^+(f)) = \partial_{\mathbf{X}}^+(\xi \blacktriangleright (a \blacktriangleright f)), \quad f \in \text{Hom}_{\mathbb{k}}((K_H)_r, \mathbf{X}), \xi \in H_1^*, a \in A_0.$$

Let $u = u_1 \otimes u_2 \otimes u_3 \in (K_H)_{r+1}$, $r \geq 1$, where $u_1, u_3 \in H_1$ and $u_2 \in (K_H)_{r-1}$. Then

$$\begin{aligned} (\xi \blacktriangleright (a \blacktriangleright \partial_{\mathbf{X}}^+(f)))(u) &= \xi(u_3)a_{(1)}\partial_{\mathbf{X}}^+(f)(u_1 \triangleleft a_{(2)} \otimes u_2 \triangleleft a_{(3)}) \\ &= (-1)^r \xi(u_3)a_{(1)}(u_1 \triangleleft a_{(2)})f(u_2 \triangleleft a_{(3)}) \\ &= (-1)^r \xi(u_3)u_1 a_{(1)}f(u_2 \triangleleft a_{(2)}) \\ &= (-1)^r u_1 (\xi \blacktriangleright (a \blacktriangleright f))(u_2 \otimes u_3) \\ &= \partial_{\mathbf{X}}^+(\xi \blacktriangleright (a \blacktriangleright f))(u). \end{aligned}$$

The claim follows. □

4.2. We have the following isomorphisms of \mathbb{k} -vector spaces

$$(\text{soc l})^{\bullet}(\mathbf{X}) = \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\mathbb{k}}((K_H)_r, \mathbf{X}) \cong \bigoplus_{r \in \mathbb{Z}} (K_H)_r^* \otimes_{\mathbb{k}} \mathbf{X} \cong H^1 \otimes_{\mathbb{k}} \mathbf{X},$$

where we used [2, §2.8] for the last identification. The canonical isomorphism

$$\Theta : (K_H)_r^* \otimes_{\mathbb{k}} \mathbf{X} \rightarrow \text{Hom}_{\mathbb{k}}((K_H)_r, \mathbf{X})$$

is given by $\xi \otimes x \mapsto \Theta_{\xi \otimes x}$, where we have $\Theta_{\xi \otimes x}(u) = \xi(u)x$, $x \in \mathbf{X}$, $\xi \in (K_H)_r^*$ and $u \in (K_H)_r$.

Following [2, Section 2.12] and [16, Section 2.4], denote by $\mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$ the full subcategory of $\mathcal{K}^+(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$ which consists of all complexes (\mathbf{X}^\bullet, d) satisfying $\mathbf{X}^j_i = 0$ for all $j+i \gg 0$. Similarly, denote by $\mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$ the full subcategory of $\mathcal{K}^-(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$ which consists of all complexes $(\mathbf{Y}^\bullet, \partial)$ satisfying $\mathbf{Y}^j_i = 0$ for all $j+i \ll 0$. We note that our notation is consistent with that in [16, Section 2.4] and is opposite to that in [2, Section 2.12].

Proposition. *For any $\mathbf{X} \in \text{Gr}_A^- \mathcal{C}$, the complex $(\text{soc } I)^\bullet(\mathbf{X})$ identifies with the complex $(\mathbf{Y}^\bullet, \partial_{\mathbf{X}}^+) \in \mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$, where*

$$\mathbf{Y}^j = (A^\otimes \otimes_{A_0} \Pi_j(\mathbf{X}))\langle -j \rangle.$$

In particular, $(\text{soc } I)^\bullet$ extends to a functor

$$(\text{Soc}^+ I)^\bullet : \mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C})) \rightarrow \mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C})).$$

Proof. The isomorphisms of \mathbb{k} -vector spaces

$$A^\otimes \otimes_{A_0} \mathbf{X} \cong H^1 \otimes_{\mathbb{k}} A_0 \otimes_{A_0} \mathbf{X} \cong H^1 \otimes_{\mathbb{k}} \mathbf{X}$$

induce a left A^\otimes -module structure on $H^1 \otimes_{\mathbb{k}} \mathbf{X}$ (cf. §3.1) which is given by

$$a \triangleright (\xi \otimes x) = (a_{(2)} \triangleright \xi) \otimes a_{(1)} \triangleright x, \quad \xi' \triangleright (\xi \otimes x) = \xi' \xi \otimes x,$$

for all $a \in A_0$, $\xi, \xi' \in H^1$ and $x \in \mathbf{X}$. Using this and the identification $(K_H)_r^* \cong H_r^!$ we obtain

$$\begin{aligned} \Theta_{a \triangleright (\xi \otimes x)}(u) &= \Theta_{a_{(2)} \triangleright \xi \otimes a_{(1)} \triangleright x}(u) = (a_{(2)} \triangleright \xi)(u) a_{(1)} \triangleright x \\ &= \xi(u \triangleleft a_{(2)}) a_{(1)} \triangleright x = (a \blacktriangleright \Theta_{\xi \otimes x})(u), \end{aligned}$$

while

$$\Theta_{\xi' \triangleright (\xi \otimes x)}(u) = \Theta_{\xi' \xi \otimes x}(u) = \xi(u_1) \xi'(u_2) x = (\xi' \blacktriangleright \Theta_{\xi \otimes x})(u)$$

for all $a \in A_0$, $\xi, \xi' \in H^1$, $u = u_1 \otimes u_2 \in K_H$ and $x \in \mathbf{X}$.

Thus, we have $(\text{soc } I)^\bullet(\mathbf{X}) \cong \bigoplus_{r,j \in \mathbb{Z}} A_r^\otimes \otimes_{A_0} \mathbf{X}_j$ as an A^\otimes -module. Set

$$\mathbf{Y}^j = (A^\otimes \otimes_{A_0} \Pi_j(\mathbf{X}))\langle -j \rangle.$$

Then $(\text{soc } I)^\bullet(\mathbf{X}) \cong \bigoplus_{j \in \mathbb{Z}} \mathbf{Y}^j$ and $\mathbf{Y}^j_r = A_{r+j}^\otimes \otimes_{A_0} \Pi_j(\mathbf{X})$. Clearly, $\mathbf{Y}^j \in \text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C})$. Since $\partial(A_{r+j}^\otimes \otimes_{A_0} \Pi_j(\mathbf{X})) \subset A_{r+j+1}^\otimes \otimes_{A_0} \Pi_{j+1}(\mathbf{X})$ under our identifications, it follows that $\partial_{\mathbf{X}}^+ : \mathbf{Y}^j \rightarrow \mathbf{Y}^{j+1}$ is a morphism in $\text{Gr}_{A^\otimes}^+ \mathcal{C}$. Thus, $(\mathbf{Y}^\bullet, \partial_{\mathbf{X}}^+)$ is a complex in $\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C})$. Since $\mathbf{X}_j = 0$ if $j \gg 0$, we conclude that $(\mathbf{Y}^\bullet, \partial)$ is bounded above. It is easy to see that we even have $(\mathbf{Y}^\bullet, \partial) \in \mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$.

It remains to note that, similarly to the classical Koszul duality, the natural extension $(\text{Soc}^+ \mathbf{I})^\bullet$ of $(\text{soc} \mathbf{I})^\bullet$ maps $\mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$ to $\mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$. \square

4.3. Now we consider the dual picture. Retain the assumptions of §3.6. We will decorate all functors related to $\text{Gr}_{A^\otimes}^\pm \mathcal{C}$ with \otimes to distinguish them from those related to $\text{Gr}_A^\pm \mathcal{C}$. Let \mathbf{Y} be an object in $\text{Gr}_{A^\otimes}^- \mathcal{C}$. Then

$$\mathbf{P}^{\otimes -r}(\mathbf{Y}) = A^{\otimes} \otimes_{A_0} (K_{A^{\otimes}})_r \otimes_{A_0} \mathbf{Y} \cong H^! \otimes_{\mathbb{k}} A_0 \otimes_{A_0} (K_{H^!})_r \otimes_{\mathbb{k}} A_0 \otimes_{A_0} \mathbf{Y} \cong H^! \otimes_{\mathbb{k}} (K_{H^!})_r \otimes_{\mathbb{k}} \mathbf{Y},$$

hence

$$(\text{top } \mathbf{P}^{\otimes})^{-r}(\mathbf{Y}) \cong (K_{H^!})_r \otimes_{\mathbb{k}} \mathbf{Y}.$$

Since $(H^!)^! \cong H$, we have that $(K_{H^!})_r$ identifies with H_r^* and so

$$(\text{top } \mathbf{P}^{\otimes})^{-r}(\mathbf{Y}) = H_r^* \otimes_{\mathbb{k}} \mathbf{Y} \cong \text{Hom}_{\mathbb{k}}(H_r, \mathbf{Y})$$

where the isomorphism is given by $\xi \otimes y \mapsto (h \mapsto \xi(h)y)$. Thus, $(\text{top } \mathbf{P}^{\otimes})^\bullet(\mathbf{Y})$ identifies with $\mathbf{l}^0(\mathbf{Y})$ where we regard \mathbf{Y} as an A -module via the canonical epimorphism $A \twoheadrightarrow A_0$.

Proposition. *For any $\mathbf{Y} \in \text{Gr}_{A^\otimes}^+ \mathcal{C}$, we have that $(\text{top } \mathbf{P}^{\otimes})^\bullet(\mathbf{Y})$ is isomorphic to $\mathbf{l}^0(\mathbf{Y})$ as a left A -module and the A action commutes with the differential $\partial^{\otimes \bar{\vee}}$. In particular, $(\text{top } \mathbf{P}^{\otimes})^\bullet(\mathbf{Y})$ identifies with $(\mathbf{X}^\bullet, \partial^{\otimes \bar{\vee}}) \in \mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$, where $\mathbf{X}^j = \mathbf{l}^0(\Pi_j(\mathbf{Y}))\langle j \rangle$ and so $(\text{top } \mathbf{P}^{\otimes})^\bullet$ extends to a functor*

$$(\text{Top}^- \mathbf{P}^{\otimes})^\bullet : \mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C})) \rightarrow \mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C})).$$

Proof. We have

$$\mathbf{l}^0(\mathbf{Y}) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{k}}(H_i, \mathbf{Y}) \cong (\text{top } \mathbf{P}^{\otimes})^\bullet(\mathbf{Y}),$$

so we only need to check that the A -action on $\mathbf{l}^0(\mathbf{Y})$ commutes with the differential induced by d . For this we observe that the action of H on $(K_{H^!})_r$ which yields the A action is induced by the action of H_1 by left contractions, while the differential is given by the action of the rightmost factor of the tensor product on \mathbf{X} (cf. §3.7). The rest of the argument repeats the one in the proof of Proposition 4.2 and is omitted. \square

4.4. Now we can establish the main property of functors constructed above in Proposition 4.2 and Proposition 4.3.

Proposition. *Let \mathcal{C} be a category of left A_0 -modules satisfying (I)–(IV) for $A = A_0 \ltimes H$. Then*

$$(a) \quad (\text{Top}^- \mathbf{P}^{\otimes})^\bullet \circ (\text{Soc}^+ \mathbf{I})^\bullet \cong \mathbf{I}^\bullet \text{ as endofunctors on } \mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C})).$$

(b) $(\text{Soc}^+ \mathbb{1})^\bullet \circ (\text{Top}^- \mathbb{P}^\otimes)^\bullet \cong \mathbb{P}^{\otimes \bullet}$ as endofunctors on $\mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$.

Proof. Let \mathcal{X} denote the smallest triangulated subcategory of $\mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$ containing \mathcal{C} (which is identified with a subcategory of $\text{Gr}_A^- \mathcal{C}$ via the full exact embedding gr_0 , see §2.3) and closed under the degree shift. Then every object in $\mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$ can be considered as a limit of objects in \mathcal{X} in the usual way (by cutting increasing finite pieces both in position and gradings). Similarly, every object in $\mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$ is a limit of objects in the category \mathcal{Y} , where the latter is defined as the smallest triangulated subcategory of $\mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$ containing \mathcal{C} and closed under the degree shift. Therefore it suffices to prove both assertions for an object X in \mathcal{C} .

Using Proposition 4.2, we conclude that $(\text{Soc}^+ \mathbb{1})^\bullet(X)$ identifies with the trivial complex of $\mathbb{P}^{\otimes 0}(X)$ (cf. §2.4). Furthermore,

$$(\text{Top}^- \mathbb{P}^\otimes)^\bullet \circ (\text{Soc}^+ \mathbb{1})^\bullet(X) = (\text{top } P^\otimes)^\bullet(\mathbb{P}^{\otimes 0}(X)) = (\mathbf{Z}^\bullet, d),$$

where, by §4.2–§4.4, we have

$$\begin{aligned} \mathbf{Z}^j &= \mathbb{I}^0(\Pi_j(\mathbb{P}^{\otimes 0}(X)))\langle j \rangle = \mathbb{I}^0(A_j^\otimes \otimes_{A_0} X)\langle j \rangle = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A_0}(A_i, A_j^\otimes \otimes_{A_0} X)\langle j \rangle \\ &\cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{k}}(H_i, H_j^! \otimes_{\mathbb{k}} X)\langle j \rangle \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{k}}((H^!)^*_j \otimes_{\mathbb{k}} H_i, X) \\ &\cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathbb{k}}((K_H)_j \otimes_{\mathbb{k}} H_i, X) \cong \mathbb{I}^j(X) \end{aligned}$$

as objects in $\text{Gr}_A^- \mathcal{C}$. Furthermore, note that on \mathbf{Z}^\bullet the differential d_{-r} is given by the map $\bar{m}_{r,i}^! \otimes \text{id}_X$, where $\bar{m}_{r,i}^! : (K_{H^!})_r \otimes_{\mathbb{k}} H^!_i \rightarrow (K_{H^!})_{r-1} \otimes_{\mathbb{k}} H^!_{i+1}$ was defined in §3.6. Since $\text{Hom}_{\mathbb{k}}((K_H)_j \otimes_{\mathbb{k}} H_i, X) \cong ((K_H)_j \otimes_{\mathbb{k}} H_i)^* \otimes_{\mathbb{k}} X$ and, under that identification, $(\partial_X^+ f) \otimes x = f \circ \bar{m}_{j+1,i-1} \otimes x$, $f \in ((K_H)_j \otimes_{\mathbb{k}} H_i)^*$, it follows from Lemma 3.6 that $(\mathbb{I}^\bullet(X), \partial_X^+)$ is naturally isomorphic to (\mathbf{Z}^\bullet, d) . This proves (a).

Similarly, we identify $(\text{Top}^- \mathbb{P}^\otimes)^\bullet(X)$ with the trivial complex of $\mathbb{I}^0(X)$. Then

$$(\text{Soc}^+ \mathbb{1})^\bullet \circ (\text{Top}^- \mathbb{P}^\otimes)^\bullet(X) = (\text{soc } \mathbb{1})^\bullet(\mathbb{I}^0(X)) = (\mathbf{Y}^\bullet, \partial),$$

where

$$\begin{aligned} \mathbf{Y}^j &= (A^\otimes \otimes_{A_0} \Pi_j(\mathbb{I}^0(X)))\langle -j \rangle \cong H^! \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}}(H_j, X)\langle -j \rangle \\ &\cong H^! \otimes_{\mathbb{k}} (H_j)^* \otimes_{\mathbb{k}} X = \mathbb{P}^{\otimes -j}(X). \end{aligned}$$

It is now easy to see that the differential ∂ coincides with $\partial_X^{\otimes -}$. \square

4.5. The following theorem is the main result of this paper and provides an analogue of Koszul duality for our setting.

Theorem. *Let A_0 be a \mathbb{k} -bialgebra. Let H be a right A_0 -module algebra with $H_0 = \mathbb{k}$, which is quadratic with H_1 finite dimensional. Let $A = A_0 \ltimes H$, $A^{\otimes} = H^! \ltimes A_0^{\text{cop}}$ and let \mathcal{C} be a category of left A_0 -modules satisfying (I)–(IV). Then the following are equivalent:*

- (a) $(\text{Top}^- \mathbf{P}^{\otimes})^{\bullet} \circ (\text{Soc}^+ \mathbf{I})^{\bullet}$ is isomorphic to the identity functor on $\mathcal{K}^{\uparrow}(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$.
- (b) $(\text{Soc}^+ \mathbf{I})^{\bullet} \circ (\text{Top}^- \mathbf{P}^{\otimes})^{\bullet}$ is isomorphic to the identity functor on $\mathcal{K}^{\downarrow}(\text{Proj}(\text{Gr}_{A^{\otimes}}^+ \mathcal{C}))$.
- (c) For all X in \mathcal{C} , $X \cong \mathbf{I}^{\bullet}(X)$ in $\mathcal{D}^+(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$.
- (d) For all X in \mathcal{C} , $X \cong \mathbf{P}^{\otimes \bullet}(X)$ in $\mathcal{D}^-(\text{Proj}(\text{Gr}_{A^{\otimes}}^+ \mathcal{C}))$.
- (e) H is Koszul.

In view of this theorem, it is natural to call either of the dual properties (c) and (d) the \mathcal{C} -Koszulity. Note that, due to Lemma 2.10, we always have for X in \mathcal{C}

$$\mathbf{I}^{\bullet}(X) \in \mathcal{K}^{\uparrow}(\text{Inj}(\text{Gr}_A^- \mathcal{C})) \quad \text{and} \quad \mathbf{P}^{\otimes \bullet}(X) \in \mathcal{K}^{\downarrow}(\text{Proj}(\text{Gr}_{A^{\otimes}}^+ \mathcal{C})). \quad (4.2)$$

Proof. By Proposition 4.4, (a) is equivalent to (c) and (b) is equivalent to (d). Furthermore, identifying X with its trivial complex (cf. §2.4) we have $\mathbf{H}_0(X) = X \cong \mathbf{H}_0(\mathbf{I}^{\bullet}(X))$ by Lemma 2.10(a). Then by Lemma 2.10(d), part (c) is equivalent to the exactness of the complex $\mathbf{I}^{\bullet}(X)$. By (II)–(I), $\mathbf{I}^{\bullet}(X)$ is exact if and only if the right Koszul complex $K_H \otimes_{\mathbb{k}} H$ is exact, which by [2, Theorem 2.6.1] is equivalent to H being Koszul. Thus, (c) \iff (e). Similarly, (d) is equivalent to the exactness of $\mathbf{P}^{\otimes \bullet}(X)$ which is equivalent to the exactness of the left Koszul complex for $H^!$ which happens if and only if $H^!$ is Koszul. By [2, Proposition 2.9.1], if H is Koszul then so is $H^!$. Since in our case $(H^!)^! \cong H$, it follows that $H^!$ is Koszul if and only if H is Koszul. Thus, (d) \iff (e). \square

Note that, in the case A_0 is semi-simple, the above theorem reduces to the classical Koszul duality from [2]. The other extreme case $H = H_0 = \mathbb{k}$ provides a trivial generalization of the classical Koszul duality for a semi-simple algebra (concentrated in the zero component) to the case of a non semi-simple algebra concentrated in the zero component.

4.6. As an immediate corollary of Theorem 4.5 we obtain the following analogue of [2, Theorem 2.12.5], [16, Theorem 30] and [14, Theorem 4.3.1] for our setting.

Define $\mathcal{D}^\uparrow(\underline{\text{Inj}}(\text{Gr}_A^- \mathcal{C}))$ as the isomorphism closure of $\mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C}))$ inside the category $\mathcal{D}^\uparrow(\underline{\text{Inj}}(\text{Gr}_A^- \mathcal{C}))$. Similarly, we define $\mathcal{D}^\downarrow(\overline{\text{Proj}}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$ as the isomorphism closure of $\mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$ inside $\mathcal{D}^\downarrow(\overline{\text{Proj}}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$. By definition, we have equivalences

$$\begin{aligned} \mathcal{D}^\uparrow(\underline{\text{Inj}}(\text{Gr}_A^- \mathcal{C})) &\cong \mathcal{K}^\uparrow(\text{Inj}(\text{Gr}_A^- \mathcal{C})) \\ \mathcal{D}^\downarrow(\overline{\text{Proj}}(\text{Gr}_{A^\otimes}^+ \mathcal{C})) &\cong \mathcal{K}^\downarrow(\text{Proj}(\text{Gr}_{A^\otimes}^+ \mathcal{C})). \end{aligned}$$

Note that if \mathcal{C} is semisimple then $\text{Gr}_A^- \mathcal{C}$ and $\text{Gr}_{A^\otimes}^+ \mathcal{C}$ are abelian and $\mathcal{D}^\uparrow(\text{Gr}_A^- \mathcal{C})$, $\mathcal{D}^\downarrow(\text{Gr}_{A^\otimes}^+ \mathcal{C})$ can be defined as in [2, §2.12] and [16, §2.4].

Corollary. *Suppose that H is Koszul. Then:*

- (a) *the categories $\mathcal{D}^\uparrow(\underline{\text{Inj}}(\text{Gr}_A^- \mathcal{C}))$ and $\mathcal{D}^\downarrow(\overline{\text{Proj}}(\text{Gr}_{A^\otimes}^+ \mathcal{C}))$ are equivalent;*
- (b) *if \mathcal{C} is semisimple then $\mathcal{D}^\uparrow(\text{Gr}_A^- \mathcal{C}) \cong \mathcal{D}^\downarrow(\text{Gr}_{A^\otimes}^+ \mathcal{C})$.*

Proof. The first assertion is immediate from the definitions and [Theorem 4.5](#). The second assertion follows from [Theorem 4.5](#) and (4.2). \square

5. EXAMPLES

5.1. Let \mathfrak{g} be a semi-simple Lie algebra and V a finite dimensional \mathfrak{g} -module. Consider the generalized Takiff Lie algebra $\mathfrak{g} \ltimes V$ as in [§3.3](#). Take $\widehat{\mathcal{C}} = \mathcal{C}$ to be the (abelian) category of finite dimensional \mathfrak{g} -modules. Clearly, \mathcal{C} satisfies the conditions (I)–(IV). This is exactly the case considered in [7] (where V was the adjoint representation of \mathfrak{g}) and then in [8].

Let $H = S(V)$ which we regard as a *right* $U(\mathfrak{g})$ -module algebra. Since $U(\mathfrak{g})$ is cocommutative and $H^1 = \bigwedge V^*$ as a *left* $U(\mathfrak{g})$ -module algebra, our construction establishes a duality between a category of graded modules with graded component in \mathcal{C} over the generalized Takiff Lie algebra $\mathfrak{g} \ltimes V$ and a category of graded modules with graded components in \mathcal{C} over the Lie superalgebra $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$ with $\mathfrak{s}_0 = \mathfrak{g}$ and $\mathfrak{s}_1 = V$, see [§3.3](#) for the definition. Note that we should take V and not V^* as a \mathfrak{g} -module in the definition of \mathfrak{s} .

5.2. The previous example admits a non-trivial generalization. For the same \mathfrak{g} , V and $\mathfrak{g} \ltimes V$ as in the previous example, choose as $\widehat{\mathcal{C}}$ the Bernstein-Gelfand-Gelfand category $\mathcal{O}_{\mathfrak{g}}$ ([1]). Then \mathcal{C} can be taken to be one of the following categories

- $\text{Proj}(\mathcal{O}_{\mathfrak{g}})$;

- $\text{Inj}(\mathcal{O}_{\mathfrak{g}})$;
- the category of tilting modules in $\mathcal{O}_{\mathfrak{g}}$ (see [9] or [11, Chapter 11]).

Conditions (I) and (II) are clear for all these subcategories while conditions (III) and (IV) follow from the observation that all of them are closed under tensoring with finite dimensional \mathfrak{g} -modules.

Our results again establish Koszul duality between the category of graded modules with graded component in \mathcal{C} over $\mathfrak{g} \times V$ and the category of graded modules with graded components in \mathcal{C} over the corresponding generalized Takiff Lie superalgebra from §3.3.

5.3. Another family of examples can be obtained in the framework proposed in [4, 20]. Let A_0 be the quantized enveloping algebra $U_q(\mathfrak{g})$ and let $H = S_q(V_q)$ where V_q is a right $U_q(\mathfrak{g})$ -module and $S_q(V_q)$ is its braided symmetric algebra as defined in [4]. If $S_q(V_q)$ is a flat deformation of the symmetric algebra $S(V)$ of the $q = 1$ limit V of V_q , then it is Koszul and its quadratic dual is the braided exterior algebra of V_q^* which again is regarded as a left module. Our construction thus establishes a duality between a category of graded modules with finite dimensional graded pieces over the semidirect products of $U_q(\mathfrak{g})$ with the braided symmetric algebra and with the braided exterior algebra, respectively. It should be noted that \mathfrak{g} -modules V for which $S_q(V_q)$ is a flat deformation of $S(V)$ are rather rare. All simple modules with this property were classified in [19], and it should be noted that the adjoint representation of \mathfrak{g} is not among them. The first non-simple example of V for which $S_q(V_q)$ is flat was constructed in [3] where the corresponding semidirect product is one of the key ingredients of the construction. The relationship between generalized Takiff algebras and semidirect products $U_q(\mathfrak{g}) \times S_q(V_q)$ was studied in [20].

REFERENCES

- [1] I. N. Bernšteĭn, I. M. Gel'fand, and S. I. Gel'fand, *A certain category of \mathfrak{g} -modules*, Funkcional. Anal. i Priložen. **10** (1976), no. 2, 1–8 (Russian).
- [2] A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.
- [3] A. Berenstein and J. Greenstein, *Quantum folding*, Int. Math. Res. Not. **21** (2011), 4821–4883.
- [4] A. Berenstein and S. Zwicknagl, *Braided symmetric and exterior algebras*, Trans. Amer. Math. Soc. **360** (2008), no. 7, 3429–3472.
- [5] V. Chari, *Minimal affinizations of representations of quantum groups: the rank 2 case*, Publ. Res. Inst. Math. Sci. **31** (1995), no. 5, 873–911.
- [6] V. Chari and J. Greenstein, *Current algebras, highest weight categories and quivers*, Adv. Math. **216** (2007), no. 2, 811–840.

- [7] ———, *A family of Koszul algebras arising from finite-dimensional representations of simple Lie algebras*, Adv. Math. **220** (2009), no. 4, 1193–1221.
- [8] V. Chari, A. Khare, and T. Ridenour, *Faces of polytopes and Koszul algebras*, J. Pure Appl. Algebra **216** (2012), no. 7, 1611–1625.
- [9] D. H. Collingwood and R. S. Irving, *A decomposition theorem for certain self-dual modules in the category \mathcal{O}* , Duke Math. J. **58** (1989), no. 1, 89–102.
- [10] P. Freyd, *Representations in abelian categories*, Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, pp. 95–120.
- [11] J. E. Humphreys, *Representations of semisimple Lie algebras in the BGG category \mathcal{O}* , Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, RI, 2008.
- [12] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102.
- [13] A. N. Kirillov and N. Yu. Reshetikhin, *Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **160** (1987), 211–221.
- [14] D. Madsen, *On a common generalization of Koszul duality and tilting equivalence*, Adv. Math. **227** (2011), no. 6, 2327–2348.
- [15] V. Mazorchuk and V. Miemietz, *Cell 2-representations of finitary 2-categories*, Compos. Math. **147** (2011), no. 5, 1519–1545.
- [16] V. Mazorchuk, S. Ovsienko, and C. Stroppel, *Quadratic duals, Koszul dual functors, and applications*, Trans. Amer. Math. Soc. **361** (2009), no. 3, 1129–1172.
- [17] S. B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.
- [18] S. J. Takiff, *Rings of invariant polynomials for a class of Lie algebras*, Trans. Amer. Math. Soc. **160** (1971), 249–262.
- [19] S. Zwicknagl, *R-matrix Poisson algebras and their deformations*, Adv. Math. **220** (2009), no. 1, 1–58.
- [20] ———, *Equivariant quantizations of symmetric algebras*, J. Algebra **322** (2009), no. 12, 4247–4282.

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