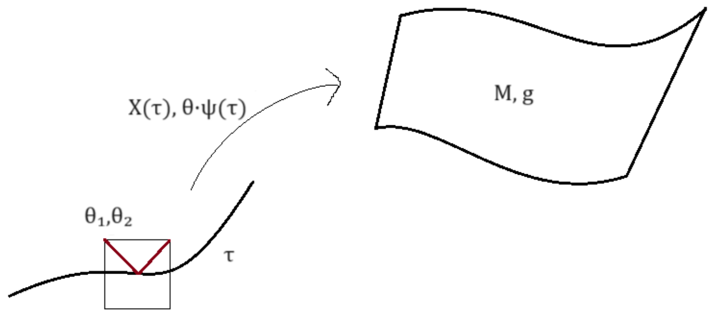
BRST of spinning particles and BV target space field theory



Spinning particles are free relativistic massless point particles moving along their $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \dots$ graded worldine $\mathbb{R}^{1|\mathcal{N}}$. We will be discussing $\mathcal{N} = 1, 2$. Their positions are coordinates for the target space manifold M:

$$\mathbb{X} : \mathbb{R}^{1|\mathcal{N}} \to M.$$

As customary in graded geometry, the structure sheaf of the graded manifold $\mathbb{R}^{1|\mathcal{N}}$ is isomorphic to

$$C^\infty(\mathbb{R}) \otimes S^\bullet(\mathbb{R}^{0|\mathcal{N}}),$$

so:

$$\mathbb{X} = \overset{e}{X^\mu}(\tau) + \overset{o}{\psi_i^\mu}(\tau)\theta^i + \dots . \tag{1}$$

We will not need higher polynomials. The superscripts denote the even or odd intrinsic parity.

The physical theory we are going to study is invariant under super-reparametrizations, $(\tau, \theta) \mapsto (\tau'(\tau), \theta'(\theta, \tau))$ that do not mix the parity; equivalently, we shall call them *super-diffeomorphisms* of the line. The action functional, upon Berezinian integration[1], is [Brink-Di Vecchia-Howe '76] [Sorokin review '00]

$$S_{\mathcal{N}=1}[X, \psi, e, \chi, P] = \int_{\mathbb{R}^1} P_\mu \mathrm{d}X^\mu + \psi_\mu \mathrm{d}\psi^\mu - \big(e\frac{P^2}{2} + \chi\psi^\mu P_\mu\big)\mathrm{d}\tau\,. \tag{2}$$

Therefore e, χ are Lagrange multipliers for the constraints $P^2 = 0 = \psi \cdot P$.

BRST:

- In the action above there is implicit use of a canonical symplectic structure on

  $$\underset{X,\psi}{M} \cong \underset{X,\mathrm{d}X}{T^*N}$$

  given by the Poisson brackets

  $$\{X^\mu, P_\nu\} = \delta^\mu_\nu, \quad \{\psi_i^\mu, \psi_j^\nu\} = 2\delta_{ij} g^{\mu\nu} = \{\psi_j^\nu, \psi_i^\mu\}.$$

- Thus the action of $Diff(\mathbb{R}^{1|\mathcal{N}})$ lifted to M is Hamiltonian. It produces the Hamiltonian functions

  $$P^2, \quad \psi_i \cdot P.$$

- Moreover, the Hamiltonian vector fields give rise to a map $\Phi : M \to Lie\,Diff(\mathbb{R}^{1|\mathcal{N}})^*$ and the action of the Lie algebra on its dual and M is equivariant:

  $$\begin{aligned}\{\Phi_g, \Phi_h\} &= \Phi_{[g,h]},\\ \{\psi_i \cdot P, \psi_j \cdot P\} &= 2\delta_{ij}P^2.\end{aligned} \tag{3}$$

- We are interested in the level set of the moment map $\Phi^{-1}(0)$, although 0 is not a regular value. We believe that this issue is solved by doing canonical quantization (see later).

[1] indeed the right object is

$$S_{\mathcal{N}=1} = -\int \mathrm{d}\tau \mathrm{d}\theta \frac{D\mathbb{X}\partial_\tau \mathbb{X}}{2\mathbb{E}}$$

where D is the superderivative of the line and $\mathbb{E}$ is the supereinbein.

- In the ideal situation that 0 is regular, then, $\Phi^{-1}(0)/Diff \equiv M//Diff$ inherits a symplectic structure [Marsden-Weinstein, symplectic reduction, Kirillov-Kostant-Soriau for coadjoint orbits].
- Cranking the *Koszul resolution*, we can know the invariant functions $C^\infty(\Phi^{-1}(0))^{Diff}$.
- For the invariant functions on M, we can rely on the fundamental theorem of BRST:

$$C^\infty(M//Diff) \cong H^0_Q(\mathcal{C}). \tag{4}$$

- We shall explain the RHS in the above statement, adapted to the case of super-reparametrization invariance of the line.
  - The cochains are

$$\mathcal{C}^{p,q} := \underset{c,\gamma,\bar{\gamma}}{S^p LieDiff^*[1]} \otimes \underset{b,\beta,\bar{\beta}}{S^q LieDiff[-1]} \otimes C^\infty(M).$$

    However the right cohomological degree is $n = p - q$, the ghost number (obtained by subtracting the number of antighosts to the number of ghosts).
  - The BRST charge is

$$Q^{\mathcal{N}=1} = cP^2 + \gamma\psi \cdot P - \gamma^2 b (= \Phi^{-1}(0)_i e^i + f_{ij}{}^k e^i e^j \hat{e}_k)$$

    and by Poisson action (adjoint action via Poisson brackets) it increases the ghost number, as expected from a coboundary operator:

$$Q : \mathcal{C}^n \to \mathcal{C}^{n+1}.$$

Canonical quantization. We are interested to move on to the first quantized setting, where Poisson brackets on the smooth functions are replaced by commutators of operators of a Hilbert/Fock space:

$$(\{-,-\}, C^\infty(M)) \Longrightarrow \big(-\frac{\mathrm{i}}{\hbar}[-,-], \mathcal{O} : \mathcal{H} \to \mathcal{H}\big).$$

A *polarization* must be chosen (correspondingly, a Lagrangian submanifold will be singled out). For our "ground state" $|0\rangle$ we will always set

$$P_\mu |0\rangle = 0 = \psi^{\mu_{1/2}} |0\rangle, \quad b|0\rangle = 0,$$

(where the notation 1/2 in superscript denotes the fact that half of the ψ's become annihilators) but for the Weyl algebra of the γ, β system we keep our options open. This gives rise to an extra filtration labeled by the *picture number*.

1. $c^{p_1}\gamma^{p_2}\beta^q F(X,\psi) \in \mathcal{C}^{p,q}_{pic=0}$ with $p = p_1 + p_2$, which corresponds to the polarization $\bar{\beta}|0\rangle = 0 = \bar{\gamma}|0\rangle$;
2. $c^p \delta^{(q_1)}(\gamma)\beta^{q_2} F(X,\psi) \in \mathcal{C}^{p,q}_{pic=1}$ with $q = q_1 + q_2$, for the polarization $\gamma|0\rangle = 0 = \bar{\gamma}|0\rangle$. Note that in this representation the polynomials in γ have been traded for (derivatives of) the Dirac distribution in the same variable [Belopolsky] [Castellani-Catenacci-Grassi];
3. $c^{p_1}\delta^q(\gamma)\delta^{(p_2)}(\beta)F(X,\psi) \in \mathcal{C}^{p,q}_{pic=2}$ with $p = p_1 + p_2$ and the polarization given by $\gamma|0\rangle = 0 = \beta|0\rangle$.

We have been able to show the following [B.-Grassi-Hulík-Sachs]:

Result 1 ($\mathcal{N} = 1$ BRST cohomology)**.** *In the* superform *case, noting that*

$$Q^{\mathcal{N}=1} = c\Box + \gamma(\mathrm{d} + \mathrm{d}^\dagger) - \gamma^2 \partial_c,$$

the cohomology is ($n \in \mathbb{N}^*$)

$$H^n_{pic=0}(\mathcal{C}) \begin{cases} \neq \emptyset, \; n = 0, 1, \\ = \emptyset, \; otherwise. \end{cases} \tag{5}$$

When there is some cohomology, the latter is that of $\mathrm{d}+\mathrm{d}^\dagger$ *closed* ***covariant*** *multiforms(=functions of M) which are not* $\mathrm{d} + \mathrm{d}^\dagger$ *exact, or equivalently it corresponds to Dirac spinors. Furthermore,* $H^n_{pic=0}(\mathcal{C}) = H^n_{pic=1}(\mathcal{C})$.

For results on the cohomology at negative ghost degree see [Getzler '15]. Moreover, we cross-checked our result using the Hilbert-Poincaré series. On our Fock space V_l graded by the ghost degree (thus finite dimensional at each l), the Hilbert-Poincaré series reads:

$$\mathbb{P}(q,s) := \sum (-1)^l \mathrm{dim} V_{k,l}\, q^k s^l, \quad s \text{ cohomological ''fugacity''}.$$

Fact: $\mathbb{P}(r,s) = (-1)^l b_{k,l}\, q^k s^l$ where b are the Betti numbers of the ring. Therefore, this computes a partition function. Then, with reference to (1):

	ψ	γ	c
ghost number s	0	1	1
"scaling" number q	1	-1	-2
	q	sq^{-1}	sq^{-2}

Table 1: Assignments of fugacities to our algebra.

$$\begin{aligned} \mathbb{P}^{\mathcal{N}=1}(q,s) &= \frac{(1-sq^{-2})(1+q)^{D/2}}{1+sq^{-1}} \\ &\overset{s=1}{=} (1-q^{-1})(1+q)^{D/2} \end{aligned} \tag{6}$$

counts the number of d.o.f.'s of the two sets of $D/2$-multiforms that are in BRST cohomology (ghost degree 0 and 1). Note that from the get-go s is twisted by a negative sign, which amounts to have a twisted partition function, and then set to the unit for clarity of exposition.

Result 2 ($\mathcal{N}=2$ BRST cohomology)**.** *Focusing on the picture zero sector,*

$$Q^{\mathcal{N}=2} = c\Box + \partial_\beta \mathrm{d} + \gamma \mathrm{d}^\dagger - \gamma\partial_\beta\partial_c,$$

and noting that there is a $U(1)$-charge R

$$R = \psi\cdot\partial_\psi + \gamma\partial_\gamma + \beta\partial_\beta, \quad [Q,R]=0,$$

which further filters $\mathcal{C}^n_{pic=0} = \bigoplus_r \mathcal{C}^n_{pic=0,r}$, the cohomology is:

$$H^0_{pic=0,r}(\mathcal{C}) \begin{cases} \textit{Klein-Gordon}, \ r=0, \\ \textit{Maxwell (with auxiliary field)}, \quad r=1 \textit{ (see also [Dai-Huang-Siegel])}, \\ \textit{EM with higher forms}, \ r>1 \end{cases}$$

For picture 2 there holds $H^0_{pic=2,r'}(\mathcal{C}) = H^0_{pic=0,r}(\mathcal{C})$ (possibly after some shift of r by ± 1). This can be seen from two concurring arguments:

- By applying a Picture Changing Operator Y, defined to be:

  $$Y : \mathcal{C}^n_{pic=0,r} \to \mathcal{C}^{n'}_{pic=1,r'}, \quad [Q,Y]=0, \quad Y \neq [Q,-]\,.$$

  In fact, since Y is a cocycle, it does not affect the cohomology;

- By applying a Hodge star operator defined by

  $$\star : \mathcal{C}^n_{pic=0,r} \xrightarrow{\sim} \mathcal{C}^{n'}_{pic=2,r'}.$$

  Several options for the isomorphism $\star$ are available. We could find one for which $\star Q^{\mathcal{N}=2}\star = Q^{\mathcal{N}=2}$.

The same results hold for the cohomology at picture 1. However, in picture 1 there are sectors inaccessible by PCOs, namely those with negative $r < -1$. Then, $H^0_{pic=1,r<-1}(\mathcal{C})$ is compatible with *Chern-Simons* (flat connections).

BV in target space. We shall now see how BRST cohomology lends a free BV field theory in the target space M. Then we will explain how to obtain an interacting one. This enhancement is known already for bosonic strings [Zwiebach, Witten] and partially for superstrings [Sen], building on a BV structure on the moduli space of punctured Riemann surfaces. Here, with worldlines, the multiproducts of a homotopy algebra are instead not granted to exist, and if they do, are they connected by L_∞ morphisms to those of YM theory? Recall that

$$\mu_3(A, A, A) \propto [A, *[A, A]_{\mathfrak{su}(n)}]_{\mathfrak{su}(n)}$$

is the highest multiproduct in YM theory.

Observations:

- Taking for concreteness the cochains $\mathcal{C}_1^n$ in picture 0, note that

$$\underset{C(X)\beta|0\rangle}{(\mathcal{C}^{-1})} \overset{Q}{\to} \underset{\begin{pmatrix} A_\mu(X)\psi^\mu\,|0\rangle \\ \phi(X)c\beta\,|0\rangle \end{pmatrix}}{\mathcal{C}^0} \overset{Q}{\to} \underset{\begin{pmatrix} A^*_\mu(X)\psi^\mu\,|0\rangle \\ \phi^*(X)\gamma\,|0\rangle \end{pmatrix}}{\mathcal{C}^1} \overset{Q}{\to} \underset{C^*(X)c\gamma|0\rangle}{(\mathcal{C}^2)} \,. \tag{7}$$

  This is a cochain complex for Maxwell/Yang-Mills in BV. We already explained what the cohomology in ghost degree zero is.

- There is a natural BV-pairing:

$$\int_{T^*(N\times LieDiff)} \langle 0|\,(\star\omega)\,\omega\,|0\rangle\,, \quad \omega \in \mathcal{C}_1^n \tag{8}$$

  The integrand is the right object to integrate: a picture 2 top form of $T^*N = M$. Furthermore, Q is self-adjoint w.r.t. the BV pairing.

Fact: A BV formulation of free YM (EM) is at hand:

$$\int_{T^*(N\times LieDiff)} \langle 0|\,(\star\omega)\,Q\omega\,|0\rangle = S^{BV}_{EM}[A, \phi, C, A^*, \phi^*, \cancel{C^*}]. \tag{9}$$

In [B.-Grassi-Hulík-Sachs] we presented 2 options for the interacting theory:

1. Promote $Q \rightsquigarrow Q(\omega) = Q_0 + Q_1(\omega)$ with the understanding that $Q(\omega)\beta\,|0\rangle = \omega\,|0\rangle$. This operator-state correspondence map is just a surjection though.

$$\begin{aligned} Q(\omega) = &- c\left(p^2 + p\cdot B + B\cdot p - G_{\mu\nu}\psi^\mu\bar{\psi}^\nu - \tilde{\phi}\right) + \gamma\Pi\cdot\bar{\psi} + \bar{\gamma}\Pi\cdot\psi + C \\ &- c\bar{\gamma}\psi\cdot A^* + c\gamma\bar{\psi}\cdot A^* + \gamma\bar{\gamma}\phi^* + c\gamma\bar{\gamma}C^* + \gamma\bar{\gamma}b\,, \end{aligned} \tag{10}$$

   where $\Pi_\mu = p_\mu + A_\mu$ and $\tilde{\phi} = \phi + [p, B]$. The "background fields" B and $G_{\mu\nu}$ do not correspond to a state through the operator-state correspondence map. We have refrained from substituting p with the corresponding partial derivative for clarity of exposition; however this step must be performed. Then an *associative, Q_0-compatible* 2-product can be defined as

$$\mu_2(\omega_1, \omega_2) = \frac{1}{2}[Q(\omega_1), Q(\omega_2)].$$

   Eventually $S_{free+int}$ is given by:

$$S_{free+int}[\omega] = \int_{T^*(N\times LieDiff)} \langle 0|\,(\star\beta Q(\omega))\,\big(\frac{1}{2}Q_0 + \frac{1}{3!}Q_1(\omega)\big)Q(\omega)\beta\,|0\rangle\,. \tag{11}$$

   Result 3. *$S_{free+int}$ is an equivalent action to YM in BV formulation: $\frac{\delta S_{free+int}}{\delta\omega} = 0 \iff$ BV YM e.o.m.'s hold.*

   See also [Meyer-Grigoriev-Sachs].

2. Another solution is the following:

$$\begin{pmatrix} \textit{assume an homotopy} \\ \textit{algebra on worldlines} \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} \textit{knowledge of the } L_3 \\ \textit{structure of } YM \end{pmatrix}$$

   In our article we have explicit formulas for the L_∞-morphisms matching the two sides.

Comment: In an upcoming preprint we have worked out an interacting BV theory for $\mathcal{N} = 1$ (Dirac spinors/de Rham closed and co-closed multiforms).